PRACTICE PROBLEMS FOR MIDTERM EXAM
Linear Algebra II
Nagoya University
G30 Program, Spring 2013

The following problems should be used as practice for the midterm exam. The actual exam problems will be different but these can be considered as examples of what might appear. The time limit for the midterm exam will be 90 minutes. No tools will be allowed except pen and paper.

Problem 1. Let

$$A = \begin{bmatrix} -1 & 2 & -2 & 1 \\ 2 & -4 & 1 & 4 \\ 1 & -2 & -2 & 7 \end{bmatrix}.$$

(a) Find a basis in the image of A.

(b) Find a basis in the kernel of A.

Problem 2. Let P_2 be the linear space of all polynomials of degree at most 2. Determine which of the following subsets of P_2 are subspaces, and for each such subspace find its dimension.

(a) $\{p(x) \in P_2 \mid p(1) = p(-1)\}$.

(b) $\{p(x) \in P_2 \mid (p(1))^2 = (p(-1))^2\}$.

(c) $\{p(x) \in P_2 \mid p(1) < p(2)\}$.

(d) $\{p(x) \in P_2 \mid p''(x) = 0\}$.

Problem 3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & -1 & -1 \end{bmatrix}$$

and $\mathfrak{B}$ the basis of $\mathbb{R}^3$ consisting of the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find the $\mathfrak{B}$-matrix B of T.

Problem 4. Let P_2 be the linear space of all polynomials of degree at most 2 and

$$W = \{p(x) \in P_2 \mid p'(1) = p(1)\}.$$

(a) Show that W is a subspace of P_2.

(b) Find a basis in W.

(c) What is the dimension of W.

Problem 5. Find a basis in the space of all 2×2-matrices S such that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} S = S \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Problem 6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

and $\mathfrak{B}$ the basis of $\mathbb{R}^3$ consisting of the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Find the $\mathfrak{B}$-matrix B of T.

Problem 7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the plane given by the equation $x + 2y - 3z = 0$.

(a) Find a basis in the image $\mathrm{im}(T)$. What is the dimension of $\mathrm{im}(T)$?

(b) Find a basis in the kernel $\ker(T)$. What is the dimension of $\ker(T)$?

Problem 8. Let A be a 2×2-matrix and $(\vec{v}_1, \vec{v}_2)$ a basis of $\mathbb{R}^2$ such that $A\vec{v}_1 = \vec{v}_2$ and $A\vec{v}_2 = -\vec{v}_1$. Show that A^4 is the identity matrix I_2.