Every curve of genus not greater than eight lies on a K3 surface

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1 Introduction

Let \( C \) be a smooth irreducible complete curve of genus \( g \geq 2 \) over an algebraically closed field \( k \) of characteristic 0. An ample K3-extension of \( C \) is a K3 surface \( S \) with at worst rational double points which contains \( C \) in the smooth locus as an ample divisor. If \( C \) is contained in a smooth K3 surface, then we obtain an ample K3-extension by contracting all \((-2)\)-curves disjoint from \( C \).

The purpose of this paper is to show

**Main Theorem.** All smooth curves of genera \( 2 \leq g \leq 8 \) have ample K3-extensions. Moreover, they have smooth ample extensions except the following cases:

- \( g = 6, 7, 8 \) and \( K_C = 2D \) where \( D \) is a \( g^2 \), or
- \( g = 8 \) and \( K_C = A + 2B \) where \( A \) is a \( g^1 \) and \( B \) is a \( g^1 \).

In these exceptional cases, the canonical model \( C \subset \mathbb{P}^{g-1} \) is contained in a weighted projective variety. Rational double points come from the singularities of the weighted projective variety (Lemma 2.6).

Since the dimension of the moduli space of curves of genus \( g \) is \( 3g - 3 \) and the dimension of the moduli space of pairs \((S, C)\) of a K3 surface \( S \) and a curve \( C \subset S \) of genus \( g \) is \( 19 + g \), general smooth curves have no ample K3-extensions for \( g \geq 12 \). For \( g = 10 \), by [Muk4], general curves have no ample K3-extensions. For \( g = 11, 9 \), by [MM] and [Muk4], general curves have ample K3-extensions, but special cases are still unknown.

In section 2, we prepare some lemmas to construct ample K3-extensions, namely, double covering and Bertini type lemmas. In section 3, we study hyperelliptic curves, trigonal curves, and bielliptic curves, and construct K3
extensions which preserve the hyperelliptic pencils, trigonal pencils, and 2:1-morphisms onto the elliptic curves respectively by these lemmas. In section 4, we construct K3 extensions of remaining curves.

**Notation and conventions.** For a smooth variety $X$, we denote by $K_X$ the canonical divisor class of $X$ and by $\omega_X := \mathcal{O}_X(K_X)$ the canonical line bundle. A $g_d^r$ is a line bundle $\mathcal{L}$ of degree $d$ such that $h^0(\mathcal{L}) \geq r + 1$.

# 2 How to make a K3 extension

## 2.1 K3-extension as a double cover

Let $X$ be a scheme and $\mathcal{L}$ a line bundle over $X$. A global section $s \in H^0(X, \mathcal{L}^{-2})$ yields an algebra structure on $\mathcal{O}_X \oplus \mathcal{L}$. Then $\pi : Y = \text{Spec}(\mathcal{O}_X \oplus \mathcal{L}) \longrightarrow X$ is a double covering branched along $B = (s)_0$.

**Lemma 2.1.** Let $X$ be a smooth regular surface (i.e., smooth complete surface with $H^1(X, \mathcal{O}_X) = 0$). Let $B$ be a smooth member of $|-2K_X|$. Then the double cover $\pi : Y \longrightarrow X$ branched over $B$, obtained as above, is a smooth K3 surface.

**Proof.** The double covering $Y$ is obviously smooth, and has the irregularity $h^1(Y, \mathcal{O}_Y) = h^1(X, \mathcal{O}_X \oplus \mathcal{O}_X(K_X)) = h^1(X, \mathcal{O}_X) + h^1(X, \mathcal{O}_X(K_X)) = 0$ by our assumption. Since the canonical divisor class $K_Y$ of $Y$ is linearly equivalent to $\pi^*K_X + R$ where $R$ is the ramification divisor class, and $R$ is linearly equivalent to $\pi^*\mathcal{O}_X(-K_X)$ in this situation, we conclude that $K_Y$ is linearly equivalent to zero. \hfill \square

## 2.2 Bertini type lemmas for smooth extension

Let $S$ be a surface in $\mathbb{P}^g$ and $C$ a hyperplane section of $S$, then we have a commutative diagram;

$$
\begin{array}{ccc}
S & \subset & \mathbb{P}^g \\
\cup & \cup & \text{hyperplane section} \\
S \cap \mathbb{P}^{g-1} = C & \subset & \mathbb{P}^{g-1}.
\end{array}
$$

**Lemma 2.2** ([Reid] 3.3). Assume that $S \subset \mathbb{P}^g$ is a surface with at worst rational double points. Then the following conditions are equivalent;

(i) $S$ is a K3 surface embedded by a very ample complete linear system.
(ii) Every smooth hyperplane section is a canonical curve of genus $g$.

(iii) One smooth hyperplane section is a canonical curve of genus $g$.

According to this lemma, we only need to show that the extension $S$ is smooth or $S$ has at worst rational double points as its singularities for our main theorem. We shall often use Bertini’s theorem which guarantees us the existence of smooth extensions; if $\Lambda$ is a base point free linear system on a smooth variety $X$, then every general member of $\Lambda$ is smooth ([GH], p.137). The same holds true under the weaker assumption that there exists a member which is smooth at $p$ for every base point $p$ of $\Lambda$.

**Lemma 2.3 (Bertini type lemma for complete linear sections).** Let $\Lambda$ be a linear system of dimension $n$ on $X$. Assume that the base locus $B$ of the system $\Lambda$ is smooth of codimension $n + 1$, i.e., $B$ is a complete intersection of basis divisors of $\Lambda$, then the general member of $\Lambda$ is smooth.

**Proof.** The general member $D$ of a linear system $\Lambda$ is smooth away from the base locus. Since $B$ is smooth complete intersection of $D$ and $n$ divisors of $\Lambda$, $D$ is also smooth around $B$. $\square$

**Lemma 2.4 (Bertini type lemma for two divisors).** Let $W$ be a smooth divisor and $\mathcal{L}$ a line bundles on $X$. Let $D \subset W$ be a smooth member of the linear system $|\mathcal{L}|_{W}$. Assume that $H^{1}(X, \mathcal{L}(-W)) = 0$ and the linear system $|\mathcal{L}(-W)|$ is base point free. Then $D$ has a smooth extension, i.e., there is a smooth divisor $\tilde{D} \in |\mathcal{L}|$ on $X$ which satisfies $\tilde{D} \cap W = D$.

**Proof.** Since $H^{1}(X, \mathcal{L}(-W)) = 0$, the restriction map
\[ H^{0}(X, \mathcal{L}) \longrightarrow H^{0}(W, \mathcal{L}|_{W}) \]
is surjective, and therefore there is a divisor $\mathcal{D} \in |\mathcal{L}|$ such that $\mathcal{D} \cap W = D$.

Consider the linear subsystem
\[ \Lambda = \langle \mathcal{D}, |\mathcal{L}(-W)| + W \rangle \subset |\mathcal{L}| \]
gerated by $\mathcal{D}$ and the members of $|\mathcal{L}(-W)| + W$. Since $|\mathcal{L}(-D)|$ is base point free, the base locus of $\Lambda$ is $D \cap W = D$. By Bertini’s theorem, there is a divisor $\tilde{D} \in \Lambda$ which is smooth away from $D = \tilde{D} \cap W$. Since $D = \tilde{D} \cap W$ is smooth complete intersection, $\tilde{D}$ is smooth around $D$, hence smooth everywhere. $\square$
Lemma 2.5 (Bertini type lemma for more divisors). Let $D_1, \ldots, D_s$, and $W$ be divisors on $X$. Assume that $C := W \cap D_1 \cap \cdots \cap D_s$ is smooth complete intersection, and $D_i \cap Bs|D_i - W| = \emptyset$ for $i = 1, \ldots, s$. Then there exist divisors $\tilde{D}_1, \ldots, \tilde{D}_s$, $\tilde{D}_i \sim D_i$, such that $S := \tilde{D}_1 \cap \cdots \cap \tilde{D}_s$ is smooth and $S \cap W = C$.

Proof. We prove the case $s = 2$. Induction goes for $s \geq 2$.

First, consider the linear system
$$
\Lambda_1 = \langle D_1, |D_1 - W| + W \rangle \subset |D_1|
$$
on $X$. Since $D_1 \cap Bs|D_1 - W| = \emptyset$, we have $Bs(\Lambda_1) = D_1 \cap W$. Let $\tilde{D}_1$ be a general member of $\Lambda_1$, then $\tilde{D}_1$ is smooth away from $D_1 \cap W = \tilde{D}_1 \cap W$.

Next, consider the linear system
$$
\Lambda_2 = ((D_2, |D_2 - W| + W))|_{\tilde{D}_1} \subset |(D_2)|_{\tilde{D}_1}|
$$
on $\tilde{D}_1$. Since $D_2 \cap Bs|D_2 - W| = \emptyset$, we have $Bs(\Lambda_2) = \tilde{D}_1 \cap D_2 \cap W = C$ which is a smooth complete intersection. Therefore a general member $D'_2 \in \Lambda_2$ satisfies $D'_2 \cap W = \tilde{D}_1 \cap D_2 \cap W = C$ and is smooth away from $Sing(\tilde{D}_1) \cup Bs(\Lambda_2) \subset (W \cap \tilde{D}_1) \cup C$. Since $D'_2$ meets $W$ only at $C$, $D'_2$ is smooth away from $C$.

It is clear, from the definition of $\Lambda_2$, that there exist an extension $\tilde{D}_2 \in |D_2|$ of $D'_2$, i.e., $\tilde{D}_2 \cap \tilde{D}_1 = D'_2$. Since $S = \tilde{D}_1 \cap \tilde{D}_2 = D'_2$ is smooth away from $C = W \cap \tilde{D}_1 \cap \tilde{D}_2$, $S$ is smooth everywhere.

A weighted projective variety $X \subset \mathbb{P}(a_1 : a_2 : \cdots : a_n)$ is said to be quasi-smooth if its affine cone $\text{Cone}(X) \subset \mathbb{A}(a_1 : a_2 : \cdots : a_n) = \mathbb{A}^n$ is smooth outside the vertex $0 \in \mathbb{A}^n$. If a weighted projective variety $X$ is quasi-smooth, then $X$ has at worst cyclic quotient singularities.

Lemma 2.6 (Bertini type lemma for weighted projective varieties).
Let $X$ be a quasi-smooth weighted projective variety. Assume that $C$ is a smooth complete intersection of divisors in $X$, and satisfies the same assumptions as in Lemmas 2.3, 2.4, or 2.5. Then there is an extension $S$ of $C$ which has at worst cyclic quotient singularities. Moreover, if $C$ is smooth curve and $X$ is Gorenstein, then the extension $S$ has at worst rational double points.

Proof. Since $C$ is smooth, its affine cone $\text{Cone}(C)$ is smooth outside the vertex. By Bertini type lemmas, we can construct an extension $\text{Cone}(S)$.
of Cone($C$), which is smooth outside the vertex. Therefore $S$ has at worst cyclic quotient singularities.

If $C$ is a curve and $X$ is Gorenstein, then the extension $S$ is a surface with at worst Gorenstein cyclic quotient singularities. Therefore these singularities are rational double points.

3 Curves with very special linear systems

The main tool in this section is the rational normal scrolls $F = F(a_1, \ldots, a_n)$. We denote by $H$ (instead of $M$ in [Reid]) the pull back of the hyperplane section divisor under the natural projective morphism $F \to \mathbb{P}^N$ ($N = \sum (a_i + 1) - 1$), and by $L$ the fiber (class) of the projection $F \to \mathbb{P}^1$. As in [Reid], we denote by $F_i$ the $i$-th coordinate divisor $\{x_i = 0\}$, which is a divisor of class $H - a_i L$.

3.1 Hyperelliptic cases

Let $C$ be a smooth hyperelliptic curve of genus $g$. Then the canonical divisor $K_C$ defines a two-to-one map $\Phi|_{K_C}$ from $C$ onto a rational normal curve $\overline{C}$ of degree $g - 1$ in $\mathbb{P}^{g-1}$. The morphism $\Phi|_{K_C} : C \to \overline{C}(\subset \mathbb{P}^{g-1})$ is branched over $2g - 2$ points $P_1, \ldots, P_{2g+2}$. Since $C$ is smooth, these points are distinct.

We consider a commutative diagram

$$
\begin{array}{ccc}
F & \hookrightarrow & \mathbb{P}^g \\
\cup & & \cup \\
C & \twoheadrightarrow & \overline{C} \hookrightarrow \mathbb{P}^{g-1},
\end{array}
$$

where $F$ is the two-dimensional rational normal scroll of degree $g - 1$ and $\overline{C}$ is embedded as a hyperplane section. The canonical divisor of $F$ is $K_F = -2H + (g - 3)L$. We take

$$
\begin{cases}
F(\frac{g-1}{2}, \frac{g-1}{2}) & \text{if } g \text{ is odd}, \\
F(\frac{g}{2}, \frac{g}{2} - 1) & \text{if } g \text{ is even}.
\end{cases}
$$

as $F$.

Proposition 3.1. If $2 \leq g \leq 9$, there is a smooth curve $B \in |{-2K_F}|$ which passes through $P_1, \ldots, P_{2g+2}$.

Proof. Since $-2K_F \sim 4H - 2(g - 3)L$ and $\overline{C} \sim H$, we have an exact sequence

$$
0 \to \mathcal{O}_F(3H - 2(g - 3)L) \to \mathcal{O}_F(-2K_F) \to \mathcal{O}_{\overline{C}}(-2K_F) \to 0.
$$

5
Since the degree of $\mathcal{O}_C(-2K_F)$ is
\[
(4H - 2(g - 3)L)H = 4H^2 - 2(g - 3)HL = 4(g - 1) - 2(g - 3) = 2g + 2
\]
on $C \cong \mathbb{P}^1$, we have $\mathcal{O}_C(-2K_F) \cong \mathcal{O}_{\mathbb{P}^1}(2g + 2)$ and $P_1 + \cdots + P_{2g+2}$ is a smooth member of the system $|\mathcal{O}_C(-2K_F)|$.

If $g$ is odd, we have
\[
H^1(\mathbb{P}, \mathcal{O}_F(3H - 2(g - 3)L)) = H^1(\mathbb{P}, (\text{Sym}^3(\mathcal{O}_{\mathbb{P}^1}(\frac{2g-1}{2}) \oplus \mathcal{O}_{\mathbb{P}^1}(\frac{g-1}{2}))))(-2(g - 3)))
\]
and this vanishes for $g \leq 11$. Moreover, since
\[
3H - 2(g - 3)L = 3(H - \frac{g-1}{2}L) + (\frac{g-2}{2}L),
\]
the linear system $|3H - 2(g - 3)L|$ is base point free for $g \leq 9$. Therefore there is a smooth extension $B \in | - 2K_F|$ of $P_1 + \cdots + P_{2g+2} \in | - 2K_F|_C$ by Lemma 2.4.

If $g$ is even, since
\[
\pi_* \mathcal{O}_F(3H - 2(g - 3)L) \cong \mathcal{O}_{\mathbb{P}^1}(6 - \frac{g}{2}) \oplus \mathcal{O}_{\mathbb{P}^1}(5 - \frac{g}{2}) \oplus \mathcal{O}_{\mathbb{P}^1}(4 - \frac{g}{2}) \oplus \mathcal{O}_{\mathbb{P}^1}(3 - \frac{g}{2}),
\]
$H^1(\mathbb{P}, \mathcal{O}_F(3H - 2(g - 3)L))$ vanishes for $g \leq 8$, and the linear system $|3H - 2(g - 3)L|$ is base point free for $g \leq 6$. Therefore, by Lemma 2.4, there is a smooth extension $B \in | - 2K_F|$ for $g = 2, 4, 6$.

If $g = 8$, the system $|3H - 2(g - 3)L| = |3H - 10L|$ has $F_1 \sim H - 4L$ as its base component, and the system $|3H - 10L - F_1| = |2(H - 3L)|$ is base point free. We may assume that $P$ does not intersect $F_1$, since there is an action of $\text{PGL}(1)$ on $C \cong \mathbb{P}^1$. Let $B \subset F$ be an extension of the 18 branch points $P = P_1 + \cdots + P_{18} \subset C$ such that $F_1 \not\subset B$. We now consider the linear system
\[
\Lambda = \langle B, |3H - 10L| + C \rangle = \langle B, |2(H - 3L)| + F_1 + C \rangle.
\]

By Lemma 2.4, we can choose $B$ so general that $B$ is smooth outside $B \cap F_1$. Since $F_1 \cong \mathbb{P}^1$ is smooth, general members of $\Lambda$ are smooth at $B \cap F_1$. Hence general members of $\Lambda$ are smooth everywhere. \qed
3.2 Trigonal cases

Let $C$ be a smooth non-hyperelliptic trigonal curve of genus $g \geq 5$. Then $C$ is contained in a 2-dimensional rational normal scroll $F = F(a_1, a_2)$ of degree $d = a_1 + a_2 = g - 2$, and $C$ is a divisor linearly equivalent to $3H - (g - 4)L$. By [Sch], we have a bound
\[
\frac{2g - 2}{3} \geq a_1 \geq a_2 \geq \frac{g - 4}{3}.
\]

If $g = 5$, $C$ is contained in $F = F(2, 1)$ and $C$ is a divisor of class $3H - L$. There is a commutative diagram
\[
\tilde{F} := F(1, 1, 1) \xrightarrow{\phi} \mathbb{P}^5 \\
C \subset F := F(2, 1) \xrightarrow{\psi} \mathbb{P}^4,
\]
and $F$ is a divisor linearly equivalent to the hyperplane section $\tilde{H}$ on $\tilde{F}$. Since $2\tilde{H} - \tilde{L} = 2(\tilde{H} - \tilde{L}) + L$, the system $|2\tilde{H} - \tilde{L}|$ is base point free. We have
\[
H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(2\tilde{H} - \tilde{L})) = H^1(\mathbb{P}^1, \text{Sym}^2(\mathcal{O}_{\mathbb{P}^1}(1))^{\oplus 2})(-1)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 6}) = 0,
\]
and therefore by Lemma 2.4, there is a smooth surface $S$ of class $3\tilde{H} - \tilde{L}$ in $\tilde{F}$. Thus $C$ has a smooth K3 extension.

For a smooth trigonal curve of genus $g$, what we have to do is;

1. classify the type $(a_1, a_2)$ of $F$ and find a type $(b_1, b_2, b_3)$ of $\tilde{F}$ suitable for extension,
2. check the vanishing of $H^1(\mathbb{P}^1, (\text{Sym}^2 \tilde{E})(4 - g))$, and
3. check the freeness of the system $|2\tilde{H} - (g - 4)\tilde{L}|$.

where $\tilde{E} = \mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b_2) \oplus \mathcal{O}_{\mathbb{P}^1}(b_3)$.

The table below is the answer to (1). The condition (2) holds for $5 \leq g \leq 9$, and (3) holds for $g = 5, 6, 8$.

For $g = 7$, since $H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}(2\tilde{H} - 3\tilde{L})) = 0$, there is an extension $S' \in |3\tilde{H} - 3L|$ of $C$. The linear pencil
\[
\Lambda = \langle S', |2\tilde{H} - 3\tilde{L}| + F \rangle
\]
has the base locus $Bs\Lambda = (S' \cap F) \cup (S' \cap B) = (S' \cap F_1 \cap F_2)$.
<table>
<thead>
<tr>
<th>genus</th>
<th>$F$</th>
<th>$\bar{F}$</th>
<th>base locus</th>
<th>vanishing of $H^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>(2,1)</td>
<td>(1,1,1)</td>
<td>$\emptyset$</td>
<td>$H^1(\mathbb{P}^1, (\text{Sym}^2 E)(-1)) = 0$</td>
</tr>
<tr>
<td>6</td>
<td>(3,1)</td>
<td>(2,1,1)</td>
<td>$\emptyset$</td>
<td>$H^1(\mathbb{P}^1, (\text{Sym}^2 \tilde{E})(-2)) = 0$</td>
</tr>
<tr>
<td>7</td>
<td>(4,1)</td>
<td>(2,2,1)</td>
<td>$F_1 \cap F_2$</td>
<td>$H^1(\mathbb{P}^1, (\text{Sym}^2 \tilde{E})(-3)) = 0$</td>
</tr>
<tr>
<td>8</td>
<td>(4,2)</td>
<td>(2,2,2)</td>
<td>$\emptyset$</td>
<td>$H^1(\mathbb{P}^1, (\text{Sym}^2 \tilde{E})(-4)) = 0$</td>
</tr>
<tr>
<td>9</td>
<td>(5,2)</td>
<td>(3,2,2)</td>
<td>$F_1 \cap F_2$</td>
<td>$H^1(\mathbb{P}^1, (\text{Sym}^2 \tilde{E})(-5)) = 0$</td>
</tr>
<tr>
<td>10</td>
<td>(6,2)</td>
<td>(4,2,2)</td>
<td>$F_1 \cap F_2$</td>
<td>$h^1(\mathbb{P}^1, (\text{Sym}^2 E)(-6)) = 1$</td>
</tr>
<tr>
<td></td>
<td>(5,3)</td>
<td>(3,3,2)</td>
<td>$F_1 \cap F_2$</td>
<td>$h^1(\mathbb{P}^1, (\text{Sym}^2 \tilde{E})(-6)) = 0$</td>
</tr>
<tr>
<td></td>
<td>(4,4)</td>
<td>(4,3,1)</td>
<td>$F_1 \cap F_2$</td>
<td>$h^1(\mathbb{P}^1, (\text{Sym}^2 \tilde{E})(-6)) = 1$</td>
</tr>
</tbody>
</table>

We can choose the linear embedding $\mathbb{F} \subset \mathbb{F}(2,2,1)$ so that $C$ does not contain $F_1 \cap F_2 \cap \mathbb{F}$. Therefore $S'$ does not contain $F_1 \cap F_2 \cong \mathbb{P}^1$. Since $S'$ and $F_1 \cap F_2$ have the intersection number

$$(S')(F_1)(F_2) = (3\tilde{H} - 3\tilde{L})(\tilde{H} - 2\tilde{L})^2$$

$$= 3\tilde{H}^3 - 15\tilde{H}^2\tilde{L}$$

$$= 3 \cdot 5 - 15 \cdot 1 = 0,$$

we conclude that $S' \cap F_1 \cap F_2$ is empty. Hence a general member $S$ of $\Lambda$ is smooth by Lemma 2.4. Thus $C$ has a smooth K3-extension $S$.

### 3.3 Bielliptic cases

Let $C \subset \mathbb{P}^{g-1}$ be a smooth bielliptic canonical curve of genus $g$. By definition, there is a two-to-one morphism $f : C \rightarrow E$ from $C$ onto an elliptic curve $E$. For any point $p$ in $E$, set $f^*(p) = q_1 + q_2$, and define the line $l_p$ in $\mathbb{P}^{g-1}$ as follows:

$$l_p = \begin{cases} 
\text{the line passing through } q_1 \text{ and } q_2 & \text{if } q_1 \neq q_2, \\
\text{the tangent line to } C \text{ at } q_1 & \text{if } q_1 = q_2.
\end{cases}$$

Let $p, p'$ be points in $E$ and set $f^*(p) = q_1 + q_2$ and $f^*(p') = q'_1 + q'_2$. Then

$$h^0(C, \mathcal{O}_C(q_1 + q_2 + q'_1 + q'_2)) = h^0(E, \mathcal{O}_E(p + p')) = 2,$$
and therefore \( q_1, q_2, q'_1, \) and \( q'_2 \) are all lie in a 2-plane by the geometric version of Riemann-Roch theorem ([ACGH]). Since \( C \) is non-degenerate, this implies that all the lines \( l_p \)'s pass through a common point \( p \in \mathbb{P}^{g-1} \setminus C \). The projection from \( p \) gives a two-to-one map \( \pi_p : C \rightarrow E_{g-1} \) from \( C \) onto an elliptic curve \( E_{g-1} \subset \mathbb{P}^{g-2} \) of degree

\[
\deg E_{g-1} = \frac{1}{2} \deg C = g - 1.
\]

Every elliptic curves \( E := E_{g-1} \) of degree \( g - 1 \) in \( \mathbb{P}^{g-2} \), where \( 5 \leq g - 1 \leq 8 \), is smoothly extended to del Pezzo surfaces \( S := S_{g-1} \) of degree \( g - 1 \) in \( \mathbb{P}^{g-1} \). The extension \( S \) is the blowing-up \( \pi : S \rightarrow \mathbb{P}^2 \) at \( 9 - (g - 1) \) points, and the elliptic curve \( E \) is the strict transform of a nonsingular cubic curve which passes through all the center of the blowing-up.

Let \( B = B_1 + \cdots + B_{2g-2} \) be the branch locus of \( \pi_p : C \rightarrow E \), and \( R = R_1 + \cdots + R_{2g-2} \) be the ramification locus. Then \( K_C \sim \pi_p^*(K_E) + R \sim R \) since \( E \) is elliptic. We distinguish the ambient spaces \( \mathbb{P}^{g-1} \) of \( C \) and \( S \), and denote them by \( \mathbb{P}^{g-1}_1 \) and \( \mathbb{P}^{g-1}_2 \) respectively. Let \( H_i(i = 1, 2) \) be the hyperplane divisor classes of \( \mathbb{P}^{g-1}_i \). Then \( H_1|_C = K_C \sim R \) and hence

\[
2H_2|_E \sim \pi_p^*H_1|_C \sim \pi_p^*R \sim B.
\]

On the other hand, we have \( H_2|_E \sim -K_S|_E \), thus we conclude that

\[
B \sim (-2K_S)|_E.
\]

**Proposition 3.2.** There is a smooth curve \( X \in |-2K_S| \) on \( S \) which passes through \( B_1, \cdots, B_{2g-2} \).

**Proof.** Let \( h \in \text{Pic}(S) \) be the pull-back of a line of \( \mathbb{P}^2 \) and \( e = e_1 + \cdots + e_{10-g} \) be the sum of all the exceptional divisors. Since \( K_S \sim -3h + e \) and \( E \sim 3h - e \sim -K_S \), there is an exact sequence

\[
0 \rightarrow \mathcal{O}_S(-K_S) \rightarrow \mathcal{O}_S(-2K_S) \rightarrow \mathcal{O}_E(-2K_S) \rightarrow 0.
\]

Since \( -K_S \sim H_2|_S \), the system \( |-K_S| = |\mathcal{O}_S(H_2)| = |\mathcal{O}_S(1)| \) is base point free and \( H^1(\mathcal{O}_S(-K_S)) = H^1(\mathcal{O}_S(1)) \) vanishes. Therefore, by Lemma 2.5, \( B \in |-2K_S|_E \) extends to a smooth curve \( X \in |-2K_S| \).

\[
\square
\]

**4 Curves without very special linear systems**

**4.1 Genus \( \leq 5 \)**

Every curve of genus 2 is hyperelliptic, so we have done before. Every non-hyperelliptic curve of genus 3 is a plane quartic, every non-hyperelliptic
curve of genus 4 is a complete intersection of hypersurfaces of degree three and four in $\mathbb{P}^3$, and every non-hyperelliptic non-trigonal curve of genus 5 is a complete intersection three quadric hypersurfaces. Hence they are K3 by Lemma 2.5.

4.2 Genus 6

Let $C$ be a smooth non-hyperelliptic, non-trigonal, non-bielliptic canonical curve of genus 6. There are two cases remaining:

1. $C$ is not plane quintic, and
2. $C$ is smooth plane quintic.

Case 1. In this case, by [Muk2], there is a commutative diagram

$$
G = \text{Grass}(5,2) \subset \mathbb{P}^9 \subset \mathbb{P}^5 \subset C \subset S_5 = G \cap \mathbb{P}^5 \subset \mathbb{P}^5,
$$

where $S_5$ is a quintic del Pezzo surface and $C$ is a hyperquadric section of $S_5$.

Let $H_1, H_2, H_3, H_4$ be the hyperplanes and $Q$ the hyperquadric in $\mathbb{P}^9$ such that $C = G \cap H_1 \cap H_2 \cap H_3 \cap H_4 \cap Q$. Then the systems $|H_i - H_1|_{\mathcal{O}_G}$ and $|(Q - H_1)|_{\mathcal{O}_G} = |\mathcal{O}_G(1)|$ are base point free and $H^1(\mathcal{O}_G) = H^1(\mathcal{O}_G(1)) = 0$. Therefore there are extensions $\tilde{H}_2, \tilde{H}_3, \tilde{H}_4$ and $\tilde{Q}$ such that $S := G \cap \tilde{H}_2 \cap \tilde{H}_3 \cap \tilde{H}_4 \cap \tilde{Q}$ is a smooth surface. Thus $C$ has a smooth K3 extension.

Case 2. If $C$ has a $g^2_5$, then there is an isomorphism from $C$ onto a smooth plane quintic $C_5 = \{f(x_0, x_1, x_2) = 0\} \subset \mathbb{P}^2$, and the canonical model is the image of $C_5$ under the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^9$.

Let $L = \{l(x_0, x_1, x_2) = 0\} \subset \mathbb{P}^2$ be a line which meets $C_5$ transversally at 5 distinct points. Let $S \rightarrow \mathbb{P}^2$ be the blowing-up at $L \cap C_5$, and $\tilde{L}$ and $\tilde{C}_5$ be the strict transform of $L$ and $C_5$ respectively. Then $\tilde{L} + \tilde{C}_5$ is a smooth member of $|-2K_S|$, and therefore the double covering $X \rightarrow S$ is the smooth K3 surface which contains a curve isomorphic to $C$.

Remark. The pull back of $L$ is a $(-2)$-curve on the smooth K3 surface $X$. Collapsing this and we get a weighted projective ample K3-extension $\tilde{X} = \{l(x)y^2 + f_3(x) = 0\} \subset \mathbb{P}(1:1:1:2)$.  

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4.3 Genus 7

Let $C$ be a smooth non-hyperelliptic non-trigonal non-bielliptic curve of genus 7. There are three cases remaining:

1. $C$ has a $g^1_4$ but no $g^2_6$.
2. $C$ has a $g^2_6$ but is not bielliptic.
3. $C$ is non-tetragonal (i.e., $C$ has no $g^1_4$’s)

For Case 3, our main theorem is immediate from the Bertini type lemma 2.3 and the Mukai linear section theorem.

**Theorem 4.1 ([Muk3]).** A curve $C$ of genus 7 is a transversal linear section of the 10-dimensional orthogonal Grassmannian $X \subset \mathbb{P}^{15}$ if and only if $C$ is not tetragonal.

**Case 1.** Let $\alpha$ be a $g^1_4$ and $\beta := \omega C \alpha^{-1}$ its Serre adjoint. Then $\beta$ is a $g^3_8$ by the Riemann-Roch theorem. Since $C$ has no $g^2_6$ the morphism $\Phi_{|\alpha|} : C \rightarrow \mathbb{P}^3 = \mathbb{P}^* H^0(\beta)$ is an embedding and the multiplication map

$$\mu : H^0(\alpha) \otimes H^0(\beta) \rightarrow H^0(\omega_C)$$

is surjective by [Muk3]. Hence we have a linear embedding

$$\mu^* : \mathbb{P}^6 = \mathbb{P}^*(H^0(\omega_C)) \rightarrow \mathbb{P}^*(H^0(\alpha) \otimes H^0(\beta))$$

and there is a commutative diagram

$$\begin{array}{ccc}
\mathbb{P}^1 \times \mathbb{P}^3 & \xrightarrow{\text{Segre}} & \mathbb{P}^7 \\
\Phi_{|\alpha|} \times \Phi_{|\beta|} & & \\
\downarrow & & \uparrow \mu^* \\
C & \xrightarrow{\text{canonical}} & \mathbb{P}^6.
\end{array}$$

By [Muk3], $C$ is a complete intersection of divisors of bidegrees (1,1), (1,2) and (1,2) in $\mathbb{P}^1 \times \mathbb{P}^3$. Let $W = (\mathbb{P}^1 \times \mathbb{P}^3) \cap \mu^*(\mathbb{P}^6)$ be the divisor of bidegree (1,1) and $D_1, D_2$ the divisors of degree (1,2) such that $C = W \cap D_1 \cap D_2$. Since $|D_i - W| = |O_{\mathbb{P}^1 \times \mathbb{P}^3}(0,1)|$ is base point free for $i = 1, 2$ and $H^1(O_{\mathbb{P}^1 \times \mathbb{P}^3}(0,1)) = 0$, by Lemma 2.5, we have extensions $\tilde{D}_1$ and $\tilde{D}_2$ of $D_1$ and $D_2$ respectively such that $S = \tilde{D}_1 \cap \tilde{D}_2$ is a smooth surface. Thus $C$ has a K3 extension.
Case 2. Let $\alpha$ be a $g^2_6$ and $\beta = \omega_C \alpha^{-1}$ its Serre adjoint. Then $\beta$ is also a $g^2_6$ by the Riemann-Roch theorem. If $\alpha$ is not isomorphic to $\beta$, we have a commutative diagram

$$\begin{array}{ccc}
P^2 \times P^2 & \xrightarrow{\text{Segre}} & P^8 \\
\phi_{|\alpha|} \times \phi_{|\beta|} & \uparrow & \alpha^* \\
C & \xrightarrow{\text{canonical}} & P^6.
\end{array}$$

By [Muk3], all morphisms in the diagram are embeddings, and $C$ is a complete intersection of divisors of bidegrees $(1,1)$, $(1,1)$ and $(2,2)$ in $P^2 \times P^2$. Let $H_1$ and $H_2$ be divisors of bidegree $(1,1)$ and $D$ a divisor of bidegree $(2,2)$ such that $C = H_1 \cap H_2 \cap D$. Then the systems $|H_2 - H_1| = |O_{P^2 \times P^2}|$ and $|D - H_1| = |O_{P^2 \times P^2}(1,1)|$ are base point free and $H^1(O_{P^2 \times P^2}) = H^1(O_{P^2 \times P^2}(1,1)) = 0$. Therefore, by Lemma 2.5, we have extensions $\tilde{H}_2$ and $\tilde{D}$ such that $S := \tilde{H}_2 \cap \tilde{D}$ is a smooth K3-extension of $C$.

If $\alpha$ is isomorphic to $\beta$, then by [Muk3] the canonical embedding $C \hookrightarrow P^6$ factors through the weighted projective space $P(1 : 1 : 1 : 2)$, and $C$ is a complete intersection of two divisors $D_3$ and $D_4$ in $P(1 : 1 : 1 : 2)$ of degree 3 and 4 respectively. By Lemma 2.6, we can extend these divisors to $\tilde{D}_3$ and $\tilde{D}_4$ in $P(1 : 1 : 1 : 2 : 2)$ of degree 3 and 4 such that $S = \tilde{D}_3 \cap \tilde{D}_4$ has at worst cyclic quotient singularities. These singularities are Gorenstein since $P(1 : 1 : 1 : 2 : 2)$ is so. Thus $S$ has only rational double points as its singularities and $S$ is an ample K3-extension of $C$.

4.4 Genus 8

Let $C$ be a non-hyperelliptic, non-trigonal, non-bielliptic smooth curve of genus 8. We have one of the following:

1. $C$ has a $g^1_4$ but has no $g^2_6$,
2. $C$ has a $g^2_6$ but is not bielliptic,
3-1 $C$ has a $g^2_7$ such that $\alpha^2 \not\sim \omega_C$, but $C$ has no $g^1_4$,
3-2 $C$ has a $g^2_7$ such that $\alpha^2 \cong \omega_C$, but $C$ has no $g^1_4$, or
4. $C$ has no $g^2_7$.

For Case 4, it is immediate from Bertini type lemma 2.3 and the Mukai linear section theorem.
Theorem 4.2 ([Muk2]). A curve $C$ of genus 8 is a transversal linear section of the 8-dimensional Grassmanian variety $Gr(2, 6) \subset \mathbb{P}^{14}$ if and only if it has no $g^2_7$.

Case 1. In this case we have

Theorem 4.3 ([Muk1],[MI]). The canonical curve $C$ is the complete intersection of four divisors in $\mathbb{P}^1 \times \mathbb{P}^4$ of bidegrees $(1, 1), (1, 1), (1, 2)$ and $(0, 2)$.

Let $X$ be the unique irreducible divisor of bidegree $(0, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^4$ which contains $C$. Let $D'_1, D'_2$, and $E'$ be the divisors on $X$ of bidegrees $(1, 1), (1, 1)$ and $(1, 2)$ respectively, such that $C = D'_1 \cap D'_2 \cap E'$ in $X$. Since $|E' - D'_2|$ and $|D'_1 - D'_2|$ are base point free linear systems and since $H^1(O_X(D'_1 - D'_2)) = 0$, there are divisors $D'_0$ and $E'_0$ of bidegrees $(1, 1)$ and $(1, 2)$ such that $S = D'_0 \cap E'_0$ is smooth away from the singular locus $\text{Sing}(X)$ of $X$.

If $X$ is $\mathbb{P}^1 \times \mathbb{P}(1 : 1 : 2 : 2)$, then $\dim \text{Sing}(X) = 2$ and we can choose $D'_0$ and $E'_0$ so general that $S = D'_0 \cap E'_0$ has at worst ordinally double points as its singularities.

If $X$ is $\mathbb{P}^1 \times \text{Cone}(\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3)$ or $\mathbb{P}^1 \times$ (smooth quadric), then $\dim \text{Sing}(X) \leq 1$ and therefore a general intersection $S = D'_0 \cap E'_0$ does not meet $\text{Sing}(X)$.

Hence $S$ is smooth.

Case 2. By [Muk1], the canonical curve $C$ is the complete intersection of two divisors in $X$ of classes $| - K_X |$ and $| - \frac{1}{2} K_X |$, where $X$ is a blowing-up of $\mathbb{P}^3$ at a one point. Then $| - \frac{1}{2} K_X |$ is very ample and therefore $C$ is a hyperplane section of $D$. Since $| - \frac{1}{2} K_X | = | 2h - e |$ is base point free, $C$ has a smooth extension $\tilde{D} \in | - K_X |$ by Lemma 2.5.

Case 3. Let $\alpha$ be a $g^2_7$ and $\beta = \omega_C \alpha^{-1}$ its Serre adjoint. By the Riemann-Roch theorem, $\beta$ is also a $g^2_7$.

Case 3-1. If $\alpha$ is not isomorphic to $\beta$, then by [MI], the canonical curve $C$ is the complete intersection of three divisors in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegrees $(1, 1), (1, 2)$ and $(2, 1)$.

Let $W = (\mathbb{P}^2 \times \mathbb{P}^2) \cap \mathbb{P}^7$ be the unique divisor of bidegree $(1,1)$, and $D_1, D_2$ divisors of bidegrees $(1,2)$ and $(2,1)$ respectively such that $C = W \cap D_1 \cap D_2$. Then $|D_i - W|$ is base point free, $H^0(D_i - W) \neq 0$, and $H^1(D_i - D_2) = 0$. Therefore, by the Lemma 2.5, there are divisors $\tilde{D}_1, \tilde{D}_2$ of bidegrees $(1,2)$ and $(2,1)$ such that $S := \tilde{D}_1 \cap \tilde{D}_2$ is smooth and $\tilde{D}_1 \cap \tilde{D}_2 \cap W = C$. Thus $S$ is a smooth K3 extension of $C$. 

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Case 3-2. If $\alpha$ is isomorphic to $\beta$, then the canonical embedding factors through a weighted projective space

$$\mathbb{P}(1 : 1 : 2 : 2) = \mathbb{P}(1^3 : 2^2) = \text{Proj} \mathbb{k}[x_0, x_1, x_2, y_0, y_2],$$

where \{\(x_0, x_1, x_2\}\} is a basis of $H^0(\alpha)$ and \{\(y_0, y_1, \text{Sym}^2(x)\}\} that of $H^0(\alpha^2) = H^0(\omega_C)$.

$$C \hookrightarrow \mathbb{P}(1^3 : 2^2) \hookrightarrow \mathbb{P}(2^6 : 2^2) \cong \mathbb{P}^7 = \mathbb{P}^*H^0(\omega_C).$$

Theorem 4.4 ([MI]). The canonical model $C$ is the complete linear section of the weighted Grassmann $G := \text{Gr}(2, (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2})) \subset \mathbb{P}(1^3 : 2^6 : 3^3)$,

$$[C \subset \mathbb{P}(1^3 : 2^2)] = [G \subset \mathbb{P}(1^3 : 2^6 : 3^3)] \cap \mathbb{P}(1^3 : 2^2).$$

Since $C$ is smooth, its affine cone

$$\text{Cone}(C) = \text{Cone}(G) \cap \mathbb{A}(1 : 1 : 1 : 2) \subset \mathbb{A}(1^3 : 2^6 : 3^3),$$

is smooth away from the vertex. By the Bertini type lemma 2.6, there is a general 5-dimensional plane $\mathbb{P}(1 : 1 : 1 : 2 : 2)$ containing $\mathbb{P}(1 : 1 : 1 : 2 : 2)$ such that $S := G \cap \mathbb{P}(1 : 1 : 1 : 2 : 2)$ has at worst rational double points. Therefore $C$ has an ample K3-extension.

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References


