Analytic Geometry

A *matrix* is a rectangular array of real numbers. If the array has m rows and n columns we call it an $m \times n$ matrix. A matrix consisting of a single column is called a column vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

It represents an arrow pointing from the origin to the point with coordinates $(v_1, v_2, \ldots, v_n)$ in n-dimensional space. Given another vector

$$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix},$$

and a real number c we define new vectors

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix} \text{ and } c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$

Moreover, we define the dot product of $\vec{v}$ and $\vec{w}$ to be the real number

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

We can capture the geometric concepts of lengths and angles using this operation. Firstly the length of $\vec{v}$ is defined to be

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}.$$

Secondly, the angle between two non-zero vectors $\vec{v}$ and $\vec{w}$ is the angle θ which satisfies

$$\vec{v} \cdot \vec{w} = ||\vec{v}|| \cos\theta ||\vec{w}||.$$

Problem 1 Calculate

(a) the vector $2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) the length of $2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) the angle between $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 2 Find all vectors

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

that point at the intersection of the following three planes.

$$\begin{aligned} 2x - 3y + 5z &= 4, \\ -4x + 5y - 7z &= -10, \\ 3x - 4y + 6z &= 7. \end{aligned}$$

Problem 3 Consider the following two lines ℓ_1 and ℓ_2 given by the vectors

$$\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

respectively (where s and t vary over all real numbers). Find all vectors that point from the origin to a point that is both on ℓ_1 and ℓ_2. Are the lines parallel? Do the lines intersect?

Problem 4 A box has dimensions 20 cm, 30 cm, and 60 cm. What is the longest distance between any two points on the box. Hint: Find a vector that point from one corner of the box to the opposite corner.

Problem 5 Find the shortest distance from the point $(4, -2, 7)$ to the plane

$$x + y + 3z = 1.$$

Problem 6 Find the shortest distance between the lines given by

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

Problem 7 Two objects with masses $m_1 = 3$ and $m_2 = 2$ (in kg) collide at velocities

$$\vec{v_1} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \text{ and } \vec{v_2} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \text{ (in m/s)}.$$

After collision the objects stick together. At what velocity are the objects traveling after the collision.