GEIGLE-LENZING WEIGHTED PROJECTIVE SPACES VIA ORDERS

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I gave this talk given on the 14th of Feb 2014 at Warwick university as part of the Warwick-Nagoya seminar on McKay correspondence and related topics.

In this talk I will present very recent research that was conducted at Nagoya university. In particular I will focus on 3 very recent papers – so recent in fact 2 of these have not yet been written:

(1) "Geigle-Lenzing spaces and d-canonical algebras" – Herschend, Iyama, Minamoto and Oppermann.
(2) "Tilting bundles on orders on $\mathbb{P}^d$" – Iyama and me.
(3) Title pending – Iyama, Oppermann and me.

The first section is motivation the next 3 sections focus on the above 3 papers one at a time.

1. Motivation

The motivation for this research stems from representation theory of finite dimensional algebras. We fix a field k and finite dimensional k-algebra Λ. Suppose gl.dim $\Lambda = 1$, or equivalently, Λ is the path algebra of a quiver with no orientated cycles. What properties does $\mathsf{mod}\,\Lambda$ satisfy? We know it is:

- abelian, k-linear
- Hom / Ext-finite
- has global dimension 1; i.e. $\mathrm{Ext}^i(-,-) = 0$ for all $i > 1$
- has a tilting object.

Theorem 1.1 (Happel, 2001)**.** *Let $\mathcal{A}$ be a connected abelian category satisfying the above 4 conditions. Then $\mathcal{A}$ is derived equivalent to either* $\mathsf{mod}\ \Lambda$ *or* $\mathsf{coh}\,\mathbb{X}$

I will explain what $\mathsf{coh}\,\mathbb{X}$ is in the next section. The motivation of for the research I will present lies in finding other abelian categories, of higher global dimension, with tilting objects and studying the corresponding endomorphism algebras of these tilting objects.

2. Geigle-Lenzing weighted projective spaces

I will straight away give a more general construction then the one given by Geigle and Lenzing. This was done by HIMO. If one sets, $d = 1$ in the following, then the GL definition is recovered.

We prepare the following:

- We work on $\mathbb{P}^d_{T_0:\dots:T_d}$
- Fix $n \geq 2$ hyperplanes H_i given by a linear equation $\ell_i(T_0, \cdots, T_d) = 0$. Assume these are in general position.
- Weights $p_i \geq 2$ for $i \in \{1, \cdots, n\}$.

Using this data, define

$$R := \frac{k[T_0, \cdots, T_d, X_1, \cdots, X_n]}{\langle X_i^{p_i} - \ell_i(T_0, \cdots, T_d)\rangle_{1\leq i\leq n}}$$

(If $n = 0$ then $R = k[T_0, \cdots, T_d]$) Furthermore, let

$$\mathbb{L} := \frac{\langle \vec{x}_1, \cdots, \vec{x}_n, \vec{c}\rangle}{\langle p_i\vec{x}_i - \vec{c}\rangle_{1\leq i\leq n}}$$

Note $\mathbb{L}$ is partially ordered - this will be important shortly. Give R an $\mathbb{L}$-grading by declaring deg $X_i = \vec{x}_i$ and $\deg T_i = \vec{c}$. The category of coherent sheaves on a weighted projective space is then defined to be

$$\mathsf{coh}\,\mathbb{X} = \frac{\mathsf{mod}^{\mathbb{L}} R}{\mathsf{mod}^{\mathbb{L}}_0 R}$$

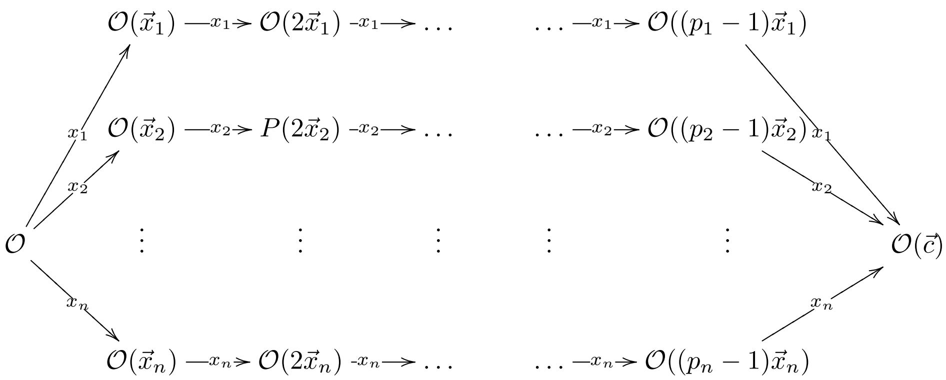
i.e. $\mathbb{L}$-graded R-modules modulo those which are finite dimensional. The image of R in this category is denoted by $\mathcal{O}$. As I mentioned earlier, if $d = 1$ this was done by GL.

Theorem 2.1 (GL (in $d = 1$ case), HIMO)**.**

$$T = \bigoplus_{\vec{x} \in [0, d\vec{c}]} \mathcal{O}(\vec{x})$$

is a tilting object in $\mathsf{coh}\,\mathbb{X}$*.*

Example 2.2. If $d = 1$ then End T is given by



with relations

$$x_i^{p_i} = \ell_i(x_1^{p_1}, x_2^{p_2})$$

for $i \geq 2$. The algebras are called "canonical algebras" and were introduced by Ringel.

3. Orders

We continue with the notation from previous section. We will call the following sheaf algebras an *order* on $\mathbb{P}^d$:

$$A := \bigotimes_{i=1}^{n} \begin{bmatrix} \mathcal{O} & \mathcal{O}(-H_i) & \cdots & \mathcal{O}(-H_i) & \mathcal{O}(-H_i) \\ \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O}(-H_i) & \mathcal{O}(-H_i) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O}(-H_i) \\ \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \end{bmatrix}$$

$$= \bigotimes_{i=1}^{n} A_i$$

where $\mathcal{O} = \mathcal{O}_{\mathbb{P}^d}$ In our paper, Iyama and I prove:

Theorem 3.1.

$$A\,\mathsf{mod} \simeq \mathsf{coh}\,\mathbb{X}$$

Theorem 3.2. *There exists a tilting object in* $A\,\mathsf{mod}$

Of course the second theorem follows from the first and the result in HIMO, however we give a new geometric proof, showing the advantage of working in $A\,\mathsf{mod}$ rather than in $\mathsf{coh}\,\mathbb{X}$. In particular we show the that the existence of a tilting object in $A\,\mathsf{mod}$ depends on the existence of one on $\mathsf{coh}\,\mathbb{P}^n$ for $1 \leq n \leq d$ a fact that was established by Beilinson.

4. Generalisations

One clear advantage of working in $A\,\mathsf{mod}$ over $\mathsf{coh}\,\mathbb{X}$ is that there is a naive way to attempt to generalise: simply let $\mathcal{O} = \mathcal{O}_X$ for some variety X and let the H_i's be prime divisors, and ask: does $A\,\mathsf{mod}$ have a tilting object. We give a sufficient condition for the answer to be "yes" and in fact construct a tilting object. The tilting object will be constructed from tilting objects on $X, H_i, H_i \cap H_j, \ldots$ so for one, we need to assume existence of those.

Before stating the general theorem, I will give an example with just one weight.

Example 4.1. Suppose we have only one weight. I this case

$$A := \begin{bmatrix} \mathcal{O} & \mathcal{O}(-H) & \cdots & \mathcal{O}(-H) & \mathcal{O}(-H) \\ \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O}(-H) & \mathcal{O}(-H) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O}(-H) \\ \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \end{bmatrix}$$

where $\mathcal{O} = \mathcal{O}_X$ for some variety X and $H$is a prime divisor. We have to 2 global idempotents

$$e = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}, \quad f = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \in H^0(A)$$

In general, for any $\mathcal{O}_X$-algebra A and global idempotent e we have the following functors

$$\frac{A}{\langle e \rangle} \,\mathsf{mod} \xrightarrow{\iota} A \,\mathsf{mod} \xrightarrow{\pi} eAe \,\mathsf{mod}$$

where ι is just the inclusion and π is left multiplication by e. Note that $R_\pi = \mathrm{Hom}_{eAe}(eA, -)$ is the right adjoint of π. Applying this to our specific A we get

$$\bar{A} = \begin{bmatrix} \mathcal{O}_H & 0 & 0 & \dots & 0 & 0 \\ \mathcal{O}_H & \mathcal{O}_H & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathcal{O}_H & \mathcal{O}_H & \mathcal{O}_H & \dots & \mathcal{O}_H & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \mathsf{mod} \xrightarrow{\iota} \begin{bmatrix} \mathcal{O} & \mathcal{O}(-H) & \cdots & \mathcal{O}(-H) & \mathcal{O}(-H) \\ \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O}(-H) & \mathcal{O}(-H) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O}(-H) \\ \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \end{bmatrix} \mathsf{mod} \xrightarrow{\pi} \mathsf{coh}\, X$$

Note also that in this case, $R_\pi = Af \otimes -$.

Now suppose $U \in \mathsf{coh}\, X$ is tilting and $T \in \mathsf{coh}\, H$ are titling. Then $\bar{A} \otimes T$ is tilting in $\bar{A}\,\mathsf{mod}$.

Proposition 4.1. $\iota(\bar{A} \otimes T) \oplus R_\pi U = \frac{A}{\langle e \rangle} \otimes T \oplus Af \otimes U$ *is titling of* $\mathrm{Ext}^i_H(U|_H, T) = 0$ *for all* $i > 0$.

Now back to the general setup: $A = \bigotimes A_i$, where each A_i is of the form defined earlier. Each A_i has global idempotents $e_i, f_i \in H^0(A_i)$ the same way A did in the previous example.

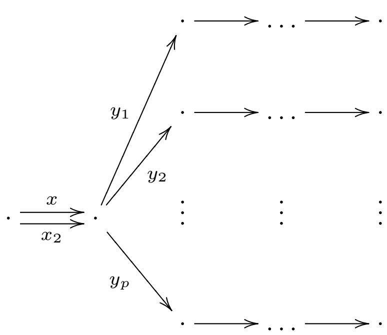
Theorem 4.2. *Let* $A = \bigotimes A_i$ *be an order on* X. *Suppose that for each* $I \in \mathcal{P}\{1, \dots, n\}$ *(the power set of* $\{1, \dots, n\}$*) there exists a tilting object* T_I *in* $\cap_{i \in I} H_i$. *Then*

$$\bigoplus_{I \in \mathcal{P}} \bigotimes_{i \notin I} A_i f_i \otimes \bigotimes_{i \in I} \frac{A_i}{\langle e_i \rangle} \otimes T_I$$

is titling in $A\,\mathsf{mod}$ *if for all* $I, J \in \mathcal{P}$ *with* $I \cap J = \emptyset$ *we have*

$$\mathrm{Ext}^i_{\cap_{i \in I \cup J} H_i}(T_I|_{\cap_{i \in I \cup J} H_i}, T_{I \cap J}) = 0, \ \textit{for all } i > 0.$$

Example 4.3. What does this general construction of a tilting object give when we apply to $\mathbb{P}^1$. I remind you that before we got the canonical algebra. This time End T is



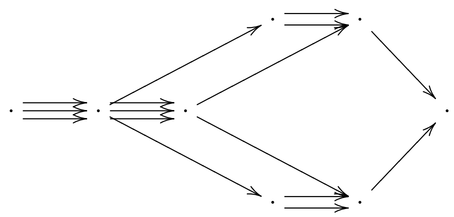
plus relations. This is known as the "squid" algebra.

Since our construction yields the squid algebra in the $d = 1$ cases, it is only natural to call the analogous algebras on other $\mathbb{P}^d$'s d-dimensional squids

Example 4.4. In the $d = 2$ with only two weights, both of weight 2. In this case

$$A = \begin{bmatrix} \mathcal{O} & \mathcal{O}(-H_1) \\ \mathcal{O} & \mathcal{O} \end{bmatrix} \otimes \begin{bmatrix} \mathcal{O} & \mathcal{O}(-H_2) \\ \mathcal{O} & \mathcal{O} \end{bmatrix}$$

where $\mathcal{O} = \mathcal{O}_{\mathbb{P}^2}$ and End T is



plus relation. We can clearly see the tilting bundle from $\mathbb{P}^2$ present in the above quiver, as well ass the tilting bundles for H_1 and H_2 and the finally the tilting bundle for the point of intersection.

Happy Valentine's day.