Sets and Numbers

A set is a collection of distinct objects called elements of the set. Two sets are equal if they have the same elements. We write $x \in A$ to indicate that x is an element of A. The sets of all natural numbers, integers, rational numbers and real numbers are denoted as follows:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$\mathbb{Z} = \{\ldots - 3, -2, -1, 0, 1, 2, 3, \ldots\}$$
$$\mathbb{Q} = \{\tfrac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}$$
$$\mathbb{R} = \{\text{infinite decimal expansions}\}$$

For sets A, B we say that A is a subset of B if every element of A is also an element of B and write $A \subseteq B$. The union and intersection of two sets A and B are defined as

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\} \quad \text{and} \quad A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

respectively.

Problem 1 Write down all elements in the following sets.

(a) $\{x \in \mathbb{Z} \mid 3 \leq x \leq 9\}$

(b) $\{y \in \mathbb{N} \mid y + 2 < 8\}$

(c) $\{q \in \mathbb{Q} \mid 2q \in \mathbb{N} \text{ and } q < 3\}$

(d) $\{a \in \mathbb{N} \mid a = 2k + 1 \text{ for some } k \in \{0, 1, 2, 3\}\}$

(e) $\{x \in \mathbb{R} \mid x(x^2 - 1)(x^2 + 1) = 0\}$

(f) $\{n \in \mathbb{N} \mid n^2 \leq 9\}$

(g) $\{n \in \mathbb{Z} \mid n^2 \leq 9\}$

(h) $\{n \in \mathbb{Z} \mid 4 \leq n^2 \leq 25\}$

(i) $\{0, 1, 2, 3, 4\} \cup \{k \in \mathbb{N} \mid 4 \leq k \leq 8\}$

(j) $\{x \in \mathbb{R} \mid x^3 \leq 8\} \cap \mathbb{N}$

(k) $\{x \in \mathbb{R} \mid x(x^2 - 1) = 0\} \cup \{n \in \mathbb{Z} \mid n^2 \leq 4\}$

(l) $\{x \in \mathbb{R} \mid x(x^2 - 1) = 0\} \cap \{n \in \mathbb{Z} \mid n^2 \leq 4\}$

(m) $\{A \mid A \subseteq \{1, 2, 3\}\}$

Problem 2 Determine if either of the sets A and B is a subset of the other.

(a) $A = \{5, 6, 7\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$

(b) $A = \{-14, -1, 3, 7\}$ and $B = \mathbb{N}$

(c) $A = \{x \in \mathbb{N} \mid x^2 \leq 1000\}$ and $B = \{1, 2, 3\}$

(d) $A = \{x \in \mathbb{R} \mid -10 \leq x \leq 10\}$ and $B = \{x \in \mathbb{R} \mid x^2 \leq 10\}$

(e) $A = \{x \in \mathbb{R} \mid x(x^2 - 1) = 0\}$ and $B = \{n \in \mathbb{Z} \mid n^2 \leq 4\}$

(f) $A = \{x \in \mathbb{R} \mid x(x^2 - 1) = 0\}$ and $B = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\} \cap \mathbb{Z}$

Problem 3 Write the following subsets of $\mathbb{R}$ as an interval.

(a) $\{x \in \mathbb{R} \mid x^2 \leq 4\}$

(b) $[2, 5] \cup [4, 14]$

(c) $[-1, 3] \cup [-10, 10]$

(d) $[2, 5] \cap [4, 10)$

(e) $(-1, 3) \cap [-10, 10]$

(f) $[5, 10) \cap (-2, 9]$

Problem 4 Prove the following statements for all sets A, B and C.

(a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(b) $A \subseteq B$ if and only if $A \cap B = A$.

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.