Ricci Flow on 3-dim Lie groups and 4-dim Ricci flat manifolds

Kensuke Onda (Nagoya University)

1 Introduction

Def 1. • g :Ricci Flow

$$\Leftrightarrow \frac{\partial}{\partial t} g(t)_{ij} = -2\mathrm{Ric}[g]_{ij}$$

• g :Backward Ricci Flow

$$\Leftrightarrow \frac{\partial}{\partial t} g(t)_{ij} = 2\mathrm{Ric}[g]_{ij}$$

Remark 2. • $h := \frac{k}{2} g$, where k>0, then

$$\frac{\partial}{\partial t} h(t)_{ij} = (\frac{k}{2})(-2\mathrm{Ric}[g]_{ij}) = -k\mathrm{Ric}[h]_{ij},$$

because $\mathrm{Ric}[h] = \mathrm{Ric}[\frac{k}{2} g] = \mathrm{Ric}[g]$.

• $g(t)$:Ricci Flow
$\Leftrightarrow g(-t)$: Backward Ricci Flow

Def 3. .
(M, g) :cohomogeneity one for a group G
$\Leftrightarrow$
(1)$G \subset Isom(M, g)$, and
(2) codimension of principal orbit of M for G equals 1.

3-dim unimodular simply-connected Lie groups

Def 4. left-invariant movig frame $\{F_i\}_{i=1}^3$
$\{F_i\}_{i=1}^3$: milnor frame $\Leftrightarrow$

$$[F_2, F_3] = 2n_1 F_1$$
$$[F_3, F_1] = 2n_2 F_2,$$
$$[F_1, F_2] = 2n_3 F_3,$$
$$n_1 \leq n_2 \leq n_3,\ n_i \in \{\pm 1, 0\}$$

Signature	Lie group	description
(-1, -1, -1), (+1, +1, +1)	SU(2)	simle
(-1, -1, 0), (+1, +1, 0)	$\widetilde{\mathrm{Isom}(\mathbf{E^2})}$	solvable
(-1, -1, +1), (-1, +1, +1)	$\widetilde{\mathrm{SL}(2,\mathbb{R})}$	simle
(-1, 0, 0), (+1, 0, 0)	nil	nilpotent
(-1, 0, +1)	$\widetilde{\mathrm{Isom}(\mathbf{E^1_1})}$	solvable
(0, 0, 0)	$\mathrm{R}\otimes\mathbb{R}\otimes\mathbb{R}$	commutative

Table 1: 3-dim unimodular simply-connected Lie groups

Example 5. left-invariant metric g_3 on $SU(2)$

Milnor frame $\{F_i\}_{i=1}^3$,
$[F_i, F_j] = -2F_k$ $((i,j,k) : (1,2,3) cyclic)$
$\{\omega^i\}$: dual coframe of $\{F_i\}_{i=1}^3$

$$g_3 = A(\omega^1)^2 + B(\omega^2)^2 + C(\omega^3)^2$$

Ricci tensor of g_3

$$R_{11} = 2\frac{A^2 - (B-C)^2}{BC},$$

$$R_{22} = 2\frac{B^2 - (C-A)^2}{CA},$$

$$R_{33} = 2\frac{C^2 - (A-B)^2}{AB}.$$

Ricci Flow equation

$$\frac{d}{dt}A = \frac{(B-C)^2 - A^2}{BC}$$

$$\frac{d}{dt}B = \frac{(C-A)^2 - B^2}{CA}$$

$$\frac{d}{dt}C = \frac{(A-B)^2 - C^2}{AB}$$

Prop 6 ('01 D.Knopf)**.** Solution of R.F. equation exists on $(-\infty, T)$, and

$g_3 \longrightarrow$ standard metric on round sphere $(t \longrightarrow T)$

(M^4, g) :cohomogeneity one for $SU(2)$

$$g = dt^2 + a(t)^2(\omega^1)^2 + b(t)^2(\omega^2)^2 + c(t)^2(\omega^3)^2$$

Example 7. Taub-Nut metric

$$g = \left(\frac{r+m}{r-m}\right)dr^2 + (r^2 - m^2)\{(\omega^1)^2 + (\omega^2)^2\} + 4m\left(\frac{r-m}{r+m}\right)(\omega^3)^2$$

$$dt = hdr, a = b = (r^2 - m^2)^{\frac{1}{2}}, c = 2m\left(\frac{r-m}{r+m}\right)^{\frac{1}{2}}, h = 2\left(\frac{r+m}{r-m}\right)^{\frac{1}{2}}$$

,

This metric is Ricci flat and

$$\frac{d}{dt}a = \frac{a^2 - (b-c)^2}{bc}, \frac{d}{dt}b = \frac{b^2 - (c-a)^2}{ca}, \frac{d}{dt}c = \frac{c^2 - (a-b)^2}{ab}.$$

This equation are Backward Ricci Flow equation!

Theorem 8 (G. W. Gibbons, '98)**.** $SU(2)$

$$g_3 = A(\omega^1)^2 + B(\omega^2)^2 + C(\omega^3)^2$$

Ricci Flow equation

$$\frac{d}{dt}A = \frac{(B-C)^2 - A^2}{BC}$$

$$\frac{d}{dt}B = \frac{(C-A)^2 - B^2}{CA}$$

$$\frac{d}{dt}C = \frac{(A-B)^2 - C^2}{AB}$$

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cohomogeneity one for $SU(2)$

$$g = dt^2 + a(t)^2(\omega^1)^2 + b(t)^2(\omega^2)^2 + c(t)^2(\omega^3)^2$$

$$\frac{d}{dt}a = \frac{(b-c)^2 - a^2}{bc}$$

$$\frac{d}{dt}b = \frac{(c-a)^2 - b^2}{ca}$$

$$\frac{d}{dt}c = \frac{(a-b)^2 - c^2}{ab}.$$

g : (Backward) Ricci Flow $\Rightarrow$ g : Ricci flat metric.

Remark 9. Eguchi-Hanson metric

$$dt = hdr, a = b = r, c = r\Big(1-(\frac{m}{r})^4\Big)^{\frac{1}{2}}, h = \frac{1}{\Big(1-(\frac{m}{r})^4\Big)^{\frac{1}{2}}},$$

$$g = \frac{dr^2}{1-(m/r)^4} + r^2\{(\omega^1)^2 + (\omega^2)^2\} + r^2(1-(m/r)^4)(\omega^3)^2$$

is a Ricci flat metric too, but

$$\frac{d}{dt}a = \frac{(b-c)^2 - a^2}{bc} + 2,$$
$$\frac{d}{dt}b = \frac{(c-a)^2 - b^2}{ca} + 2,$$
$$\frac{d}{dt}c = \frac{(a-b)^2 - c^2}{ab} + 2.$$

Ricci flow + (? - ?) = modified? ricci flow?

2 preparation

(M^4, g): codimension one 4-dim manifold for Lie group G
$\{F_i\}_{i=1}^{3}$: Milnor frame of G.
$\{\omega_i\}$: dual coframe of $\{F_i\}$.

Metric

$$g = dt^2 + a(t)^2(\omega^1)^2 + b(t)^2(\omega^2)^2 + c(t)^2(\omega^3)^2$$

Ricci tensor

$$Rc(F_0, F_0) = R_{00} = -\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c}$$

$$Rc(F_1, F_1) = R_{11} = -a\ddot{a} - a\dot{a}\frac{\dot{b}}{b} - a\dot{a}\frac{\dot{c}}{c} - 2\frac{(n_2b^2 - n_3c^2)^2 - n_1^2a^4}{b^2c^2}$$

$$Rc(F_2, F_2) = R_{22} = -b\ddot{b} - b\dot{b}\frac{\dot{c}}{c} - b\dot{b}\frac{\dot{a}}{a} - 2\frac{(n_3c^2 - n_1a^2)^2 - n_2^2b^4}{c^2a^2}$$

$$Rc(F_3, F_3) = R_{33} = -c\ddot{c} - c\dot{c}\frac{\dot{a}}{a} - c\dot{c}\frac{\dot{b}}{b} - 2\frac{(n_1a^2 - n_2b^2)^2 - n_3^2c^4}{a^2b^2}$$

$$Rc(F_i, F_j) = Rc_{ij} = 0 \qquad (i \neq j)$$

3 Nil-geometry

3 dimensional nilpotent manifold

$\{F_i\}$: Milnor frame, $[F_2, F_3] = -2F_1$. $\{\omega_i\}$: dual coframe of $\{F_i\}$.

Metric

$$g = a(\omega^1)^2 + b(\omega^2)^2 + c(\omega^3)^2$$

.

Ricci Flow equation are

$$\frac{d}{dt}a = -\frac{a^2}{bc}, \tag{2a}$$

$$\frac{d}{dt}b = \frac{a}{c}, \tag{2b}$$

$$\frac{d}{dt}c = \frac{a}{b}. \tag{2c}$$

['01 D.Knopf]

$$\begin{cases} a(t) = a_0^{\frac{2}{3}} b_0^{\frac{1}{3}} c_0^{\frac{1}{3}} (3t + b_0 c_0/a_0)^{-\frac{1}{3}}, \\ b(t) = a_0^{\frac{1}{3}} b_0^{\frac{2}{3}} c_0^{-\frac{1}{3}} (3t + b_0 c_0/a_0)^{\frac{1}{3}}, \\ c(t) = a_0^{\frac{1}{3}} b_0^{-\frac{1}{3}} c_0^{\frac{2}{3}} (3t + b_0 c_0/a_0)^{\frac{1}{3}}. \end{cases} \tag{3}$$

Lemma 10 ('01 D.Knopf). • Solution of R.F. equation exists on $(-T, +\infty)$, where $T = b_0 c_0 / 3a_0$

- If $t \to -T$, then $a \to +\infty, b \to 0, c \to 0$.
- If $t \to +\infty$, then $a \to 0, b \to +\infty, c \to +\infty$.

□

(M^4, g) : codimension one for nilpotent manifold G

$$g = dt^2 + a(t)^2(\omega^1)^2 + b(t)^2(\omega^2)^2 + c(t)^2(\omega^3)^2$$

$F_0 = \frac{\partial}{\partial t} \Rightarrow \{F_i\}$: orthogonal frame.

$$R_{00} = -\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c}$$
$$R_{11} = -a\ddot{a} - a\dot{a}\frac{\dot{b}}{b} - a\dot{a}\frac{\dot{c}}{c} + 2\frac{a^4}{b^2c^2}$$
$$R_{22} = -b\ddot{b} - b\dot{b}\frac{\dot{c}}{c} - b\dot{b}\frac{\dot{a}}{a} - 2\frac{a^2}{c^2}$$
$$R_{33} = -c\ddot{c} - c\dot{c}\frac{\dot{a}}{a} - c\dot{c}\frac{\dot{b}}{b} - 2\frac{a^2}{b^2}$$

Theorem 11 (G. W. Gibbons, '98)**.** Let a, b, c satisfy Ricci flow equation,

$$\frac{d}{dt}a = -\frac{a^2}{bc},$$
$$\frac{d}{dt}b = \frac{a}{c},$$
$$\frac{d}{dt}c = \frac{a}{b},$$

then g becomes a Ricci flat metric. □

Sectional curvature are

$$K_{01} = 2\frac{a^2}{b^2c^2}, K_{02} = 2\frac{a^2}{b^2c^2}, K_{03} = -4\frac{a^2}{b^2c^2},$$
$$K_{12} = -4\frac{a^2}{b^2c^2}, K_{13} = 2\frac{a^2}{b^2c^2}, K_{23} = 2\frac{a^2}{b^2c^2}.$$

Prop 12. $a_0b_0c_0 > 0$

- Solution of equation exists on $(-T, \infty)$, where $T = b_0c_0/3a_0$.
- If $t \to -T$,
  then $a \to +\infty, b \to 0, c \to 0$.
  $\Rightarrow K_{ij} \to \pm\infty$.
- If $t \to +\infty$,
  $a \to 0, b \to +\infty, c \to +\infty$.
  $\Rightarrow K_{ij} \to 0$.

4 Sol-Geometry

$\{F_i\}_{i=1}^3$: milnor frame of $\widetilde{\mathrm{Isom}(\mathbf{E}_1^1)}$,
$[F_1, F_2] = -2F_3, [F_3, F_1] = 0, [F_2, F_3] = 2F_1$.
$\{\omega_i\}$: dual coframe of $\{F_i\}$.
Metric

$$g = a(\omega^1)^2 + b(\omega^2)^2 + c(\omega^3)^2.$$

Ricci Flow equation

$$\begin{cases} \dfrac{d}{dt}a = \dfrac{c^2 - a^2}{bc}, \\ \dfrac{d}{dt}b = \dfrac{(a+c)^2}{ac}, \\ \dfrac{d}{dt}c = \dfrac{a^2 - c^2}{ab}. \end{cases}$$

Lemma 13 ('01. D.Knopf)**.** • Solution of Ricci Flow equation exists on $(-T, +\infty)$, where $T > 0$.

- $ac = a_0 c_0, b(c-a) = b_0(c_0 - a_0)$
- $\rho := a/c$,

$$\Rightarrow \frac{d}{dt} b = \frac{(1+\rho)^2}{\rho},$$

$$\frac{d}{dt}\rho = 2\frac{1-\rho^2}{b}.$$

- $\rho_0 < \rho_\infty \leq 1$ or $1 \leq \rho_\infty < \rho_0$.
- If $\rho \neq 1$,

$$b = k_0 \frac{\sqrt{\rho}}{|1-\rho|},$$

where $k_0 = b_0 \frac{|1-\rho_0|}{\sqrt{\rho_0}}$

- $\rho \to 1, b \to \infty \ (t \to +\infty)$.
- $a_\infty = c_\infty = \sqrt{a_0 c_0}$

(M^4, g) : codimension one for $\widetilde{\mathrm{Isom}(\mathbf{E_1^1})}$

$\{F_i\}_{i=1}^3$: milnor frame of $\widetilde{\mathrm{Isom}(\mathbf{E_1^1})}$,

$[F_2, F_3] = -2F_1, [F_3, F_1] = 0, [F_1, F_2] = 2F_3.$

$\{\omega_i\}$: dual coframe of $\{F_i\}$.

Metric

$$g = dt^2 + a(t)^2(\omega^1)^2 + b(t)^2(\omega^2)^2 + c(t)^2(\omega^3)^2.$$

$F_0 = \frac{\partial}{\partial t} \Rightarrow \{F_i\}$: orthogonal frame.

$$R_{00} = -\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c},$$

$$R_{11} = -a\ddot{a} - a\dot{a}\frac{\dot{b}}{b} - a\dot{a}\frac{\dot{c}}{c} + 2\frac{a^4 - c^4}{b^2c^2},$$

$$R_{22} = -b\ddot{b} - b\dot{b}\frac{\dot{c}}{c} - b\dot{b}\frac{\dot{a}}{a} - 2\frac{(a^2 + c^2)^2}{a^2c^2},$$

$$R_{33} = -c\ddot{c} - c\dot{c}\frac{\dot{a}}{a} - c\dot{c}\frac{\dot{b}}{b} + 2\frac{c^4 - a^4}{a^2b^2}.$$

Let a, b, c satisfy Ricci Flow equation,

$$\begin{cases} \dfrac{d}{dt}a = \dfrac{c^2 - a^2}{bc}, \\ \dfrac{d}{dt}b = \dfrac{(a+c)^2}{ac}, \\ \dfrac{d}{dt}c = \dfrac{a^2 - c^2}{ab}. \end{cases}$$

Because $\frac{d}{dt}(ac) = 0$,

$$\begin{aligned} R_{22} &= -b\ddot{b} - b\dot{b}\frac{\dot{c}}{c} - b\dot{b}\frac{\dot{a}}{a} - 2\frac{(a^2+c^2)^2}{a^2c^2}, \\ &= -b\ddot{b} - b\dot{b}\frac{\dot{c}a + \dot{a}c}{ca} - 2\frac{(a^2+c^2)^2}{a^2c^2}, \\ &= 2\frac{(c^2-a^2)^2}{a^2c^2} - 2\frac{(c^2+a^2)^2}{a^2c^2} = -8, \end{aligned}$$

and this metric is not a Ricci flat metric.

4.1 Ricci flat metric of signature $(2,2)$

Metric

$g = dt^2 + a(t)^2(\omega^1)^2 - b(t)^2(\omega^2)^2 - c(t)^2(\omega^3)^2$.

$F_0 = \frac{\partial}{\partial t} \Rightarrow \{F_i\}$: orthogonal frame.

$$
\begin{aligned}
R_{00} &= -\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c} \\
R_{11} &= -a\ddot{a} - a\dot{a}\frac{\dot{b}}{b} - a\dot{a}\frac{\dot{c}}{c} + 2\frac{a^4 - c^4}{b^2c^2} \\
R_{22} &= b\ddot{b} + b\dot{b}\frac{\dot{c}}{c} + b\dot{b}\frac{\dot{a}}{a} + 2\frac{(a^2 - c^2)^2}{a^2c^2} \\
R_{33} &= c\ddot{c} + c\dot{c}\frac{\dot{a}}{a} + c\dot{c}\frac{\dot{b}}{b} + 2\frac{a^4 - c^4}{a^2b^2}
\end{aligned}
$$

Theorem 14. Let a, b, c satisfy Ricci Flow equation,

$$
\begin{cases}
\dfrac{d}{dt}a = \dfrac{c^2 - a^2}{bc}, \\
\dfrac{d}{dt}b = \dfrac{(a+c)^2}{ac}, \\
\dfrac{d}{dt}c = \dfrac{a^2 - c^2}{ab}.
\end{cases}
$$

Then this metric becomes a Ricci flat metric. □

Sectional curvature

$$K_{01} = 2\frac{(c^2-a^2)(2a^2+ac+c^2)}{a^2b^2c^2} = 2\frac{(1-\rho^2)(2\rho^2+\rho+1)}{\rho^2b^2},$$
$$K_{02} = 2\frac{(c^2-a^2)^2}{a^2b^2c^2} = 2\frac{(1-\rho^2)^2}{\rho^2b^2},$$
$$K_{03} = -2\frac{(c^2-a^2)(a^2+ac+2c^2)}{a^2b^2c^2} = -2\frac{(1-\rho^2)(\rho^2+\rho+2)}{\rho^2b^2},$$
$$K_{12} = -2\frac{(c^2-a^2)(a^2+ac+2c^2)}{a^2b^2c^2} = -2\frac{(1-\rho^2)(\rho^2+\rho+2)}{\rho^2b^2},$$
$$K_{13} = 2\frac{(c^2-a^2)^2}{a^2b^2c^2} = 2\frac{(1-\rho^2)^2}{\rho^2b^2},$$
$$K_{23} = 2\frac{(c^2-a^2)(2a^2+ac+c^2)}{a^2b^2c^2} = 2\frac{(1-\rho^2)(2\rho^2+\rho+1)}{\rho^2b^2},$$

where $\rho := a/c$.

Prop 15. • If $a^2 = c^2$, then this metric have constant curvature 0.

- If $a^2 \neq c^2$, this metric is Ricci flat metric, but K_{ij} aren't constant.
- $a_0b_0c_0 > 0$.
  - If $t \to +\infty$, sectional curvature $K_{ij} \to 0$, because $\rho \to 1, b \to +\infty$.
  - If $t \to -T$, then $K_{ij} \to \pm\infty$, because $b \to 0$, and either of $\rho \to 0$ and $\rho \to \infty$.

5 $\widetilde{\mathrm{Isom}(\mathbf{E^2})}$

$\{F_i\}_{i=1}^3$: Milnor frame of $\widetilde{\mathrm{Isom}(\mathbf{E^2})}$,
$[F_2, F_3] = -2F_1, [F_3, F_1] = 0, [F_1, F_2] = -2F_3.$
$\{\omega_i\}$: dual coframe of $\{F_i\}$.
Metric

$$g = a(\omega^1)^2 + b(\omega^2)^2 + c(\omega^3)^2.$$

$$\begin{cases} \dfrac{d}{dt}a = \dfrac{c^2 - a^2}{bc}, \\ \dfrac{d}{dt}b = \dfrac{(a-c)^2}{ac}, \\ \dfrac{d}{dt}c = \dfrac{a^2 - c^2}{ab} \end{cases} \tag{9}$$

Lemma 16 ('01 D.Knopf). • Solution of Ricci Flow equation exists on $(-T, \infty)$.

- $ac = a_0 c_0, b(c+a) = b_0(c_0 + a_0)$
- $k := a/c$,

$$\Rightarrow \frac{d}{dt} b = \frac{(1-k)^2}{k},$$

$$\frac{d}{dt} k = 2\frac{1-k^2}{b}.$$

- $k \to 1, b \to^{\exists} b_\infty$ when $t \to \infty$.
- $$b = k_0 \frac{\sqrt{k}}{1+k},$$

$\widetilde{(M^4, g)}$: codimension one for Lie group $\mathrm{Isom}(\mathbf{E^2})$.

$\{F_i\}_{i=1}^3$: Milnor frame of $\widetilde{\mathrm{Isom}(\mathbf{E^2})}$,
$[F_2, F_3] = -2F_1, [F_3, F_1] = 0, [F_1, F_2] = -2F_3.$
$\{\omega_i\}$: dual coframe of $\{F_i\}$.

Metric
$$g = dt^2 + a(t)^2(\omega^1)^2 + b(t)^2(\omega^2)^2 + c(t)^2(\omega^3)^2.$$

$$R_{00} = -\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c},$$
$$R_{11} = -a\ddot{a} - a\dot{a}\frac{\dot{b}}{b} - a\dot{a}\frac{\dot{c}}{c} + 2\frac{a^4 - c^4}{b^2c^2},$$
$$R_{22} = -b\ddot{b} - b\dot{b}\frac{\dot{c}}{c} - b\dot{b}\frac{\dot{a}}{a} - 2\frac{(a^2 - c^2)^2}{a^2c^2},$$
$$R_{33} = -c\ddot{c} - c\dot{c}\frac{\dot{a}}{a} - c\dot{c}\frac{\dot{b}}{b} + 2\frac{c^4 - a^4}{a^2b^2}.$$

Theorem 17. If a, b, c satisfy Ricci Flow equation,

$$\begin{cases} \dfrac{d}{dt}a = \dfrac{c^2 - a^2}{bc}, \\ \dfrac{d}{dt}b = \dfrac{(a-c)^2}{ac}, \\ \dfrac{d}{dt}c = \dfrac{a^2 - c^2}{ab}, \end{cases}$$

then this metric becomes a Ricci flat metric. $\square$

Sectional curvature

$$K_{01} = 2\frac{(c^2-a^2)(2a^2-ac+c^2)}{a^2b^2c^2} = 2\frac{(1-k^2)(2k^2-k+1)}{b^2k^2},$$

$$K_{02} = 2\frac{(c^2-a^2)^2}{a^2b^2c^2} = 2\frac{(1-k^2)^2}{b^2k^2},$$

$$K_{03} = -2\frac{(c^2-a^2)(a^2-ac+2c^2)}{a^2b^2c^2} = -2\frac{(1-k^2)(k^2-k+2)}{b^2k^2},$$

$$K_{12} = -2\frac{(c^2-a^2)(a^2-ac+2c^2)}{a^2b^2c^2} = -2\frac{(1-k^2)(k^2-k+2)}{b^2k^2},$$

$$K_{13} = 2\frac{(c^2-a^2)^2}{a^2b^2c^2} = 2\frac{(1-k^2)^2}{b^2k^2},$$

$$K_{23} = 2\frac{(c^2-a^2)(2a^2-ac+c^2)}{a^2b^2c^2} = 2\frac{(1-k^2)(2k^2-k+1)}{b^2k^2},$$

where $k := a/c$

Prop 18.
- If $a^2 = c^2$, this metric has constant curvature 0.
- If $a^2 \neq c^2$, this metric is a Ricci flat metric but doesn't have constant curvature.
- $a_0b_0c_0 > 0$, If $t \to +\infty$, $K_{ij} \to 0$, because $k \to 1, b \to^{\exists} b_\infty$

6 $\widetilde{\mathrm{SL}(2,\mathbb{R})}$

$\underbrace{(M^4, g)}$: codimension one for Lie group $\mathrm{SL}(2,\mathbb{R})$.
$\{F_i\}_{i=1}^3$: milnor frame of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, $[F_1, F_2] = -2F_3$, $[F_2, F_3] = 2F_1$, $[F_3, F_1] = 2F_2$.
$\{\omega_i\}$: dual coframe of $\{F_i\}$.

$$g = dt^2 + a(t)^2(\omega^1)^2 + b(t)^2(\omega^2)^2 + c(t)^2(\omega^3)^2.$$

Ricci tensor

$$R_{00} = -\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c},$$

$$R_{11} = -a\ddot{a} - a\dot{a}\frac{\dot{b}}{b} - a\dot{a}\frac{\dot{c}}{c} - 2\frac{(b^2+c^2)^2 - a^4}{b^2c^2},$$

$$R_{22} = -b\ddot{b} - b\dot{b}\frac{\dot{c}}{c} - b\dot{b}\frac{\dot{a}}{a} - 2\frac{(a^2+c^2)^2 - b^4}{a^2c^2},$$

$$R_{33} = -c\ddot{c} - c\dot{c}\frac{\dot{a}}{a} - c\dot{c}\frac{\dot{b}}{b} - 2\frac{(a^2-b^2)^2 - c^4}{a^2b^2}.$$

If a, b, c satisfy Ricci flow equation of Lie group $\mathrm{SL}(2, \mathbb{R})$

$$\begin{cases} \dfrac{d}{dt}a = \dfrac{(b+c)^2 - a^2}{bc}, \\ \dfrac{d}{dt}b = \dfrac{(c+a)^2 - b^2}{ca}, \\ \dfrac{d}{dt}c = \dfrac{(a-b)^2 - c^2}{ab}, \end{cases} \tag{13}$$

then this metric is not a Ricci flat metric.

$\underbrace{(M^4, g)}$: codimension one for Lie group $\mathrm{SL}(2, \mathbb{R})$.

$$g = dt^2 - a(t)^2(\omega^1)^2 - b(t)^2(\omega^2)^2 + c(t)^2(\omega^3)^2.$$

This metric has a signature $(2, 2)$.

Ricci tensor

$$R_{00} = -\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c},$$

$$R_{11} = a\ddot{a} + a\dot{a}\frac{\dot{b}}{b} + a\dot{a}\frac{\dot{c}}{c} + 2\frac{(b^2 - c^2)^2 - a^4}{b^2c^2},$$

$$R_{22} = b\ddot{b} + b\dot{b}\frac{\dot{c}}{c} + b\dot{b}\frac{\dot{a}}{a} + 2\frac{(c^2 - a^2)^2 - b^4}{a^2c^2},$$

$$R_{33} = -c\ddot{c} - c\dot{c}\frac{\dot{a}}{a} - c\dot{c}\frac{\dot{b}}{b} - 2\frac{(a^2 - b^2)^2 - c^4}{a^2b^2}.$$

Ricci flow equation of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$

$$\begin{cases} \dfrac{d}{dt}a = \dfrac{(b+c)^2 - a^2}{bc}, \\ \dfrac{d}{dt}b = \dfrac{(c+a)^2 - b^2}{ca}, \\ \dfrac{d}{dt}c = \dfrac{(a-b)^2 - c^2}{ab}, \end{cases} \tag{15}$$

$a \to -a, b \to -b$

$\Rightarrow$

$$\begin{cases} \dfrac{d}{dt}a = \dfrac{(b-c)^2 - a^2}{bc}, \\ \dfrac{d}{dt}b = \dfrac{(c-a)^2 - b^2}{ca}, \\ \dfrac{d}{dt}c = \dfrac{(a-b)^2 - c^2}{ab}, \end{cases}$$

This equation are Ricci flow equation of $SU(2)$.

Theorem 19. If a, b, c satisfy Ricci flow equation of $SU(2)$, then this metric becomes a Ricci flat metric. □

7 Conclusion

We saw Ricci flat metric,

Lie group	Ricci flow equ.	metric of Signature
SU(2)	SU(2)	(4, 0)
$\widetilde{\mathrm{Isom}}(\mathbf{E^2})$	$\widetilde{\mathrm{Isom}}(\mathbf{E^2})$	(4, 0)
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	SU(2)	(2, 2)
nil	nil	(2, 2)
$\widetilde{\mathrm{Isom}}(\mathbf{E^1_1})$	$\widetilde{\mathrm{Isom}}(\mathbf{E^1_1})$	(2, 2)

Table 2: 3-dim unimodular simply-connected Lie groups

Prospects in the future

- reference of codimension 1 and ricci flow.

- Positive Einstein metric

Thank you for your kind attention. I am happy to answer any question that you might have.