PRACTICE PROBLEMS FOR FINAL EXAM – ANSWERS

Linear Algebra I

Nagoya University

G30 Program, Fall 2012

Problem 1.

(a) T is linear. The matrix of T is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

(b) T is not linear.

(c) T is linear. The matrix of T is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(d) T is not linear.

Problem 2. $X = \begin{bmatrix} s & t \\ 0 & s \end{bmatrix}$, where $s, t \in \mathbb{R}$

Problem 3.

(a) The matrix of T is $\frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$

(b) $T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \frac{1}{6}\begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right) = \frac{1}{6}\begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}$.

(c) T is not invertible.

Problem 4.

(a) The matrix of T is $A = \frac{1}{11}\begin{bmatrix} -9 & -2 & 6 \\ -2 & -9 & -6 \\ 6 & -6 & 7 \end{bmatrix}$.

(b) Since $T \circ T$ is the identity it is invertible and the matrix of T^{-1} is A.

Problem 5. Let A, B be $n \times n$-matrices. If AB is invertible then $I_n = (AB)(AB)^{-1} = A(B(AB)^{-1})$ and so A is invertible.

(a) Follows from the above since $A^2 = AA$

(b) Same argument using $A^2 - I_n = (A - I_n)(A + I_n)$.

Problem 6.

(a) $\det A = a$.

(b) $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 & 0 \\ 0 & -7 & 4 & -2 \\ 0 & -2 & 1 & 0 \\ 0 & 4 & -2 & 1 \end{bmatrix}$ for all $a \neq 0$.

(c) $\operatorname{adj}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7a & 4a & -2a \\ 0 & -2a & a & 0 \\ 0 & 4a & -2a & a \end{bmatrix}$.

Problem 7. $\det A_n = 1$ for all $n \geq 1$.

Problem 8.

(a) Since $0 = \det(A^N) = (\det A)^N$, the matrix A is not invertible. So the equation

$$A\vec{x} = 0$$

has infinitely many solutions. In particular there is one that is non-zero.

(b) Since $0 = A^N\vec{x} = k^N\vec{x}$ and $\vec{x} \neq 0$, we get $k^N = 0$ and so $k = 0$.