The University of Nagoya
School of Mathematical Sciences
G30 Calculus 1, Fall 2012-2013
FINAL EXAM
Time allowed ONE Hour and THIRTY minutes

Exercise 1. Approximate the functions $sin(x)$ and $ln(x+1)$ by the Taylor polynomial with degree 3 at $x = 1$.

Sol: $sin(x) \approx x - \frac{x^3}{6}$ by the Taylor polynomial of order 3 around 0. Then $sin(1) \approx \frac{5}{6}$.
$ln(x+1) \approx x - \frac{x^2}{2} + \frac{x^3}{3}$. Then $ln2 \approx 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

Exercise 2. Does $x_n = \sqrt{n^2+n} - n$ tend to a finite limit as $n \to \infty$? If so, calculate its limit.

Sol: Yes.
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{n^2+n} - n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} = 1/2.$$

Exercise 3. Show that the equation
$$2^x - x\ln 2 - 2 = 0$$
has exactly one root in the interval $[1, 2]$. (Hint: to show that there is not more than one root, use the Mean Value Theorem)

Sol: Let $f(x) = 2^x - x\ln 2 - 2$, then $f(x)$ is continuous on $[1, 2]$ and differentiable on $(1, 2)$. Since $f(1) = -ln2 < 0$ and $f(2) = 2 - 2ln2 > 0$. Applying the Intermediate Value Theorem, there exists at least one root in $[1, 2]$. If there exist a and b in $[1, 2]$ with $a \neq b$ such that $f(a) = f(b) = 0$. Then by Mean Value Theorem, there exists $c \in (a, b)$ such that $f'(c) = 0$. But $f'(x) = 2^x \ln 2 - \ln 2 > 0$ for all $x \in (1, 2)$. So the equation has exactly one root in $[1, 2]$.

Exercise 4. Evaluate the following integrals:
(a) $\int_0^1 e^{\sqrt{x}} dx$
(b) $\int \frac{1}{xln(x)} dx$

Sol:

(a) Let $\sqrt{x} = t$, then $\int_0^1 e^{\sqrt{x}} dx = \int_0^1 e^t dt^2 = 2\int_0^1 te^t dt = 2\int_0^1 tde^t = 2te^t|_0^1 - 2\int_0^1 e^t dt = 2e - 2e^t|_0^1 = 2.$

(b) Let $\ln x = t$, then $\int \frac{1}{x\ln x} dx = \int \frac{1}{\ln x} d\ln x = \int \frac{1}{t} dt = \ln|t| + C = \ln(|\ln x|) + C$, where C is constant.

Exercise 5. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $I = \int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

(b) $I = \int_0^1 \frac{lnx}{\sqrt{x}} dx$

Sol:

(a) Let $x = \sin t$, then $\int_u^1 \frac{1}{\sqrt{1-x^2}} dx = \int_{\arcsin u}^{\pi/2} \frac{1}{\sqrt{1-\sin^2 t}} d\sin t = \int_{\arcsin u}^{\pi/2} \frac{1}{\cos t} d\sin t = \int_{\arcsin u}^{\pi/2} dt = t|_{\arcsin u}^{\pi/2} = \pi/2 - \arcsin u$. Since

$$\lim_{u\to 0} \int_u^1 \frac{1}{\sqrt{1-x^2}} dx = \lim_{u\to 0} \pi/2 - \arcsin u = \pi/2.$$

Thus I is convergent and $I = \pi/2$.

(b) Let $\sqrt{x} = t$, then $\int_u^1 \frac{lnx}{\sqrt{x}} dx = \int_{\sqrt{u}}^1 \frac{\ln t^2}{t} dt^2 = \int_{\sqrt{u}}^1 4\ln t dt = -2\sqrt{u}\ln u - 4 + 4\sqrt{u}$. Using L'Hospital'l rule, we can obtain

$$\lim_{u\to 0} -2\sqrt{u}\ln u = 0.$$

Thus

$$I = \lim_{u\to 0} \int_u^1 \frac{lnx}{\sqrt{x}} dx = \lim_{u\to 0} -2\sqrt{u}\ln u - 4 + 4\sqrt{u} = -4.$$

So I is convergent.