ASSESSED COURSEWORK 2

Mathematics Tutorial I
Nagoya University
G30 Program, Fall 2012
Deadline: November 27, 14:45

Solutions should contain detailed arguments for all statements made. Each problem gives a maximum of 5 points. Hand in at the start of the tutorial class on November 27.

Exercise 1. Find d^2y/dx^2 in terms of x and y, if $2x^3 - 3y^2 = 8$.

Solution: Differentiate(i.e. apply $\frac{d}{dx}$ to) both sides, and we will get

$$6x^2 - 6y \cdot \frac{dy}{dx} = 0, \qquad (*)$$

which means

$$\frac{dy}{dx} = \frac{x^2}{y}.$$

Differentiate (*) again, and then we know

$$12x - 6(\frac{dy}{dx} \cdot \frac{dy}{dx} + y \cdot \frac{d^2y}{dx^2}) = 0.$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{1}{y}(2x - \frac{x^4}{y^2}).$$

Exercise 2. Find all θ such that $0 \leq \theta < 2\pi$ and

$$\cos^2\theta = \sin^2\theta.$$

Solution: There are two cases: 1. $\cos\theta = \sin\theta$ and 2. $\cos\theta = -\sin\theta$.

For case 1: $\tan\theta = 1 (0 \leq \theta < 2\pi)$, therefore

$$\theta = \frac{\pi}{4} \ \ or \ \ \frac{5\pi}{4}.$$

For case 2: $\tan\theta = -1 (0 \leq \theta < 2\pi)$, therefore

$$\theta = \frac{3\pi}{4} \ \ or \ \ \frac{7\pi}{4}.$$

So, that's all:

$$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

Exercise 3.

(a) Find the derivative of $(\cos x)^4$.
(b) Find the derivative of $\sin(x^2 + 1)$.
(c) Find all real numbers a such that the function $f(x) = \cos(ax)$ satisfies

$$f''(x) + 4f(x) = 0 \qquad (*)$$

for all $x \in \mathbb{R}$.

Solution:

(a)

$$\frac{d}{dx}\big((\cos x)^4\big) = 4(\cos x)^3 \cdot (-\sin x) = -4(\cos x)^3 \cdot \sin x.$$

(b)

$$\frac{d}{dx}\big(\sin(x^2 + 1)\big) = \cos(x^2 + 1) \cdot \frac{d}{dx}(x^2 + 1) = \cos(x^2 + 1) \cdot 2x.$$

(c)

$$f'(x) = -\sin(ax) \cdot a.$$

Then differentiate both sides:

$$f''(x) = -\cos(ax) \cdot a^2.$$

Substitute it into the above equation (*), and we will get

$$0 = -\cos(ax) \cdot a^2 + 4 \cdot \cos(ax) = \cos(ax) \cdot (4 - a^2).$$

Note that this equation holds for all $x \in \mathbb{R}$, which means $a = 2$ or -2.

Exercise 4. Solve the equation

$$\ln(x) - \ln(x - 1) = 1.$$

Solution:

$$\ln(x) - \ln(x - 1) = \ln \frac{x}{x - 1} = 1.$$

Take the exponential to both sides, we know

$$e^{(\ln \frac{x}{x-1})} = \frac{x}{x - 1} = e.$$

Therefore, we get

$$x = \frac{e}{e - 1}.$$

Exercise 5. Using L'Hospital's rule, prove that

$$\lim_{x \to 0} (1+x)^{1/x} = e.$$

Démonstration:

First, it is obvious that

$$(1+x)^{\frac{1}{x}} = (e^{\ln(1+x)})^{\frac{1}{x}} = e^{\frac{1}{x} \cdot \ln(1+x)} = e^{\frac{\ln(1+x)}{x}}$$

And we know, by L'Hospital's rule, since it is $\frac{0}{0}$-type,

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{1}{1+x} = 1.$$

Therefore,

$$\lim_{x \to 0} (1+x)^{1/x} = \lim_{x \to 0} e^{\frac{\ln(1+x)}{x}} = e^{\lim_{x \to 0} \frac{\ln(1+x)}{x}} = e.$$