Lines and planes

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In this brief text we explain how linear equations can be used to describe lines in the plane and planes in space.

1 Matrices and vectors

A *matrix* is a rectangular array of real numbers. If the array has m rows and n columns we call it an $m \times n$ matrix (pronounced "m by n matrix"). Here are examples of a 2×3, 4×2 and a 3×3 matrix:

$$\begin{bmatrix} 3 & -2 & 0 \\ 1 & 4 & 13 \end{bmatrix} \qquad \begin{bmatrix} 2 & 7 \\ 0 & -4 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 21 & 4 & -17 \\ 11 & 2 & 0 \\ 4 & 9 & 31 \end{bmatrix}$$

A matrix consisting of a single column is called a *column vector*

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Its entries v_i are called the *components* of $\vec{v}$. Similarly a *row vector* is a matrix consisting of a single row. In what follows the word vector always means column vector.

We think of the vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

as the arrow pointing from the origin to the point with coordinates $(v_1, v_2, \ldots, v_n)$ in n-dimensional space. This provides a way to visualize vectors (especially for $n = 2$ and $n = 3$) as can be seen in Figure 1.

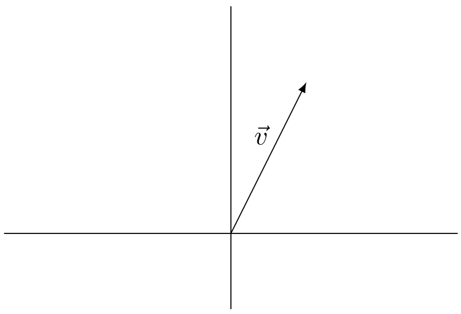


Figure 1: The vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

For any two vectors with the same number of components $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$, their *sum* is the vector $\vec{v} + \vec{w}$ defined by

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}.$$

For any real number c we define $\vec{v}$ multiplied by c as the vector $c\vec{v}$ given by

$$c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$

Adding two vectors to each other can be visualized as putting one vector after the other. Multiplying a vector $\vec{v}$ with a real number c can be visualized as scaling $\vec{v}$ by c (see Figure 2). In this context, it is therefore usual to call c a scalar.

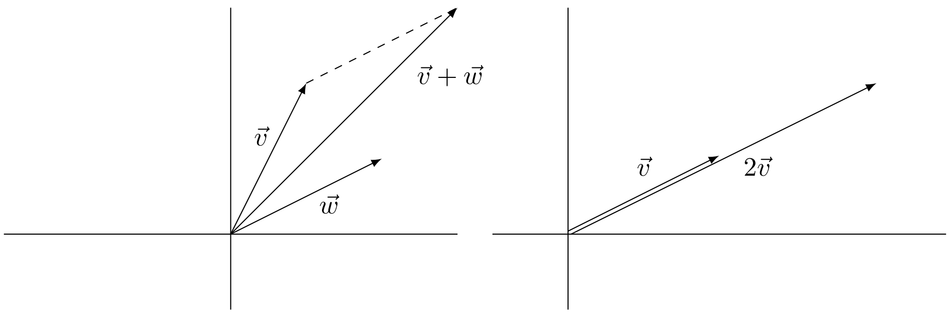


Figure 2: Vector addition, and multiplication by scalar.

The *dot product* of $\vec{v}$ and $\vec{w}$ is the real number $\vec{v} \cdot \vec{w}$ defined by

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

Notice that the dot product of two vectors is not a vector, but a real number.

In two dimensions the distance from the origin to a point (a, b) equals $\sqrt{a^2 + b^2}$ by the Pythagorean Theorem. If we let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, then $\vec{v} \cdot \vec{v} = a^2 + b^2$. So it makes sense to say that the length of $\vec{v}$ is $\sqrt{\vec{v} \cdot \vec{v}}$. This formula can be justified in three dimensions as well, by similar reasoning. Inspired by this we say (in any dimension) that the *length* of a vector $\vec{v}$ is the real number $||\vec{v}||$ defined by

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}.$$

In dimension 2 and 3, one can also show that

$$\vec{v} \cdot \vec{w} = ||\vec{v}|| \cos\theta ||\vec{w}||,$$

where θ denotes the angle between $\vec{v}$ and $\vec{w}$. The interested reader is advised to do this by using the law of cosines from trigonometry (see Exercise 4 and 5). In particular, we see that $\vec{v} \cdot \vec{w} = 0$ if and only if $\vec{v}$ and $\vec{w}$ meet at a right angle. Motivated by this we say (in any dimension) that two vectors $\vec{v}$ and $\vec{w}$ are *orthogonal* if

$$\vec{v} \cdot \vec{w} = 0.$$

2 Lines in the plane

Consider the linear equation

$$x + 2y = 0. \tag{1}$$

The left hand side can be interpreted as the dot product of the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$. Thus the solutions to equation (1) consists of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. These vectors form a line ℓ through the origin (see Figure 3).

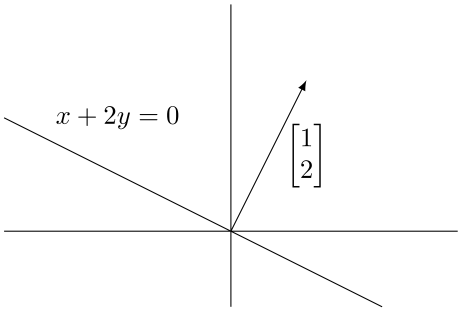


Figure 3: Solutions to $x + 2y = 0$.

Another way to present the solutions to equation (1) is to set y equal to a parameter t. Then we get a solution to equation (1) if and only if $x = -2t$, and so the solutions to (1) are precisely the vectors

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

where t varies over all real numbers.

Now consider the linear equation

$$x + 2y = 3. \tag{2}$$

If we again set $y = t$ we get $x = 3 - 2t$ and so the solutions to (2) are

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 - 2t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

where t varies over all real numbers. Geometrically we can interpret these solutions as the line ℓ' that we get by translating ℓ by the vector $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ (see Figure 4).

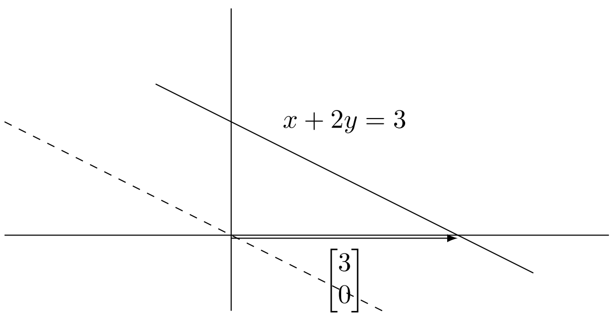


Figure 4: Solutions to $x + 2y = 3$.

Notice that we can replace $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ by any vector pointing from the origin to a point on ℓ'. For instance we can write the solutions to (2) as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

where s varies over all real numbers, since $1 \cdot 1 + 2 \cdot 1 = 3$. Another way to see this is to set $t = s + 1$.

Motivated by the above discussion we introduce some terminology. A non-zero vector that is orthogonal to a line ℓ is called a *normal vector* of ℓ and a non-zero vector that points in the same direction as ℓ is called a *direction vector* of ℓ. For instance, if ℓ' is the line corresponding to equation (2), then $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a normal vector of ℓ' and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a direction vector of ℓ'.

We now summarize the essential points of the above discussion. Let a, b and c be constants such that at least one of a and b is non-zero.

1. The solutions to the equation $ax + by = c$ form a line ℓ.
2. The vector $\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a normal vector of ℓ.
3. Any non-zero vector $\vec{v}$ satisfying $\vec{n} \cdot \vec{v} = 0$ is a direction vector of ℓ. For example we can take $\vec{v} = \begin{bmatrix} -b \\ a \end{bmatrix}$.
4. Fix a direction vector $\vec{v}$ and some solution $\vec{v}_0$ to $ax + by = c$. Then the solutions to $ax + by = c$ are given by the vectors $\vec{v}_0 + t\vec{v}$ where t varies over all real numbers.

Example 1. Find the intersection of the lines $x + 2y = 3$ and $3x + y = 4$.

The intersection is given by all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that x and y satisfy the system

$$\begin{cases} x + 2y = 3 \\ 3x + y = 4 \end{cases}$$

Solving this system we arrive at $x = 1$ and $y = 1$, so the intersection is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The lines are drawn in Figure 5.

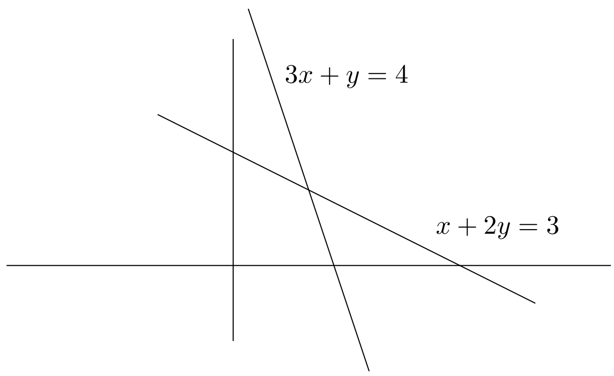


Figure 5: Intersection of the lines $x + 2y = 3$ and $3x + y = 4$.

3 Planes in space

Now we shall investigate linear equations in three dimensions. For example, let us look at the equation

$$x + 2y + 3z = 0. \tag{3}$$

As before the right hand side can be interpreted as the dot product of the vectors

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \text{ and } \begin{bmatrix}x\\y\\z\end{bmatrix}$$

and so the solutions to equation (3) are precisely the vectors that are orthogonal to

$$\vec{n} = \begin{bmatrix}1\\2\\3\end{bmatrix}.$$

These form a plane P in three dimensional space. We can also present the solutions using parameters. Set $y = s$ and $z = t$. Then we get a solution to equation (3) if and only if $x = -2s - 3t$ and so the solutions to equation (3) are the vectors

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-2s-3t\\s\\t\end{bmatrix} = s\begin{bmatrix}-2\\1\\0\end{bmatrix} + t\begin{bmatrix}-3\\0\\1\end{bmatrix}$$

where s and t vary over all real numbers. In particular, we have found two vectors

$$\vec{v} = \begin{bmatrix}-2\\1\\0\end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix}-3\\0\\1\end{bmatrix}$$

that lie in P. These vectors and the plane P are displayed in Figure 6.

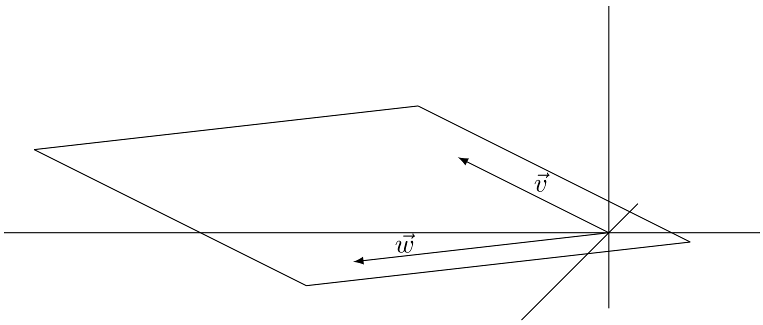


Figure 6: Solutions to $x + 2y + 3z = 0$.

Now let us look at the linear equation

$$x + 2y + 3z = 4. \tag{4}$$

To parametrize the solutions we set $y = s$ and $z = t$ as before. Then $x = 4 - 2s - 3t$ and so the solutions to equation (4) are

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}4-2s-3t\\s\\t\end{bmatrix} = \begin{bmatrix}4\\0\\0\end{bmatrix} + s\begin{bmatrix}-2\\1\\0\end{bmatrix} + t\begin{bmatrix}-3\\0\\1\end{bmatrix}.$$

It follows that the solutions to (4) form a plane P' that we get by translating P by the vector

$$\vec{u} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}.$$

See Figure 7.

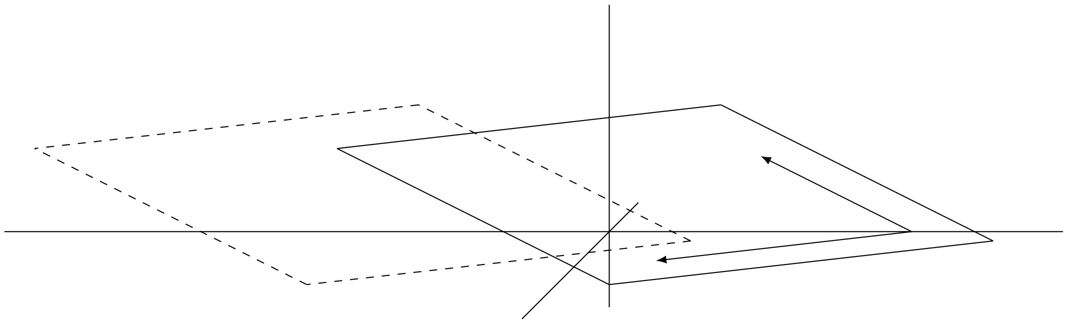


Figure 7: Solutions to $x + 2y + 3z = 4$.

Example 2. Find the intersection of the planes $x + 2y + 3z = 4$ and $3x + y + 4z = 12$.

The intersection will be given by all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that x, y and z satisfy the system

$$\begin{cases} x + 2y + 3z = 4, \\ 3x + y + 4z = 12. \end{cases}$$

Adding -3 times the first equation to the second gives

$$\begin{cases} x + 2y + 3z = 4, \\ \quad - 5y - 5z = 0, \end{cases}$$

which after dividing the second equation by -5 gives

$$\begin{cases} x + 2y + 3z = 4, \\ \quad y + z = 0, \end{cases}$$

Finally, adding -2 times the second equation to the first gives

$$\begin{cases} x \quad + z = 4, \\ \quad y + z = 0. \end{cases}$$

Now put $z = t$. Then we get $x = 4 - t$ and $y = -t$. So the intersection consists of the vectors

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 - t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

where t varies over all real numbers. We can interpret the intersection geometrically as the line passing through $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$ in direction $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

4 Exercises

Exercise 1. Calculate the following vectors and illustrate in the plane.

(a) $\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(b) $2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(c) $-3 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(d) $2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Exercise 2. Find the lengths of the following vectors.

(a) $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$

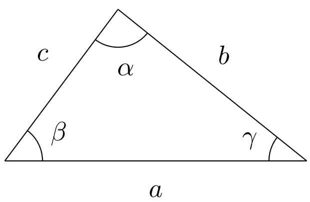
(c) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(d) $-3 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Exercise 3. Which of the following vectors are orthogonal to each other:

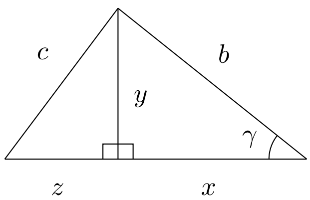
$$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Exercise 4. Prove the law of cosines



$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

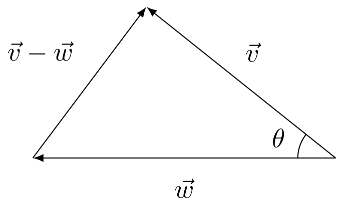
Hint: Determine x, y, and z in terms of a, b, $\sin\gamma$, and $\cos\gamma$ in the following picture. Then calculate c^2 using the Pythagorean Theorem.



Exercise 5. Use the law of cosines to prove that

$$\vec{v} \cdot \vec{w} = ||\vec{v}|| \cos\theta ||\vec{w}||,$$

for any vectors $\vec{v}$ and $\vec{w}$, where θ is the angle between them. Hint: In the picture below, calculate $||\vec{v}-\vec{w}||^2$ in to ways: first using the dot product and then using the law of cosines.



Exercise 6. Find the intersection of the following lines and interpret your result geometrically.

$$\begin{aligned} 5x + y &= 3, \\ 2x + 3y &= -4. \end{aligned}$$

Exercise 7. Find the intersection of the following lines and interpret your result geometrically.

$$\begin{aligned} 3x - 6y &= -15, \\ -x + 2y &= 5. \end{aligned}$$

Exercise 8. Find the intersection of the following lines and interpret your result geometrically.

$$\begin{aligned} -2x + 8y &= 3, \\ 3x - 12y &= 2. \end{aligned}$$

Exercise 9. Find the intersection of the following planes and interpret your result geometrically.

$$\begin{aligned} 2x + 1y + 3z &= 7, \\ 3y + 2z &= 8, \\ x + 5y + 6z &= 17. \end{aligned}$$

Exercise 10. Find the intersection of the following planes and interpret your result geometrically.

$$\begin{aligned} 2x + y + 4z &= 6, \\ 3x - 5y + 6z &= 9. \end{aligned}$$

Exercise 11. Find the intersection of the following planes and interpret your result geometrically.

$$\begin{aligned} 5x + 2y + 2z &= 4, \\ 3x - y + 4z &= 1, \\ -2x - 3y + 2z &= 0. \end{aligned}$$

Exercise 12. Find the intersection of the following planes and interpret your result geometrically.

$$\begin{aligned} 6x - 9y + 3z &= 12, \\ -8x + 12y - 4z &= -16, \\ 2x - 3y + z &= 4. \end{aligned}$$

Exercise 13. Find the intersection of the following planes and interpret your result geometrically.

$$\begin{aligned} x + y + z &= 0, \\ x + z &= 0, \\ 2x + y &= 0, \\ y + 2z &= 0. \end{aligned}$$