Determinants

For basic definitions about determinants see Bretscher chapter 6.

Problem 1 Find the determinants of the following matrices. In each case determine if the matrix is invertible or not.

(a) $A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$

(e) $A = \begin{bmatrix} 3 & 5 & 1 \\ 2 & 1 & -1 \\ 1 & 4 & 2 \end{bmatrix}$

(f) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 6 \\ 1 & 1 & 1 & 1 \\ 9 & 1 & 7 & 8 \end{bmatrix}$

Problem 2 Find all real numbers x such that the following matrix is invertible.

(a) $A = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1-x & 3 & 4 \\ 0 & 2-x & x^2 \\ 0 & 0 & 4-x \end{bmatrix}$

(c) $A = \begin{bmatrix} 1-x & 1 & 1 \\ 1 & 1-x & 1 \\ 1 & 1 & 1-x \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & x & 2 \\ 1 & 3 & 5 \\ 2 & 6 & x \end{bmatrix}$

Problem 3 For every positive integer n define the $n \times n$-matrix

$$A_n = \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots & \vdots \\ 0 & 1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 2 \end{bmatrix}$$

(a) Find the determinants of

$$A_1 = \begin{bmatrix} 2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \; A_3 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \; A_4 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(b) Show that

$$\det A_n = n + 1.$$

Hint: let $a_n = \det A_n$ and show that $a_n - a_{n-1} = 1$ for all $n \geq 2$ using Laplace expansion.