Comments on Calculus I: Precise Definition of a Limit

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One first year student asked me about the δ-ε argument in the last remedial lecture. I didn't have time to examine this problem with her. Now I would like to write down some comments about this. This is one of the most difficult concepts in Calculus I.

Definition of a Limit says that

Suppose that $f(x)$ is defined for all x in an open interval containing a (but not necessarily at $x = a$). Then

$$\lim_{x\to a} f(x) = L$$

if for all $\varepsilon > 0$, there exits $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{if} \quad 0 < |x-a| < \delta$$

Comment:

Although this definition says that for all $\varepsilon > 0$, there exits $\delta > 0$ such that $|f(x)-L| < \varepsilon$ if $0 < |x-a| < \delta$, in many cases it is sufficient to prove this for a preset number of $\varepsilon > 0$.

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Example 1: Let

$f(x) = 2x+3$, (1-1)

prove that

$$\lim_{x\to 1} f(x) = 5 \qquad (1\text{-}2)$$

using the precise definition of a limit.

Solution

Step 1: For simplicity's sake, suppose that

$0 < \varepsilon < 0.1$. (1-3)

(In this case it turns out that we don't have to limit the range of ε. You will see later.)

We want to obtain δ *as a function of* ε such that

$|f(x) - 5| < \varepsilon$. (1-4)

And

$0 < |x-1| < \delta$ (1-5)

Step 2: From (1-1) and (1-4) we get

$|2x+3-5| < \varepsilon$ (1-6)

$|2x-2| < \varepsilon$

$|2(x-1)| < \varepsilon$

We finally get

$|x-1| < \varepsilon/2$ (1-7)

$\therefore\ \delta = \varepsilon/2$ (1-8)

QED

Comment: In this case the functional form is a first order equation of x, a δ value is very easily obtained. However the next case is a little bit more complicated.

Example 2: Let

$f(x) = 2x^2-3$, (2-1)

prove that

$$\lim_{x\to 1} f(x) = -1 \qquad (2\text{-}2)$$

using the precise definition of a limit.

Solution:

Step 1: For simplicity's sake, suppose that

$0 < \varepsilon < 1$ (2-3)

We want to obtain δ *as a function of* ε such that

$|f(x) - (-1)| < \varepsilon$. (2-4)

and

$0 < |x-1| < \delta$ (2-5)

From (2-1) and (2-4) we get

$$\left|2x^2-3+1\right|<\varepsilon \quad (2\text{-}6)$$

$$\left|2x^2-2\right|<\varepsilon$$

$$\left|(x^2-1)\right|<\varepsilon/2 \quad (2\text{-}7)$$

Case 1: $x^2-1\geq 0$, (2-8)

that is, $x\geq 1$ or $x\leq -1$,

(2-7) gives us

$$0\leq x^2-1<\varepsilon/2 \quad (2\text{-}9)$$

Considering the inequality (2-8), we obtain

$$1\leq x^2<1+\varepsilon/2 \quad (2\text{-}10)$$

By taking the square roots of all terms (assuming $x>0$)

$$1\leq x<\sqrt{1+\varepsilon/2} \quad (2\text{-}11)$$

By subtracting 1 from all terms, we get

$$0\leq x-1<\sqrt{1+\varepsilon/2}-1 \quad (2\text{-}12)$$

$$0\leq x-1<\frac{(\sqrt{1+\varepsilon/2}-1)(\sqrt{1+\varepsilon/2}+1)}{\sqrt{1+\varepsilon/2}+1} \quad (2\text{-}13)$$

$$0\leq x-1<\frac{\varepsilon/2}{\sqrt{1+\varepsilon/2}+1} \quad (2\text{-}14)$$

Case 1: $x^2-1<0$, (2-15)

that is,

$$-1<x<1 \quad (2\text{-}16)$$

then

$$0<1-x^2<\varepsilon/2 \quad (2\text{-}17)$$

and this leads to

$$-\varepsilon/2<x^2-1<0 \quad (2\text{-}18)$$

Considering (2-15), (2-18) will lead to

$$-\varepsilon/2<x^2-1<0 \quad (2\text{-}19)$$

Adding 1 to all three terms, we get

$$1-\varepsilon/2<x^2<1 \quad (2\text{-}20)$$

This gives us two inequalities

$$\sqrt{1-\varepsilon/2}<x<1 \quad (2\text{-}21)$$

for $x>0$ and

$$-\sqrt{1-\varepsilon/2}>x>-1 \quad (2\text{-}22)$$

for $x<0$.

If we subtract 1 from both terms in (2-21)

$$\sqrt{1-\varepsilon/2}-1<x-1<0 \quad (2\text{-}23)$$

$$\frac{(\sqrt{1-\varepsilon/2}-1)(\sqrt{1-\varepsilon/2}+1)}{\sqrt{1-\varepsilon/2}+1}<x-1<0 \quad (2\text{-}24)$$

This is simplified as

$$\frac{-\varepsilon/2}{\sqrt{1-\varepsilon/2}+1}<x-1<0 \quad (2\text{-}15)$$

We have the following inequality

$$\sqrt{1-\varepsilon/2}+1<\sqrt{1+\varepsilon/2}+1 \quad (2\text{-}25)$$

and the reciprocal values satisfy the inequality

$$\frac{1}{\sqrt{1-\varepsilon/2}+1}>\frac{1}{\sqrt{1+\varepsilon/2}+1} \quad (2\text{-}26)$$

and

$$\frac{\varepsilon}{2\sqrt{1-\varepsilon/2}+1}>\frac{\varepsilon}{2\sqrt{1+\varepsilon/2}+1} \quad (2\text{-}27)$$

In order to determine a δ value, we take the right hand term in (2-16). The inequality relation of (2-13) will become

$$\frac{-\varepsilon/2}{\sqrt{1+\varepsilon/2}+1}<x-1<\frac{\varepsilon/2}{\sqrt{1+\varepsilon/2}+1} \quad (2\text{-}28)$$

This gives us

$$\left|x-1\right|<\frac{\varepsilon/2}{\sqrt{1+\varepsilon/2}+1}=\delta \quad (2\text{-}29)$$

QED

Comment: If ε is a very small positive number, δ is approximately equal to ε/4.

QED

Example 3: Let

$f(x) = 1/(x+3)$, (3-1)

prove that

$$\lim_{x\to 1} f(x)=1/4 \quad (3\text{-}2)$$

using the precise definition of a limit.

Solution:

Step 1:

$$\left|\frac{1}{x+3}-\frac{1}{4}\right|<\varepsilon \quad (3\text{-}3)$$

$$-\varepsilon < \frac{1}{x+3} - \frac{1}{4} < \varepsilon \quad (3\text{-}4)$$

$$-\varepsilon + \frac{1}{4} < \frac{1}{x+3} < \varepsilon + \frac{1}{4} \quad (3\text{-}5)$$

Since we are interested in the limit near $x = 1$, $x+1>0$.

$$-\varepsilon + \frac{1}{4} < \frac{1}{x+3} < \varepsilon + \frac{1}{4} \quad (3\text{-}6)$$

We consider

$$0 < \varepsilon < 1/4 \quad (3\text{-}7)$$

By taking the reciprocals of all terms

$$1/(-\varepsilon + \frac{1}{4}) > x+3 > 1/(\varepsilon + \frac{1}{4}) \quad (3\text{-}8)$$

Step 2: By subtracting 4 from all terms

$$\frac{4}{1-4\varepsilon} - 4 > (x-1) > \frac{4}{1+4\varepsilon} - 4 \quad (3\text{-}9)$$

$$\frac{4-4+16\varepsilon}{1-4\varepsilon} > (x-1) > \frac{4-4-16\varepsilon}{1+4\varepsilon} \quad (3\text{-}10)$$

$$\frac{16\varepsilon}{1-4\varepsilon} > (x-1) > \frac{-16\varepsilon}{1+4\varepsilon} \quad (3\text{-}11)$$

Since $\frac{16\varepsilon}{1-4\varepsilon} > \frac{16\varepsilon}{1+4\varepsilon}$, in order to define the narrower region, we choose $\frac{16\varepsilon}{1+4\varepsilon}$ rather than $\frac{16\varepsilon}{1-4\varepsilon}$.

Then we obtain the final condition determining δ:

$$|x-1| < \frac{16\varepsilon}{1+4\varepsilon} = \delta \quad (3\text{-}12)$$

QED

Comment: For very small , δ becomes approximately equal to 16 ε.

Example 4: Let

$$f(x) = 1/x, \quad (4\text{-}1)$$

prove that

$$\lim_{x\to 0.01} f(x) = 100 \quad (4\text{-}2)$$

using the precise definition of a limit.

Solution

Step 1: For simplicity's sake, suppose that

$$0 < \varepsilon < 10. \quad (4\text{-}3)$$

We want to obtain δ *as a function of any value* ε such that

$$|f(x) - 100| < \varepsilon . \quad (4\text{-}4)$$

and

$$|x - 0.01| < \delta \quad (4\text{-}5)$$

An explicit inequality for (4-4) is

$$\left|\frac{1}{x} - 100\right| < \varepsilon \quad (4\text{-}6)$$

This gives us

$$-\varepsilon < \frac{1}{x} - 100 < \varepsilon \quad (4\text{-}7)$$

By adding 100 for all terms, we get

$$100 - \varepsilon < \frac{1}{x} < 100 + \varepsilon \quad (4\text{-}8)$$

The reciprocal of these terms give us the inequality relation

$$\frac{1}{100-\varepsilon} > x > \frac{1}{100+\varepsilon} \quad (4\text{-}9)$$

Step 2:

By subtracting 0.01=1/100 from all of the terms in inequality (4-9)

$$\frac{1}{100-\varepsilon} - 1/100 > (x-0.01) > \frac{1}{100+\varepsilon} - 1/100 \quad (4\text{-}10)$$

$$\frac{100-100+\varepsilon}{100\ (100-\varepsilon)} > (x-0.01) > \frac{100-100-\varepsilon}{100\ (100+\varepsilon)} \quad (4\text{-}11)$$

The simplified form is

$$\frac{\varepsilon}{100\ (100-\varepsilon)} > (x-0.01) > \frac{-\varepsilon}{100\ (100+\varepsilon)} \quad (4\text{-}12)$$

Because $(100-\varepsilon) < (100+\varepsilon)$, then $1/(100-\varepsilon) > 1/(100+\varepsilon)$. Thus we obtain

$$\frac{\varepsilon}{100\ (100-\varepsilon)} > \frac{\varepsilon}{100\ (100+\varepsilon)} \quad (4\text{-}13)$$

and

$$\frac{\varepsilon}{100\ (100-\varepsilon)} > \frac{\varepsilon}{100\ (100+\varepsilon)} > (x-0.01) > \frac{-\varepsilon}{100\ (100+\varepsilon)} \quad (4\text{-}14)$$

$$\frac{\varepsilon}{100\ (100+\varepsilon)} > (x-0.01) > \frac{-\varepsilon}{100\ (100+\varepsilon)} \qquad (4\text{-}15)$$

Step 3:

From (4-15) the corresponding inequality to (4-3) is

$$|x-0.01| > \frac{\varepsilon}{100\ (100+\varepsilon)} = \delta \qquad (4\text{-}16)$$

QED

Comment: For a very small ε, δ will be approximately equal to $\varepsilon/100^2$.

Example 5: Let

$$f(x) = x^3 + 2x + 1 \qquad (5\text{-}1)$$

prove that

$$\lim_{x\to 1} f(x) = 4 \qquad (5\text{-}2)$$

using the precise definition of a limit.

Solution

Step 1: We want to obtain δ as a function of any value such that

$$|f(x)-4| < \varepsilon \qquad (5\text{-}3)$$

and

$$|x-1| < \delta \qquad (5\text{-}4)$$

An explicit inequality for (5-3) is

$$|(x^3+2x+1)-4| < \varepsilon \qquad (5\text{-}5)$$

$$|(x^3+2x-3)| < \varepsilon$$

Factoring the cubic function, (note that since $f(1) = 4$, the cubic formula $f(x)$ -4 should give us a factor $(x-1)$.

$$x^3+2x-3 = (x-1)(x^2+x+3) \qquad (5\text{-}6)$$

$$|(x-1)(x^2+x+3)| < \varepsilon \qquad (5\text{-}7)$$

For all x the term (x^2+x+3) is always positive and this term is minimum of 11/4 at $x = -1/2$.

For $x > 0$ $(x^2 + x + 3) > 3$.

From (5-7) we get

$$-\varepsilon < (x-1)(x^2+x+3) < \varepsilon \qquad (5\text{-}7)$$

By dividing all terms by the positive term (x^2+x+3)

$$-\frac{\varepsilon}{(x^2+x+3)} < (x-1) < \frac{\varepsilon}{x^2+x+3} \qquad (5\text{-}8)$$

By adding 1 to all of the terms in this inequality, we get

$$1-\frac{\varepsilon}{(x^2+x+3)} < x < 1+\frac{\varepsilon}{x^2+x+3} \qquad (5\text{-}9)$$

From this we obtain the range of x as

$$1-\frac{\varepsilon}{(x^2+x+3)} < x < 1+\frac{\varepsilon}{x^2+x+3} \qquad (5\text{-}10)$$

Thus the maximum range will be

$$1-\frac{\varepsilon}{3} < x < 1+\frac{\varepsilon}{3} \qquad (5\text{-}11)$$

If even ε is as large as 1 then x ranges as

$$\frac{2}{3} < x < \frac{4}{3} \qquad (5\text{-}12)$$

If ε is as large as 0.1 then x ranges as

$$1-0.0333 < x < 1.0333 \qquad (5\text{-}13)$$

The term $\frac{\varepsilon}{x^2+x+3}$ decreases with $x > 0$.

Therefore the term for $x = 4/3$

$$\frac{\varepsilon}{x^2+x+3} = \frac{\varepsilon}{16/9+4/3+3} = \frac{9\varepsilon}{55} \qquad (5\text{-}20)$$

is small enough to satisfy (5-8). Thus we obtain

$$-\frac{9\varepsilon}{55} < (x-1) < \frac{9\varepsilon}{55} \qquad (5\text{-}21)$$

$$0 < |x-1| < \frac{9\varepsilon}{55} = \delta \qquad (5\text{-}22)$$

QED

Comments:

If the functional form is simple it is rather easy to determine a δ value. However the situation is much more complicated when the inequality relation is hard to solve.

If the function $f(x)$ is a continuous function at x = a,

$\lim_{x\to a} f(x) = L = f(a)$.

You will learn later the definition of a derivative $f'(a)$, that is, the derivative of a function f at a number $x = a$, is

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists. For a sufficiently small number δ >0, $|x - a| = \delta$, ε will be determined which satisfies,

$$|f(x) - f(a)| = \varepsilon$$

then we can have an approximate relation that

$$|f'(a)| \approx \frac{\varepsilon}{\delta}$$

which gives $\delta \approx \dfrac{\varepsilon}{|f'(a)|}$

or

$$\frac{\delta}{\varepsilon} \approx \frac{1}{|f'(a)|}$$

In all Examples shown above, you will see that the ratio δ/ε approaches $\left|\dfrac{1}{f'(a)}\right|$ for small ε.