FINAL EXAM – ANSWERS
Linear Algebra I
Nagoya University
G30 Program, 28 January 2013

Time: 90 minutes. Tools: only for writing (pen, pencil and eraser). The maximum score on the exam is 50 points. Have your student ID card in front of you at all times during the exam. Use the provided answer sheets and write your name and student ID number on every sheet. All answers should be motivated clearly. Good Luck!

Problem 1. (10 points)
Find the matrix of each of the following linear transformations:

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 - x_2 \end{bmatrix}$,

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$,

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T(\vec{x})$ is the rotation of $\vec{x}$ by the angle π counterclockwise.

Answer: The matrix of T is

(a) $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$,

(b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,

(c) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Problem 2. (10 points)
Let
$$A = \begin{bmatrix} 2 & 3 & 11 & -7 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 5 & -8 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 1 & 2 & -1 & -3 \\ 1 & 3 & 0 & -2 \\ 2 & 3 & -4 & -6 \end{bmatrix}$$

(a) Find $\det A$.

(b) Find $\det B$.

(c) Find $\det AB$.

(d) Which of the matrices A, B and AB are invertible.

Answer:

(a) $\det A = 2 \times 1 \times 5 \times 3 = 30$,

(b) $\det B = 0$,

(c) $\det AB = \det A \times \det B = 0$,

(d) The matrix A is invertible, but B and AB are not.

Problem 3. (10 points)
Let a be a real number and

$$A = \begin{bmatrix} 0 & 1 & a \\ 1 & 0 & 1 \\ a & 1 & 0 \end{bmatrix}.$$

(a) Calculate $\det A$.

(b) Find A^{-1} for all values of a such that A is invertible.

Answer:

(a) $\det A = 2a$,

(b) For all $a \neq 0$, $A^{-1} = \frac{1}{2a} \begin{bmatrix} -1 & a & 1 \\ a & -a^2 & a \\ 1 & a & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\frac{1}{a} & 1 & \frac{1}{a} \\ 1 & -a & 1 \\ \frac{1}{a} & 1 & -\frac{1}{a} \end{bmatrix}$.

Problem 4. (10 points)
Let V be the plane through the origin given by the equation

$$2x - y + 2z = 0$$

and $T : \mathbb{R}^3 \to \mathbb{R}^3$ the reflection in V.

(a) Find the matrix A of T.

(b) Calculate A^2.

(c) Calculate A^{2013}.

Answer:

(a) The line ℓ through the origin with direction vector $\vec{n} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ is orthogonal to V. So for any vector $\vec{x}$,

$T(\vec{x}) = \vec{x} - 2 \cdot \mathrm{proj}_\ell(\vec{x}) = \vec{x} - 2\frac{\vec{n}\cdot\vec{x}}{\vec{n}\cdot\vec{n}}\vec{n} = \left(I_3 - \frac{2}{9} \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} \right) \vec{x} = \frac{1}{9} \begin{bmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{bmatrix} \vec{x},$

which means $A = \frac{1}{9} \begin{bmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{bmatrix}$.

(b) T is the reflection in V, so $T\big(T(\vec{x})\big) = \vec{x}$, for all $\vec{x}$. Therefore $A^2 = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) $A^{2013} = (A^2)^{1006} \cdot A = A = \frac{1}{9} \begin{bmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{bmatrix}$.

Problem 5. (10 points) Let A be a 2×2-matrix and $\vec{x}, \vec{y}$ be non-zero vectors in $\mathbb{R}^2$ such that

$$A\vec{x} = 0 \quad \text{and} \quad A\vec{y} = \vec{y}.$$

(a) Show that $\det A = 0$ and $\det(A - I_2) = 0$.

(b) Show that $A^2 = A$.

Answer:

(a) The following conditions are equivalent:
(i) There exist an $\vec{x} \neq 0$, such that $A\vec{x} = 0$.
(ii) A is not invertible.
(iii) $\det A = 0$.

Similarly, since $A\vec{y} = \vec{y}$ is equivalent to $(A - I_2)\vec{y} = 0$, we have the following equivalent conditions:
(i) There exist a $\vec{y} \neq 0$, such that $(A - I_2)\vec{y} = 0$.
(ii) $(A - I_2)$ is not invertible.
(iii) $\det(A - I_2) = 0$.

(b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
Then $0 = \det A = ad - bc$,
and $0 = \det(A - I_2) = (a-1)(d-1) - bc = ad - a - d + 1 - bc$,
which implies $ad = bc$ and $a + d = 1$.

$$A^2 = \begin{bmatrix} a^2 + bc & (a+d)b \\ (a+d)c & bc + d^2 \end{bmatrix} = \begin{bmatrix} a^2 + ad & (a+d)b \\ (a+d)c & ad + d^2 \end{bmatrix} = (a+d) \begin{bmatrix} a & d \\ b & c \end{bmatrix} = A.$$

*Alternative solution:
we claim that $\vec{x}$ and $\vec{y}$ are not parallel. (Otherwise, if $\vec{x} = \lambda\vec{y}$ ($\lambda \in \mathbb{R}$), then $0 = A\vec{x} = \lambda A\vec{y} = \lambda\vec{y} = \vec{x}$, which is a contradiction.)
So if $\vec{v} \in \mathbb{R}^2$, we can write $\vec{v} = a\vec{x} + b\vec{y}$.
Therefore $A\vec{v} = aA\vec{x} + bA\vec{y} = b\vec{y}$,
and $A^2\vec{v} = A(A\vec{v}) = Ab\vec{y} = bA\vec{y} = b\vec{y}$.
Since $A^2\vec{v} = A\vec{v}$, for all vectors $\vec{v}$, we have $A^2 = A$.