Problem set 5

Representation Theory of Associative Algebras

Solutions should contain detailed arguments for all statements made. Each problem gives a maximum of 25 points. Hand in before or during the lecture on January 17.

Problem 1. Let Q be the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

Use the Nakayama functor to compute the Auslander-Reiten translation of the following indecomposable K-linear representations of Q:

$$K \xrightarrow{0} 0 \xrightarrow{0} 0, \quad 0 \xrightarrow{0} K \xrightarrow{0} 0, \quad 0 \xrightarrow{0} 0 \xrightarrow{0} K,$$
$$K \xrightarrow{1} K \xrightarrow{0} 0, \quad 0 \xrightarrow{0} K \xrightarrow{1} K, \quad K \xrightarrow{1} K \xrightarrow{1} K.$$

Problem 2. Let A be a finite dimensional K-algebra and $f : P \to M$ an epimorphism in $\operatorname{mod} A$ with $P \in \operatorname{mod} A$ projective. Show that $f : P \to M$ is a projective cover of M if and only if f is right minimal.

Problem 3. Let A be a finite dimensional K-algebra. Recall that rad_A is the ideal in the K-category $\operatorname{mod} A$ defined by

$$\operatorname{rad}_A(M, N) = \{f : M \to N \mid 1_M - g \circ f \in \operatorname{Aut}_A(M) \text{ for all } g \in \operatorname{Hom}_A(N, M)\}$$

for all $M, N \in \operatorname{mod} A$.

Let $M, N \in \operatorname{mod} A$ be indecomposable and not isomorphic. Show that

$$\operatorname{rad}_A(M, M) = \operatorname{rad}(\operatorname{End}_A(M))$$

and

$$\operatorname{rad}_A(M, N) = \operatorname{Hom}_A(M, N).$$

Problem 4. (From Assem-Simson-Skowronski) Let A be a finite dimensional K-algebra and

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

an almost split sequence in $\operatorname{mod} A$. Show that for any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle 1_N} & & \\ 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & N & \longrightarrow & 0 \end{array}$$

where the last row is a non-split short exact sequence, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle h'} & & \uparrow{\scriptstyle k'} & & \uparrow{\scriptstyle 1_N} & & \\ 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & N & \longrightarrow & 0 \end{array}$$

such that $h'h = 1_L$ and $k'k = 1_M$.