ASSESSED COURSEWORK 1

Mathematics Tutorial I
Nagoya University
G30 Program, Fall 2012
Deadline: November 6, 14:45

Solutions should contain detailed arguments for all statements made. Each problem gives a maximum of 5 points. Hand in at the start of the tutorial class on November 6.

Exercise 1.

(a) Find the domain of $y = \frac{x+3}{4-\sqrt{x^2-9}}$.
(b) Find the range of $y = 2 + \frac{x^2}{x^2+4}$.

Solution (a) Notice that the part under root should be no less than 0 and the denominator can not be 0, so we need to solve

$$x^2 - 9 \geq 0, 4 - \sqrt{x^2-9} \neq 0.$$

Then we get the domain is $(-\infty, -5) \cup (-5, -3] \cup [3, 5) \cup (5, +\infty)$.

(b) The function can be rewritten as

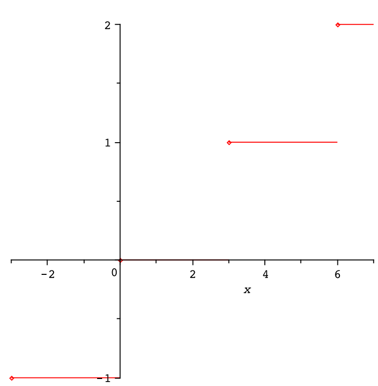
$$y = 2 + \frac{x^2}{x^2+4} = 2 + \frac{x^2+4-4}{x^2+4} = 3 - \frac{4}{x^2+4}.$$

From $x^2 \geq 0$, we know $x^2 + 4 \geq 4$ and then $0 < \frac{1}{x^2+4} \leq \frac{1}{4}$. So the range is $[2, 3)$.

Exercise 2. Let $g(x) = [[x/3]]$, where $[[x]]$ is the largest integer that is less than or equal to x.

(a) Sketch the graph of g.
(b) Evaluate each of the following limits if it exists and if does not, explain why: $\lim_{x\to 1} g(x)$; $\lim_{x\to 2} g(x)$; $\lim_{x\to 3} g(x)$.
(c) For what values of a does $\lim_{x\to a} g(x)$ exist?

Solution (a) The graph of g between $[-3, 7]$ is



Those endpoints with circles mean that they are contained in the line, otherwise not.

(b) Because the left and right limits at $x = 1$ and $x = 2$ exit and equal, i.e.,

$$\lim_{x\to 1-} g(x) = \lim_{x\to 1+} g(x) = 0, \ \lim_{x\to 2-} g(x) = \lim_{x\to 2+} g(x) = 0,$$

$\lim_{x\to 1} g(x)$ and $\lim_{x\to 2} g(x)$ exist and equal to 0. But at the point $x = 3$,

$$\lim_{x\to 3-} g(x) = 0 \neq 1 = \lim_{x\to 3+} g(x).$$

Therefore, $\lim_{x\to 3} g(x)$ does not exist.

(c) Similarly to the part (b), it is easy to show that at those points $\{3k \mid k \in \mathbb{Z}\}$, the limit $\lim_{x\to a} g(x)$ does not exist. In other words, for $a \in \mathbb{R}, a \neq 3k, k \in \mathbb{Z}$, the limit exists.

Exercise 3. Write down all elements in the following sets.

(a) $\{1, 2, 3, 4, 5\} \cap \{x \in \mathbb{Z} \mid x^2 \geq 9\}$
(b) $\{r \in \mathbb{Q} \mid 3r \in \mathbb{Z} \text{ and } 1 < r < 3\}$
(c) $\{n \in \mathbb{Z} \mid n = k^2 \text{ for some } k \in \{0, 1, 2, 3\}\}$
(d) $\{y \in \mathbb{Z} \mid (y - 3)^2 \leq 4\}$
(e) $\{(x, y) \mid x, y \in \mathbb{Z} \text{ and } 1 \leq x \leq y \leq 4\}$

Solution (a) $\{1, 2, 3, 4, 5\} \cap \{x \in \mathbb{Z} \mid x^2 \geq 9\} = \{1, 2, 3, 4, 5\} \cap \{x \in \mathbb{Z} \mid x \leq -3 \text{ or } x \geq 3\} = \{3, 4, 5\}$.

(b) $\{r \in \mathbb{Q} \mid 3r \in \mathbb{Z} \text{ and } 1 < r < 3\} = \{r \in \mathbb{Q} \mid 3r \in \mathbb{Z} \text{ and } 3 < 3r < 9\} = \{r \in \mathbb{Q} \mid 3r \in \{4, 5, 6, 7, 8\}\} = \{\frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}\}$.

(c) $\{n \in \mathbb{Z} \mid n = k^2 \text{ for some } k \in \{0, 1, 2, 3\}\} = \{0, 1, 4, 9\}$.

(d) $\{y \in \mathbb{Z} \mid (y - 3)^2 \leq 4\} = \{y \in \mathbb{Z} \mid -2 \leq y - 3 \leq 2\} = \{y \in \mathbb{Z} \mid 1 \leq y \leq 5\} = \{1, 2, 3, 4, 5\}$.

(e) $\{(x, y) \mid x, y \in \mathbb{Z} \text{ and } 1 \leq x \leq y \leq 4\}$
$= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$.

Exercise 4. Find the point on the plane given by the equation

$$x + 2y + z = 1,$$

which is closest to the point $(5, 2, 4)$.

Solution From the equation of the plane, we know its normal direction is $(1, 2, 1)$. So the vertical to the plane through the point $(5, 2, 4)$ can be written as

$$\begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, t \in \mathbb{R}.$$

In order to find its intersection point with the plane, we just put the above equation into that of the plane, which means that we solve

$$(5 + t) + 2(2 + 2t) + (4 + t) = 1.$$

Then we get $t = -2$. Therefore, the intersection point is $(3, -2, 2)$.
Remark Later we will learn an analytic method with the help of so-

called partial derivatives, similarly to Example 3 on Page 325 of the textbook.

Exercise 5. Find the following limits

(a) $\lim_{x\to\infty} \sqrt{x^2+1} - \sqrt{x^2-1}$

(b) $\lim_{x\to\infty} \frac{1-\sqrt{x}}{1+\sqrt{x}}$

(c) Using the ϵ, δ-definition of a limit, prove that

$$\lim_{x\to -5}(4 - \frac{3x}{5}) = 7.$$

Solution (a)

$$\begin{aligned} & \lim_{x\to\infty} \sqrt{x^2+1} - \sqrt{x^2-1} \\ = & \lim_{x\to\infty} \frac{(\sqrt{x^2+1} - \sqrt{x^2-1})(\sqrt{x^2+1} + \sqrt{x^2-1})}{\sqrt{x^2+1} + \sqrt{x^2-1}} \\ = & \lim_{x\to\infty} \frac{(x^2+1)-(x^2-1)}{\sqrt{x^2+1} + \sqrt{x^2-1}} = \lim_{x\to\infty} \frac{2}{\sqrt{x^2+1} + \sqrt{x^2-1}} = 0. \end{aligned}$$

(b)

$$\lim_{x\to\infty} \frac{1-\sqrt{x}}{1+\sqrt{x}} = \lim_{x\to\infty} \frac{2-(1+\sqrt{x})}{1+\sqrt{x}} = \lim_{x\to\infty} \frac{2}{1+\sqrt{x}} - 1 = -1.$$

(c) Let ϵ be a given positive number. We want to find a number δ such that

$$\text{if } 0 < |x-(-5)| < \delta, \text{ then } |4 - \frac{3x}{5} - 7| < \epsilon.$$

From $|4 - \frac{3x}{5} - 7| = |\frac{3x}{5} + 3| = \frac{3}{5}|x+5|$, we find that if we want $\frac{3}{5}|x+5| < \epsilon$, it is enough that $|x+5| < \frac{5}{3}\epsilon$. So we can choose $\delta = \frac{5}{3}\epsilon$.