PRACTICE PROBLEMS FOR FINAL EXAM
Linear Algebra I
Nagoya University
G30 Program, Fall 2012

The following problems should be used as practice for the final exam. The actual exam will contain fewer problems, which will be different but these can be considered as examples of what might appear. The time limit for the final exam will be 90 minutes. No tools will be allowed except pen and paper.

Problem 1.
Determine which of the following transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ are linear. For each that is, find its matrix.

(a) $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_2 \end{bmatrix}$,

(b) $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$,

(c) $T(\vec{x})$ is the rotation of $\vec{x}$ by the angle $\frac{\pi}{2}$ counterclockwise,

(d) $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 1 \\ x_2 + 2 \end{bmatrix}$,

Problem 2.
Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Find all 2×2 matrices X such that

$$AX = XA.$$

Problem 3.
Let V be the plane through the origin given by the equation

$$x - 2y + z = 0$$

and $T : \mathbb{R}^3 \to \mathbb{R}^3$ the orthogonal projection onto V.

(a) Find the matrix of T.

(b) Evaluate T on the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

(c) Is T invertible?

Problem 4.
Let l be the line through the origin given by the vectors

$$t \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix},$$

where $t \in \mathbb{R}$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ the reflection in l.

(a) Find the matrix of T.

(b) Show that T is invertible and find the matrix of T^{-1}.

Problem 5. Let A be an $n \times n$-matrix. Show that

(a) If A^2 is invertible, then A is invertible.

(b) If $A^2 - I_n$ is invertible, then $A - I_n$ is invertible.

Problem 6.
Let a be a real number and

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

(a) Calculate $\det A$.

(b) Find A^{-1} for all values of a such that A is invertible.

(c) Calculate the classical adjoint $\operatorname{adj}(A)$.

Problem 7. For each positive integer n find the determinant of the $n \times n$-matrix

$$A_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \ddots & \vdots \\ 1 & 0 & 1 & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 & 1 \end{bmatrix}.$$

Problem 8.
Let A be an $n \times n$ matrix and N a positive integer such that

$$A^N = 0.$$

(a) Show that there is a non-zero vector $\vec{x} \in \mathbb{R}^n$ such that

$$A\vec{x} = 0.$$

(b) Show that if there is a non-zero vector $\vec{x} \in \mathbb{R}^n$ and real number k such that

$$A\vec{x} = k\vec{x},$$

then $k = 0$.