Line Bundles and Curves on a del Pezzo Order

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Abstract

Orders on surfaces provide a rich source of examples of noncommutative surfaces. In [HS05] the authors prove the existence of the analogue of the Picard scheme for orders and in [CK11] the Picard scheme is explicitly computed for an order on $\mathbb{P}^2$ ramified on a smooth quartic. In this paper, we continue this line of work, by studying the Picard and Hilbert schemes for an order on $\mathbb{P}^2$ ramified on a union of two conics. Our main result is that, upon carefully selecting the right Chern classes, the Hilbert scheme is a ruled surface over a genus two curve. Furthermore, this genus two curve is, in itself, the Picard scheme of the order.

Throughout this paper we assume all objects and maps are defined over an algebraically closed field k of characteristic zero. We denote the dimension of any cohomology group over k by the name of the group written with a non-capital letter for e.g. $\text{ext}^i_A(M,N) := \dim_k \text{Ext}^i_A(M,N)$ and similarly for h^i and hom.

1 Introduction

The study of moduli spaces is an integral part of modern algebraic geometry and representation theory. It is thus very natural, if one is studying noncommutative surfaces, to wish to understand the various moduli spaces that can be associate to them. However, even in the commutative case, let alone the noncommutative one, very few examples have been explicitly computed and the aim of this paper is to slightly remedy this situation.

We focus our attention on studying orders on surfaces which provide a rich source of examples of noncommutative surfaces.

Definition 1.1. *Let X be a normal integral surface. An* **order** *A on X is a coherent torsion free sheaf of $\mathcal{O}_X$-algebras such that $k(A) := A \otimes_X k(X)$ is a central simple $k(X)$-algebra. X is called the* **centre** *of A.*

See [AdJ] and [Cha12] for an introduction to orders on surfaces.

Since orders are finite over their centres they are in some sense only mildly noncommutative and many classical geometric techniques can be used to study them. In this paper we first fix an order A on $\mathbb{P}^2$ ramified on a union of two conics (these orders play a special role in the Minimal Model Program for orders: see [CI05]), and study two of its moduli spaces:

(i) the moduli space of line bundles on A (see Definition 3.1), with a fixed set of Chern classes, denoted by **Pic** A, and

(ii) the moduli space of left quotients of A, with a fixed set of Chern classes, denoted by **Hilb** A.

The first moduli space should be thought of as the Picard scheme of A, but one should note that since A-line bundles are only one sided modules, this is not a group scheme. Borrowing terminology from its commutative counterparts, the second moduli space will be referred to as the Hilbert scheme of A and should be thought of as the space parametrising certain noncommutative curves on A. Not surprisingly, these two moduli spaces are intrinsically linked: in fact we will prove that **Hilb** A is a ruled surface over **Pic** A and that **Pic** A is a genus two curve. Furthermore, by analysing the universal family on **Hilb** A we will show that **Hilb** A exhibits a covering of $\mathbb{P}^2$ with branch locus being two conics and their four bitangents.

The inspiration behind this paper comes from [CK11] where the authors, Chan and Kulkarni, study the moduli space of line bundles on an order ramified on a smooth quartic. The reader is highly encouraged to read that paper in order to better understand our motivation.

We begin by using the noncommutative cyclic covering trick, described in Section 2.1.1, to construct an order A on $\mathbb{P}^2$ with maximal commutative subalgebra $Y := \mathbb{P}^1 \times \mathbb{P}^1$. Then we study A-modules by noting that they are also naturally $\mathcal{O}_Y$-modules. This allows us to talk about the Chern classes and semistability of A-modules when viewed as $\mathcal{O}_Y$-modules. In particular we will be interested in those A-line bundles with "minimal second Chern class".

We will show that it suffices to consider only two possible first Chern classes: $c_1 = \mathcal{O}_Y(-1,-1)$ with corresponding minimal $c_2 = 0$ and $c_1 = \mathcal{O}_Y(-2,-2)$ with corresponding minimal $c_2 = 2$. The former case will be rather simple and we will prove that the moduli space in that case is just one point. The latter case will be far more interesting and will be our prime focus.

Our main result is:

Theorem 1.2. *Let* **Pic** A *be the moduli space of* A*-line bundles with* $c_1 = \mathcal{O}_Y(-2,-2)$ *and* $c_2 = 2$ *and* **Hilb** A *– the Hilbert scheme of* A*, parameterising quotients of* A *with* $c_1 = \mathcal{O}_Y(1,1)$ *and* $c_2 = 2$*. Then* **Pic** A *is a smooth genus* 2 *curve and* **Hilb** A *is a smooth ruled surface over* **Pic** A*. Furthermore,* **Hilb** A *exhibits an* $8:1$ *cover of* $\mathbb{P}^2$*, ramified on a union of* 2 *conics and their* 4 *bitangents.*

In their paper, Chan and Kulkarni had a remarkably similar result concerning the moduli of line bundles with minimal c_2. They also reduced the study of their moduli space of line bundles with minimal second Chern class to two possible first Chern classes. In the first case, the moduli space was a point and in the second case, also a genus two curve.

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1.1 Outline of the rest of the paper

We begin by briefly reviewing the relevant theory of orders on surfaces. After this, the rest of the paper is primarily devoted to making sense of, and proving Theorem 1.2. In Section 3 we will define and begin studying line bundles with minimal second Chern classes on our order A. Afterwards, we will introduce the Hilbert scheme of A, compute its dimension and prove that it is smooth. It is here that we will also explore the bizarre covering of $\mathbb{P}^2$ that it exhibits and study its ramification. In the last section, we will prove that the Hilbert scheme is in fact a ruled surface over **Pic** A. Finally using the map to $\mathbb{P}^2$ we will be able to compute the self intersection of the canonical divisor of the Hilbert scheme which will allow us to compute the genus of **Pic** A.

2 Preliminaries

2.1 Orders on surfaces

We will now describe the aforementioned noncommutative cyclic covering trick which we will later use to construct the order whose moduli spaces we will be studying. This "trick" was introduced by Chan in [Cha05] and the reader is advised to look there, in particular Sections 2 and 3 for all the relevant details and proofs.

2.1.1 Noncommutative cyclic covering trick

The setup is as follows: Let W be a normal integral Cohen-Macaulay scheme and $\sigma \in \text{Aut } W$ with $\sigma^e = \text{id}$ for some minimal $e \in \mathbb{Z}^+$. Further, assume that $X := W/\langle\sigma\rangle$ is a scheme. Given any $L \in \text{Pic } W$, we can form the $\mathcal{O}_W$-bimodule L_σ such that ${}_{\mathcal{O}_W}L_\sigma \simeq L$ and $(L_\sigma)_{\mathcal{O}_W} \simeq \sigma^* L$. Suppose we have an effective Cartier divisor D and an $L \in \text{Pic } W$ such that there exists a non-zero map of $\mathcal{O}_W$-bimodules $\phi\colon L_\sigma^{\otimes e} \xrightarrow{\sim} \mathcal{O}_W(-D) \hookrightarrow \mathcal{O}_W$ satisfying the **overlap condition**; namely that the two maps $1 \otimes \phi$ and $\phi \otimes 1$ are equal on $L_\sigma \otimes_W L_\sigma^{\otimes(e-1)} \otimes_W L_\sigma$.

Then

$$A = \mathcal{O}_W \oplus L_\sigma \oplus \cdots \oplus L_\sigma^{\otimes(e-1)}$$

is an order on X with multiplication given by:

$$L_\sigma^i \otimes_W L_\sigma^j \longrightarrow \begin{cases} L_\sigma^{\otimes(i+j)}, & i+j < e \\ L_\sigma^{\otimes(i+j)} \overset{1\otimes\phi\otimes 1}{\longrightarrow} L_\sigma^{\otimes(i+j-e)}, & i+j \geq e \end{cases}$$

which is independent of any choice that needs to be made when applying the map $1 \otimes \phi \otimes 1$ due to the overlap condition. Orders constructed in this manner are called **cyclic orders**. We will almost always regard A as an $\mathcal{O}_W$-bimodule on W, in which case we pay special consideration to the fact that it is not $\mathcal{O}_W$-central.

If we want to use this method to construct an order on a specific scheme X we also need a way of finding a scheme W and an automorphism $\sigma \in \text{Aut } W$ such that $W/\langle\sigma\rangle = X$. We can do so, using the classical cyclic covering construction.

Construction 2.1. Let X be a normal integral scheme, let $E \geq 0$ be an effective divisor and $N \in \text{Pic } X$ such that $N^{\otimes e} \simeq \mathcal{O}_X(-E)$. Then

$$\pi\colon W := Spec_X(\mathcal{O}_X \oplus N \oplus \cdots \oplus N^{\otimes(e-1)}) \to X$$

is a cyclic cover of X. See Chapter 1, Section 17 of [BPVdV84] for more details. Note that if σ is the generator of $\text{Gal}\,(W/X)$ then $W/\langle\sigma\rangle = X$. To construct an order on X using the noncommutative cyclic covering trick, let $E' \geq 0$ be another effective divisor on X and let $D = \pi^* E'$. Find an $L \in \text{Pic } W$ and a non-zero morphism (if one exists) $\phi\colon L_\sigma^{\otimes e} \to \mathcal{O}_W(-D)$ satisfying the overlap condition. Then as described above, we can construct an order on X which we will denote by $A(W/X; \sigma, L, \phi)$. This order is ramified on $E \cup E'$, see [Cha05] Theorem 3.6 for a proof of this. We suppress E, E' and D from the notation.

2.2 The order we wish to study

In this section we will use the noncommutative cyclic covering trick to construct a del Pezzo order on $\mathbb{P}^2$ ramified on a union of two conics. It is the moduli space and Hilbert scheme of this order that we will be investigating for the remainder of this paper.

Construction 2.2. Let $Z = \mathbb{P}^2$ and $\pi\colon Y \to Z$ be a double cover ramified on a smooth conic $E \subset Z$ with covering involution σ. In this case $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$, Pic $Y = \mathbb{Z} \oplus \mathbb{Z}$ and we can find a basis for Pic Y such that the intersection form is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma^*(\mathcal{O}_Y(n,m)) = \mathcal{O}_Y(m,n)$. Let $E' \subset Z$ be a second smooth conic, intersecting E in 4 distinct point, let $D = \pi^* E'$ which is a smooth $(2,2)$-divisor, let $L = \mathcal{O}_Y(-1,-1) \in$ Pic Y and fix once and for all a morphism $\phi\colon L_\sigma^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_Y(-D) \hookrightarrow \mathcal{O}_Y$. Any such ϕ satisfies the overlap condition and so $A := A(Y/Z; \sigma, L, \phi)$ is a maximal order, in a division ring, on Z ramified on $E \cup E'$. See [Ler12] Chapter 1 for full proofs.

2.3 The canonical bimodule

To finish off the introduction we would like to explain in what sense our order A is del Pezzo. We begin with the definition of the canonical bimodule which is the analogue of the canonical sheaf on a scheme.

Definition 2.3. *Let X be a normal integral scheme and A an order on X. The canonical bimodule of A is defined to be*

$$\omega_A := \mathcal{H}om_{\mathcal{O}_X}(A, \omega_X).$$

Mimicking the commutative definition, we say that A is del Pezzo if $\omega_A^* := \mathcal{H}om_A(\omega_X, A)$ is ample. For more details see [CK03] Section 3.

If X is Gorenstein, then $\omega_A = \mathcal{H}om_{\mathcal{O}_X}(A, \mathcal{O}_X) \otimes_X \omega_X$. Using the reduced trace map, we can identify $\mathcal{H}om_{\mathcal{O}_X}(A, \mathcal{O}_X)$ as an A-subbimodule of $k(A)$ and so ω_A can be identified as an A-subbimodule of $k(A) \otimes_X \omega_X$. The next theorem allows us to determine, in the case where A is constructed using Construction 2.1, precisely what this subbimodule is. Knowledge of ω_A will be very valuable to us in the future for various homological computations.

Theorem 2.4. *Let X be a normal integral Gorenstein scheme. Let $A := A(W/X; \sigma, L, \phi)$ be an order on X as described in Construction 2.1 and let $R \subset W$ be the reduced pullback of E to W.*

Then

$$\begin{aligned}\omega_A &= A \otimes_W L_\sigma \otimes_W \mathcal{O}_W\left((e-1)R + D\right) \otimes_X \omega_X \\ &= A \otimes_W L_\sigma \otimes_W \mathcal{O}_W(D) \otimes_W \omega_W\end{aligned}$$

in $k(A) \otimes_W \omega_W$.

Proof. From Lemma 17.1 of [BPVdV84] and the adjunction formula we know that

$$\omega_W = \pi^*\omega_X \otimes_W \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{O}_W, \mathcal{O}_X\right) = \pi^*\omega_X \otimes_W \mathcal{O}_W((e-1)\,R).$$

Thus, using the reduced trace map we have:

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_X}\left(A, \mathcal{O}_X\right) &= \{f \in k(A) \mid \operatorname{tr}\left(fA\right) \subseteq \mathcal{O}_X\} \subseteq k(A) \\ &= D_0 \oplus D_1 \oplus \cdots \oplus D_{e-1}\end{aligned}$$

where

$$D_{i-1} = \left\{f \in L_\sigma^{\otimes(i-1)} \otimes_W k(W) \mid \operatorname{tr}\left(fL_\sigma\right) \subseteq \mathcal{O}_X\right\} = L_\sigma^{\otimes(i-1)} \otimes_W \mathcal{O}_W\left((e-1)R + D\right)$$

for $1 \leq i \leq e$, and so:

$$\mathcal{H}om_{\mathcal{O}_X}\left(A, \mathcal{O}_X\right) = A \otimes_W L_\sigma \otimes_W \mathcal{O}_W\left((e-1)R + D\right).$$

Thus:

$$\begin{aligned}\omega_A &:= \mathcal{H}om_{\mathcal{O}_X}\left(A, \omega_X\right) \\ &= A \otimes_W L_\sigma \otimes_W \mathcal{O}_W\left((e-1)R + D\right) \otimes_W \pi^*\omega_X \\ &= A \otimes_W L_\sigma \otimes_W \mathcal{O}_W(D) \otimes_W \omega_W.\end{aligned}$$

□

Applying this theorem to our specific order A we get:

Proposition 2.5. *Let* $A := A(Y/Z; \sigma, L, \phi)$ *be as in Construction 2.2 and* H *be the pullback of a general line along* $\pi : Y \to Z$*. Then* $\omega_A = A \otimes_Y \mathcal{O}_Y(-H)$*. In particular,* A *is del Pezzo.*

Proof. We simply apply Theorem 2.4 and use the well known fact that $\omega_Y = \mathcal{O}_Y(-2,-2)$. □

From now on, unless explicitly stated otherwise, A denotes $A(Y/Z; \sigma, L, \phi)$ – the order constructed in Construction 2.2.

3 The Moduli Space of A-Line Bundles

In this section we will study line bundles on A.

Definition 3.1. *Let X be a normal integral scheme and B an order in a division ring $k(B)$ on X. Let M be a sheaf of left B-modules. We say M is a* **line bundle** *on B if M is locally projective as a B-module and* $\dim_{k(B)} k(B) \otimes_B M = 1$. *The set (not group) of isomorphism classes of B-line bundles will be denoted by* Pic B.

Let A be the order constructed earlier. The following proposition gives a very useful criterion for checking whether an A-module is in fact an A-line bundle.

Proposition 3.2. *Let M be an A-module. Then it is an A-line bundle if and only if ${}_Y M$ is a rank two locally free sheaf on Y.*

Proof. Follows from the fact that, A is locally of global dimension 2. See Proposition 2.02 in [Ler12] for a full proof. □

Example 3.3. Suppose $N \in \text{Pic } Y$. Then $A \otimes_Y N$ is an A-line bundle since it is clearly an A-module and is locally free of rank two over Y.

3.1 Chern classes of A-line bundles

In this section we study the possible Chern classes of line bundles on A. Recall that whenever we speak of Chern classes for any $M \in \text{Pic } A$ we imply that we are talking about the $\mathcal{O}_Y$-module ${}_Y M$. As it turns out, the possibilities are fairly limited.

Proposition 3.4. *Let $M \in \text{Pic } A$. Then $c_1(M) = \mathcal{O}_Y(n, n)$ for some $n \in \mathbb{Z}$. Conversely, given any such n, $A \otimes_Y \mathcal{O}_Y(0, n+1) \in \text{Pic } A$ with $c_1 = \mathcal{O}_Y(n, n)$.*

Proof. First note that we have a chain of $\mathcal{O}_Y$-submodules $M(-D) := L_\sigma^{\otimes 2} \otimes_Y M \subset L_\sigma \otimes_Y M \subset M$ which means

$$0 \to \frac{L_\sigma \otimes_Y M}{L_\sigma^{\otimes 2} \otimes_Y M} \to \frac{M}{M(-D)} \to \frac{M}{L_\sigma \otimes_Y M} \to 0$$

is an exact sequence. Let $Q = M/(L_\sigma \otimes_Y M)$. The above then becomes:

$$0 \to L_\sigma \otimes_Y Q \to M|_D \to Q \to 0.$$

Now $M|_D$ is a locally free sheaf on D of rank 2, and so $L_\sigma \otimes_Y Q$ and hence Q must be line bundles on D. Consequently, $c_1(Q) = D$ and so $c_1(M) = c_1(L_\sigma \otimes M) + D$. Hence $c_1(M) = \sigma^* c_1(M)$ and the result follows.

To see the converse, first note that by Example 3.3 we know that $M := A \otimes_Y \mathcal{O}_Y(0, n+1)$ is indeed an A-line bundle. Furthermore, $c_1(M) = c_1(\mathcal{O}_Y(0, n+1) \oplus \mathcal{O}_Y(n, -1)) = \mathcal{O}_Y(n, n)$. □

Having classified all the possible first Chern classes of A-line bundles, we move on to see what can be said about the second Chern class. As we shall see, the second Chern class has a strict lower bound analogous to Bogomolov's inequality, see [LP97].

Let X be a smooth projective surface and $\mathcal{F}$ a torsion free coherent sheaf on X with Chern classes c_1, c_2 and rank r. Fix an ample divisor H on X. The **slope** of $\mathcal{F}$ (with respect to H) is defined to be

$$\mu_H(\mathcal{F}) := \frac{c_1.H}{r}.$$

$\mathcal{F}$ is said to be μ-**semistable** if for any subsheaf $\mathcal{F}' \subset \mathcal{F}$ we have $\mu_H(\mathcal{F}') \leq \mu_H(\mathcal{F})$.

For us, H will always be a $(1,1)$-divisor on Y and we will often omit it from the notation.

Using a similar technique to [CK11] we have the following:

Proposition 3.5. *Let $M \in \operatorname{Pic} A$ and suppose $N \subseteq M$ is an $\mathcal{O}_Y$-subsheaf. Let H be a $(1,1)$-divisor. Then*

$$\mu_H(N) \leq \mu_H(M) + 1. \tag{1}$$

Proof. Note that M is locally free of rank 2 over Y. Thus the result is clear if rank N= 2 and so we assume rank $N = 1$. Observe that $c_1(L_\sigma \otimes N).H = c_1(L).H + \sigma^* c_1(N).H = -2 + c_1(N).H$. Now

$$\begin{aligned} \mu(N \oplus L_\sigma \otimes_Y N) &= \frac{c_1(N).H + c_1(L_\sigma \otimes_Y N).H}{2} \\ &= \frac{2c_1(N).H - 2}{2} \\ &= c_1(N).H - 1 \\ &= \mu(N) - 1 \end{aligned}$$

and so $\mu(N) = \mu(N \oplus L_\sigma \otimes_Y N) + 1 \leq \mu(M) + 1$. □

It is easy to see that this inequality is tight. For example the A-line bundle A has slope $\mu(A) = -1$ while the $\mathcal{O}_Y$-submodule $\mathcal{O}_Y \subset A$ has slope $\mu(\mathcal{O}_Y) = 0$.

Crucially, we still have a Bogomolov-like inequality:

Proposition 3.6. *Let $M \in \text{Pic } A$ with Chern classes c_1 and c_2. Then*

$$\Delta(M) = 4c_2 - c_1^2 \geq -2.$$

Proof. Follows immediately from Theorem 5.1 of [Lan04] with $D_1 = H$ and Proposition 3.5. $\square$

Remark 3.7. The above theorem can also be proven using rather elementary techniques, without needing the generality of [Lan04].

In Section 4.3 we will prove that $4c_2 - c_1^2 \geq -2$ is in fact a sufficient condition to guarantee that there exists am A-line bundle with these Chern classes.

Having shown that for a fixed first Chern class, the second Chern class of any A-line bundle is bounded from below, we begin studying those line bundles, with minimal second second Chern class. In particular, we would like to determine what the moduli space of such bundles is.

The existence of a projective coarse moduli scheme parametrising A-line bundles with minimal c_2 follows easily from Theorem 2.4 in [HS05] and Proposition 3.6. In fact this moduli space is smooth because A is del Pezzo. For a full explanation and proof, see [Ler12] Chapter 2.

Remark 3.8. Since the functor $\mathcal{O}_Y(nH) \otimes_Y -$ is a category autoequivalence of A-Mod, it induces an automorphism of the moduli space of A. Note that for any $M \in \text{Pic } A$, $c_1(\mathcal{O}_Y(nH) \otimes_Y M) = 2nH + c_1(M)$. Since by the previous proposition, $c_1(M) = mH$ for some $m \in \mathbb{Z}$ we me may assume that $c_1(M) = \mathcal{O}_Y(-1, -1)$ or $c_1(M) = \mathcal{O}_Y(-2, -2)$.

Before we begin our analysis of A-line bundles with minimal c_2, we need to examine the inequality (1) we met in Proposition 3.5 a little further.

Definition 3.9. *Let X be a surface and V a vector bundle on X. Fix an ample divisor H. We say V is* **almost semistable** *(with respect to H) if for any subbundle $V' \subset V$, $\mu_H(V') \leq \mu_H(V) + 1$.*

Proposition 3.10. *Let X be a surface and V a vector bundle on X.*

1. *V is almost semistable if and only if $V \otimes_X N$ is almost semistable for all $N \in \text{Pic } X$.*

2. *If V is rank 2 and almost semistable, then so is V^*.*

Proof.

1. Suppose V is almost semistable and $V' \subseteq V \otimes_X N$. Then $V' \otimes_X N^{-1} \subseteq V$ and so $\mu(V' \otimes_X N^{-1}) \leq \mu(V) + 1$ thus $\mu(V') - c_1(N).H \leq \mu(V) + 1$ and so $\mu(V') \leq (V \otimes_X N) + 1$. To see the converse simply let $N = \mathcal{O}_X$.

2. Follows from (1) and the fact that $V^* \simeq V \otimes_X (\det V)^{-1}$.

□

As we have seen in Proposition 3.5, A-line bundles are almost semistable. We will use the above proposition later on for proving various properties regarding line bundles on A.

3.2 Case 1: $c_1 = \mathcal{O}_Y(-1, -1)$

As mentioned in Remark 3.8 the problem of studying the moduli space of A-line bundles with minimal c_2 naturally breaks up into two parts $c_1 = \mathcal{O}_Y(-1, -1)$ or $\mathcal{O}_Y(-2, -2)$. In this subsection we examine the former case. By Proposition 3.6 the minimal $c_2 = 0$ and this corresponds to $\Delta = -2$, the smallest value possible. It is easy to see that the moduli space of A-line bundles with these Chern classes isn't empty for clearly A itself, regarded as a left A-module, has the desired Chern classes. As it turns out, this is in fact the only such A-line bundle.

Theorem 3.11. *Let $M \in \operatorname{Pic} A$ with $c_1 = \mathcal{O}_Y(-1, -1)$ and $c_2 = 0$. Then $M \simeq A$. In particular, the coarse moduli space of A-line bundles with these Chern classes is a reduced point.*

Proof. By the Riemann-Roch theorem $\chi(M) = 1 > 0$. On the other hand $h^2(M) = h^0(\omega_Y \otimes_Y M^*)$ and $c_1(\omega_Y \otimes_Y M^*) = \mathcal{O}_Y(-3, -3)$. As we saw in Proposition 3.5, M is almost semistable, and so by Proposition 3.10 we have that $\omega_Y \otimes_Y M^*$ is also almost semistable and so $h^2(M) = 0$. Thus $h^0(M) \neq 0$ and since M is torsion-free, $\mathcal{O}_Y \hookrightarrow M$ which gives rise to an injection of A-modules $A \otimes_Y \mathcal{O}_Y = A \hookrightarrow M$. Since their first Chern classes equal, the map must be an isomorphism.

Finally

$$\operatorname{Ext}^1_A(A, A) = \operatorname{Ext}^1_Y(\mathcal{O}_Y, \mathcal{O}_Y \oplus \mathcal{O}_Y(-1, -1)) = H^1(Y, \mathcal{O}_Y \oplus \mathcal{O}_Y(-1, -1)) = 0.$$

where the first equality follows from Proposition 2.6 of [CK11] which asserts that there is a natural isomorphism of functors $\operatorname{Ext}^i_A(A \otimes_Y N, -) \simeq \operatorname{Ext}^i_Y(N, -)$ for any $N \in \operatorname{Pic} Y$. See Chapter 3 Exercise 5.6 of [Har77] for the cohomology of $\mathbb{P}^1 \times \mathbb{P}^1$. Thus the tangent space at the point corresponding to the A-line bundle A is 0-dimensional and so the moduli space is just a reduced point. □

3.3 Case 2: $c_1 = \mathcal{O}_Y(-2,-2)$

We now study the second case mentioned in Remark 3.8: the case where $c_1 = \mathcal{O}_Y(-2,-2)$. By Proposition 3.6 the minimal $c_2 = 2$ which corresponds to $\Delta = 0$ which is its second smallest value for clearly Δ must be even. Note that $A \otimes_Y \mathcal{O}_Y(-1,0)$ is an A-line bundle by Example 3.3 and has the desired Chern classes. Thus the moduli space of such A-line bundles is not empty.

From now on **Pic** A will denote the moduli space of A-line bundles with $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$. We first establish all the possible $\mathcal{O}_Y$-module structures that such A-line bundles can have.

Theorem 3.12 ($\mathcal{O}_Y$-module structure)**.** *Let $M \in \text{Pic } A$ with $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$. Then either $M \simeq \mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$ as an $\mathcal{O}_Y$-module or $M \simeq A \otimes_Y (-F)$ as an A-module where F is either a $(1,0)$ or a $(0,1)$-divisor.*

Proof. The beginning of this proof is very similar to the proof of Theorem 3.11 so we skip some details which we have already explained there. Let $M_1 = \mathcal{O}_Y(1,1) \otimes_Y M$. Then $c_1(M_1) = 2c_1(\mathcal{O}_Y(1,1)) + c_1(M) = 0$ and $c_2(M_1) = c_2(M) + c_1(M).c_1(\mathcal{O}_Y(1,1)) + c_1(\mathcal{O}_Y(1,1))^2 = 2 - 4 + 2 = 0$. Thus by the Riemann-Roch theorem $\chi(M_1) = 2 > 0$. However, by duality $h^2(M_1) = h^0(\omega_Y \otimes_Y M_1^*)$ whilst $\omega_Y \otimes_Y M_1^*$ is almost semistable with slope -4 and so $h^0(\omega_Y \otimes_Y M_1^*) = h^2(M_1) = 0$ and so $h^0(M_1) \neq 0$. Thus we know $\mathcal{O}_Y(-1,-1) \hookrightarrow M$. Now if there exists a bigger $\mathcal{O}_Y$-line bundle (ordered by inclusion) which embeds into M then $\mathcal{O}_Y(-F)$ embeds into M where F is either a $(1,0)$ or a $(0,1)$-divisor. This extends to an embedding $A \otimes_Y \mathcal{O}_Y(-F) \hookrightarrow M$ of A-line bundles and so comparison of the first Chern classes guarantees that $M \simeq A \otimes_Y \mathcal{O}_Y(-F)$. Suppose on the other hand that $\mathcal{O}_Y(-1,-1)$ is the biggest line bundle which embeds into M. Let the quotient be Q and note that it is torsion free. By Proposition 5 (ii) in [Fri98] $Q = L' \otimes_Y \mathcal{I}_Z$ for some $L' \in \text{Pic } Y$ and $\mathcal{I}_Z$ being the ideal sheaf of some 0-dimensional subscheme. Computing Chern classes we see that $L' = \mathcal{O}_Y(-1,-1)$ and $Z = 0$. Finally, $\text{Ext}^1_Y(\mathcal{O}_Y(-1,-1), \mathcal{O}_Y(-1,-1)) = 0$ and so we see that as an $\mathcal{O}_Y$-module $M \simeq \mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$. □

This result is very different to what Chan and Kulkarni encountered in [CK11]. In their example if an A-module was split as an $\mathcal{O}_Y$-module then they prove that the module must be of the form $A \otimes_Y N$ for some $N \in \text{Pic } Y$. Furthermore, any rank two vector bundle on Y could be given at most two A-module structures. In our case, as the above theorem at least suggests, the $\mathcal{O}_Y$-vector bundle $\mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$ can be given an infinite number of non-isomorphic A-module structures. In the following proposition, we prove that this is indeed the case.

Proposition 3.13. *The tangent space to* **Pic** A *at the point corresponding to* $A \otimes_Y \mathcal{O}_Y(0,-1)$ *and* $A \otimes_Y \mathcal{O}_Y(-1,0)$ *has dimension* 1.

Proof. The dimension of the tangent space is given by:
$\mathrm{ext}^1_A(A \otimes_Y \mathcal{O}_Y(-1,0), A \otimes_Y \mathcal{O}_Y(-1,0))$
$= \mathrm{ext}^1_Y(\mathcal{O}_Y(-1,0), \mathcal{O}_Y(-1,0) \oplus \mathcal{O}_Y(-1,-2)) = 1$. The other case is identical. □

Thus at least one connected component of this moduli space is a smooth curve with all, except at most 2 points, corresponding to A-modules with the underlying $\mathcal{O}_Y$-module structure being $\mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$.

We finish off the section with an algebraic description of the A-line bundles.

Proposition 3.14. *Let* $M \in \mathrm{Pic}\, A$ *with* $c_1 = \mathcal{O}_Y(-2,-2)$ *and* $c_2 = 2$. *Then* $\mathrm{Hom}_A(M, A) = 2$. *Further, if* $0 \neq \varphi \in \mathrm{Hom}_A(M, A)$ *then* φ *is injective.*

Proof. We consider all the possibilities from Theorem 3.12. If $M \simeq A \otimes_Y \mathcal{O}_Y(-F)$ then

$$\begin{aligned}\mathrm{hom}_A(M, A) &= \mathrm{hom}_A(A \otimes_Y \mathcal{O}_Y(-F), A)\\ &= \mathrm{hom}_Y(\mathcal{O}_Y(-F), \mathcal{O}_Y \oplus \mathcal{O}_Y(-1,-1)) = 2.\end{aligned}$$

If, on the other hand, $M \simeq \mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$ as an $\mathcal{O}_Y$-module then:

$$\begin{aligned}\mathrm{hom}_A(M, A) &= \mathrm{ext}^2_A(A, \omega_A \otimes_A M)^*\\ &= \mathrm{ext}^2_Y(\mathcal{O}_Y, \mathcal{O}_Y(-H) \otimes_Y M)^*\\ &= h^2(Y, \mathcal{O}_Y(-2,-2) \oplus \mathcal{O}_Y(-2,-2))\\ &= h^0(Y, \mathcal{O}_Y \oplus \mathcal{O}_Y) = 2.\end{aligned}$$

Since M and A are torsion free, any non zero map $M \to A$ must be injective. □

To understand better how M sits inside A we need to understand the all the possible cokernels. We do so, in the next theorem.

Theorem 3.15. *Let* $M \in \mathrm{Pic}\, A$ *with* $c_1 = \mathcal{O}_Y(-2,-2)$ *and* $c_2 = 2$. *Then for any* $0 \neq \varphi \in \mathrm{hom}_A(M, A)$ *there exists an exact sequence of* A*-modules*

$$0 \longrightarrow M \xrightarrow{\varphi} A \longrightarrow Q \longrightarrow 0$$

where:

1. *if* $M \simeq A \otimes_Y \mathcal{O}_Y(-1,0)$ *(respectively* $M \simeq A \otimes_Y \mathcal{O}_Y(0,-1)$*) then* $Q \simeq A \otimes_Y \mathcal{O}_F$ *where* F *is a* $(1,0)$ *(respectively* $(0,1)$*) divisor;*

2. *if* $M \simeq \mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$ *then* $Q \simeq \mathcal{O}_C$ *as an* $\mathcal{O}_Y$*-module, where* C *is a* $(1,1)$*-divisor.*

Proof. From the previous proposition, we know $\varphi\colon M \to A$ is injective. Let us compute the cokernel.

1. We prove only the case where $M \simeq A \otimes_Y \mathcal{O}_Y(-1,0)$ because the other is similar. Note that $\hom_Y(\mathcal{O}_Y(-1,0), \mathcal{O}_Y) = 2 = \hom_A(M, A)$ and so all A-module morphisms arise from an $\mathcal{O}_Y$-module morphism $\mathcal{O}_Y(-1,0) \to \mathcal{O}_Y$ via $A \otimes_Y -$. Since any non zero morphism $\mathcal{O}_Y(-1,0) \to \mathcal{O}_Y$ gives rise to the following exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-1,0) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_F \longrightarrow 0$$

for some $(1,0)$-divisor F and because A is flat over Y, the result follows.

2. Note that with respect to the $\mathcal{O}_Y$-module decomposition

$$\begin{aligned} M &= \mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1) \\ A &= \mathcal{O}_Y \oplus \mathcal{O}_Y(-1,-1) \end{aligned}$$

we have

$$\begin{aligned} \operatorname{Hom}_Y(M, A) &= \operatorname{Hom}_Y(\mathcal{O}_Y(-1,-1)) \oplus \mathcal{O}_Y(-1,-1), \mathcal{O}_Y \oplus \mathcal{O}_Y(-1,-1)) \\ &= \begin{pmatrix} H^0(Y, \mathcal{O}_Y(-1,-1)^*) & \operatorname{End}_Y(\mathcal{O}_Y(-1,-1)) \\ H^0(Y, \mathcal{O}_Y(-1,-1)^*) & \operatorname{End}_Y(\mathcal{O}_Y(-1,-1)) \end{pmatrix}. \end{aligned}$$

Thus any $\mathcal{O}_Y$-module morphism $\varphi\colon M \to A$ is given by $X = \begin{pmatrix} \varphi_1 & \lambda_1 \\ \varphi_2 & \lambda_2 \end{pmatrix}$ where $\varphi_1, \varphi_2 \in \mathcal{O}_Y(-1,-1)^*$ and $\lambda_1, \lambda_2 \in \operatorname{End}_Y(\mathcal{O}_Y(-1,-1)) = k$ which acts as right multiplication on the row vector $\mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$. For this to be in fact an A-module morphism further conditions on X need to be imposed. In particular φ needs to be injective and so λ_1, λ_2 are not both zero.

We claim that

$$Q = \frac{\mathcal{O}_Y}{\operatorname{im}(\lambda_2\varphi_1 - \lambda_1\varphi_2)}$$

and that we have the following exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi} A \xrightarrow{\psi} Q \longrightarrow 0$$

with $\psi\colon A \to Q$ given by right multiplication by

$$\begin{cases} \begin{pmatrix} \lambda_1 + \lambda_2 \\ -(\varphi_1 + \varphi_2) \end{pmatrix} & \text{if } \lambda_1 + \lambda_2 \neq 0 \\ \begin{pmatrix} \lambda_1 \\ -\varphi_1 \end{pmatrix} & \text{if } \lambda_1 + \lambda_2 = 0. \end{cases}$$

Since $M \to A$ must be injective, $\operatorname{im}(\lambda_2\varphi_1 - \lambda_1\varphi_2) \neq 0$ and so, Q is isomorphic, as an $\mathcal{O}_Y$-module, to $\mathcal{O}_C$ for some $(1,1)$-divisor C. The proof of this claim is just a routine local computation and is done in Lemma 2.4.5 in [Ler12].

□

The above theorem suggests that we should study quotients of A. In particular, we should try to better understand the component(s) of the Hilbert scheme of A containing the A-modules whose underlying $\mathcal{O}_Y$-module structure is $\mathcal{O}_C$ where C is a $(1,1)$-divisor. We do this in the following section.

4 The Hilbert Scheme of A

In this section we will study the Hilbert scheme of A – the moduli space of left sided quotients of A with a fixed set of Chern classes. This is a closed subscheme of the classical Quot scheme of A, which is projective provided we fix a Hilbert polynomial. See Chapter 3 in [Ler12] for all the details.

Mimicking the commutative case, one should think of a quotient of A, which is supported on a curve on Y, as a noncommutative curve lying on A. As mentioned at the end of the last section, we are primarily interested in those quotients of A which are supported on a $(1,1)$-divisor on Y.

4.1 Properties of Hilb A

Recall that in Theorem 3.15 we saw a link between the moduli space of A-line bundles with with $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$ and quotients of A, or noncommutative curves on A, with $c_1 = \mathcal{O}_Y(1,1)$ and $c_2 = 2$.

Proposition 4.1. *Let S be a scheme. Let $\mathcal{F}$ be a flat family of quotients of A (considered as A-modules) on S with Chern classes $c_1 = \mathcal{O}_Y(1,1)$ and $c_2 = 2$. Let $I := \ker(A_S \to \mathcal{F})$. Then I is a flat family of A-line bundles on S with Chern classes $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$.*

Proof. I is flat over S because A_S and $\mathcal{F}$ are. Restricting to the fibre above any $p \in S$ we get

$$0 \longrightarrow I_{k(p)} \longrightarrow A \longrightarrow \mathcal{F}_{k(p)} \longrightarrow 0$$

of A-modules which is exact because $\mathcal{F}$ is flat over S and so $Tor^1_{\mathcal{O}_S}(\mathcal{F}, k(p)) = 0$. Since

$$c_1(A) = \mathcal{O}_Y(-1,-1), \quad c_2(A) = 0, \quad c_1(\mathcal{F}_{k(p)}) = \mathcal{O}_Y(1,1), \quad c_2(\mathcal{F}_{k(p)}) = 2$$

we see that $c_1(I_{k(p)}) = \mathcal{O}_Y(-2,-2)$ and $c_2(I_{k(p)}) = 2$. $I_{k(p)}$ is torsion free and so $I^{**}_{k(p)} \in \mathrm{Pic}\, A$ because it is reflexive and hence locally free over Y. By Proposition 3.6 we have $c_2(I^{**}_{k(p)}) = 2$ and so $I^{**}_{k(p)} = I_{k(p)}$. □

Having established a relationship between flat families of A-line bundles and flat families of quotients of A, we now use Theorem 3.12 to classify all the possible $\mathcal{O}_Y$-module structures that quotients of A may possess. As we shall see, some (and in fact most) must also be quotients of $\mathcal{O}_Y$.

Corollary 4.2. *Let Q be a quotient of A with $c_1 = \mathcal{O}_Y(1,1)$ and $c_2 = 2$. Then either:*

- *$Q \simeq A \otimes_F \mathcal{O}_F$ (as an A-module) where F is either a $(1,0)$ or $(0,1)$-divisor; or*
- *$Q \simeq \mathcal{O}_C$ (as an $\mathcal{O}_Y$-module) for some σ-invariant $(1,1)$-divisor $C \subset Y$.*

Proof. The above proposition asserts that the kernel of $A \to Q$ is an A-line bundle with $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$. We have already classified all such line bundles and their respective cokernels in Proposition 3.12 and Theorem 3.15. The fact that C must be σ invariant follows from the fact that in order to be an A-module there must be a non-zero map $L_\sigma \otimes_Y \mathcal{O}_C \to \mathcal{O}_C$ which is only possible if $\sigma^* C = C$. □

Corollary 4.3. *Let Q be a quotient of A with $c_1 = \mathcal{O}_Y(1,1)$ and $c_2 = 2$. If the support Q is smooth (i.e. the support is $\mathbb{P}^1$) then Q is also quotient of $\mathcal{O}_Y$.*

Proof. Obvious from the previous Corollary because the support of $A \otimes_Y \mathcal{O}_F$ is not smooth. □

From now on **Hilb** A will denote the Hilbert scheme of A corresponding to quotients of A with $c_1 = \mathcal{O}_Y(1,1)$ and $c_2 = 2$. We now proceed to study its properties.

Proposition 4.4. *The dimension of* **Hilb** A *at the point corresponding to* $A \otimes_Y \mathcal{O}_F$*, where* F *is a* $(1,0)$ *or* $(0,1)$*-divisor is* 2*.*

Proof. We have

$$0 \longrightarrow A \otimes_Y \mathcal{O}_Y(-F) \longrightarrow A \longrightarrow A \otimes_Y \mathcal{O}_F \longrightarrow 0.$$

Let $F' = \sigma^* F$. The dimension of the tangent space is given by:

$$\begin{aligned} \hom_A (A \otimes_Y \mathcal{O}_Y(-F), A \otimes_Y \mathcal{O}_F) &= \hom_Y (\mathcal{O}_Y(-F), A \otimes_Y \mathcal{O}_F) \\ &= \hom_Y (\mathcal{O}_Y(-F), \mathcal{O}_F \oplus \mathcal{O}_{F'}(-1)) \\ &= h^0(Y, \mathcal{O}_F \oplus \mathcal{O}_{F'}) \\ &= 2. \end{aligned}$$

□

Unfortunately, we were unable to compute the dimension of the tangent space at any other points as directly as in the above proposition. We thus proceed by first showing that **Hilb** A is smooth and later, after a considerable amount of work, that it is connected. This will of course prove that **Hilb** A is a smooth projective surface.

Theorem 4.5. Hilb A *is smooth.*

Proof. Let Q be a quotient of A corresponding to some point $p \in$ **Hilb** A. Let M the kernel of $A \to Q$. We have an exact sequence

$$0 \longrightarrow M \longrightarrow A \longrightarrow Q \longrightarrow 0 \qquad (*)$$

where by Proposition 4.1 $M \in \text{Pic } A$. Obstruction to smoothness at p is given by $\text{Ext}^1_A (M, Q)$ which we now compute. From Corollary 4.2 there are only three cases to consider:

- $M \simeq A \otimes_Y \mathcal{O}_Y(-1,0)$ and $Q \simeq A \otimes_Y \mathcal{O}_F$ where F is a $(1,0)$ divisor. Let $F' = \sigma^* F$ which is a $(0,1)$-divisor.

$$\begin{aligned} \text{ext}^1_A (A \otimes_Y \mathcal{O}_Y(-1,0), A \otimes_Y \mathcal{O}_F) &= \text{ext}^1_Y (\mathcal{O}_Y(-1,0), \mathcal{O}_F \oplus \mathcal{O}_{F'}(-1)) \\ &= h^1(Y, \mathcal{O}_F \oplus \mathcal{O}_{F'}) = 0. \end{aligned}$$

- $M \simeq A \otimes_Y \mathcal{O}_Y(0,-1)$ and $Q \simeq A \otimes_Y \mathcal{O}_F$ where F is a $(0,1)$ divisor. The proof is the same as in the case above.

- $M \simeq \mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$ as an $\mathcal{O}_Y$-module and $Q \simeq \mathcal{O}_C$ as an $\mathcal{O}_Y$-module for some $(1,1)$-divisor C. Using Serre duality, we have:

  $$\mathrm{ext}^1_A(M, \mathcal{O}_C) = \mathrm{ext}^1_A(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M).$$

  Using the local-global spectral sequence we have

  $$0 \to H^1(Y, \mathcal{H}om_A(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M)) \to \mathrm{Ext}^1_A(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M) \\ \to H^0(Y, \mathcal{E}xt^1_A(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M)).$$

  $\mathcal{H}om_A(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M) = 0$ since $\mathcal{O}_C$ is a torsion sheaf. Furthermore, $(*)$ is a locally projective A-module resolution of $\mathcal{O}_C$ and so we get

  $$0 \to \mathcal{H}om_A(A, \mathcal{O}_Y(-H) \otimes_Y M) \to \mathcal{H}om_A(M, \mathcal{O}_Y(-H) \otimes_Y M) \\ \to \mathcal{E}xt^1_A(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M) \to 0.$$

  Finally, since

  $$H^0(\mathcal{H}om_A(M, \mathcal{O}_Y(-H) \otimes_Y M)) = 0$$

  and

  $$H^1(\mathcal{H}om_A(A, \mathcal{O}_Y(-H) \otimes_Y M)) = H^1(Y, \mathcal{O}_Y(-H) \otimes_Y M) = 0$$

  we see that

  $$H^0(Y, \mathcal{E}xt^1_A(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M)) = 0$$

  and so the result follows.

□

Thus, so far we know that at least one connected component of **Hilb** A is a smooth projective surface. As mentioned earlier, in the next section we will see that in fact **Hilb** A is connected, which will prove that this must be its only component.

Corollary 4.2 says that some quotients of A are in fact also quotients of $\mathcal{O}_Y$. In particular, they are isomorphic to $\mathcal{O}_C$ where C is a σ-invariant $(1,1)$-divisor. Furthermore, the support of $A \otimes_Y \mathcal{O}_F$ is $F \cup \sigma^* F$ which is also a σ-invariant $(1,1)$-divisor. Since the tangent space at the points corresponding to $A \otimes_Y \mathcal{O}_F$ is two, whilst $\dim |F| = 1$ it must be the case that every connected component of **Hilb** A has a dense subset whose points correspond to quotients of A that are also quotients of $\mathcal{O}_Y$. We may thus expect that there is at least a rational map from the Hilbert scheme of A to the Hilbert scheme of Y. We now explore this further. Note first of all, that all σ-invariant $(1,1)$-divisors are equal to $\pi^* l$ where l is a line on Z. Furthermore, lines on Z are parameterised by $(\mathbb{P}^2)^\vee \simeq \mathbb{P}^2$. Thus we can view $(\mathbb{P}^2)^\vee$ as the parameter space of σ-invariant $(1,1)$-divisors.

Theorem 4.6. *Let $\mathcal{F}$ be the universal family of quotients of A on $Y \times_k$ **Hilb** A. There exists a regular map*

$$\Psi \colon \mathbf{Hilb}\ A \longrightarrow (\mathbb{P}^2)^\vee$$
$$p \longmapsto \operatorname{supp} \mathcal{F}_{k(p)}$$

Proof. The sheaf $\mathcal{O}_{\operatorname{supp} \mathcal{F}}$ is a family of $\mathcal{O}_Y$-quotients on **Hilb** A for as we saw, the support of every quotient of A is a σ-invariant $(1,1)$-divisor. Furthermore, it is flat over **Hilb** A since every fibre above **Hilb** A has the same Chern class and hence the same Hilbert polynomial. This gives us the map **Hilb** $A \to$ **Hilb** Y with the image being the subscheme of **Hilb** Y parameterising σ-invariant $(1,1)$-divisors. As discussed, this subscheme is just $(\mathbb{P}^2)^\vee$ and so the result follows.

□

In summary, the map Ψ does the following: every closed point on **Hilb** A corresponds to some quotient of A. There are two possibilities: either

(i) it is also a quotient of $\mathcal{O}_Y$, in which case it is isomorphic, and an $\mathcal{O}_Y$-module, to $\mathcal{O}_C$ where C is a σ-invariant $(1,1)$-divisor, or

(ii) it is not a quotient of $\mathcal{O}_Y$, then it is isomorphic, as an A-module, to $A \otimes_Y \mathcal{O}_F$ where F is either a $(0,1)$ or $(1,0)$-divisor.

The crucial point is that the support of $A \otimes_Y \mathcal{O}_F$ is also a σ-invariant $(1,1)$-divisor. Thus to every closed point on **Hilb** A one can associate a σ-invariant $(1,1)$-divisor. Since σ-invariant $(1,1)$-divisors are parameterised by $(\mathbb{P}^2)^\vee$, we get a natural set-theoretic map from (closed points of **Hilb** A) $\to$ (closed points of $(\mathbb{P}^2)^\vee$). The above theorem proves that this map is in fact morphism of schemes.

4.2 The ramification of $\Psi \colon \mathbf{Hilb}\ A \to (\mathbb{P}^2)^\vee$

We want to study the map Ψ, in particular we want to understand its ramification for then we will be able to later compute $(K_{\mathbf{Hilb}\ A})^2$. This amounts to computing the number of quotients of A which have support a σ-invariant $(1,1)$-divisor and $c_2 = 2$. Corollary 4.2 implies that this question will be answered provided we can understand the number of A-module structures that $\mathcal{O}_C$ can be given, where C is a σ-invariant $(1,1)$-divisor.

To give a coherent sheaf $\mathcal{G}$ on Y an A-module structure amounts to giving a left $\mathcal{O}_Y$-module morphism $\varphi : A \otimes_Y \mathcal{G} \to \mathcal{G}$ satisfying the necessary associativity condition. Two such morphisms φ, φ' give rise to isomorphic A-modules provided there exists $\psi \in \mathrm{Aut}_Y \mathcal{G}$ such that

$$\begin{array}{ccc} A \otimes_Y \mathcal{G} & \xrightarrow{\varphi} & \mathcal{G} \\ {\scriptstyle \mathrm{id}\otimes\psi}\downarrow\wr & & \wr\downarrow{\scriptstyle \psi} \\ A \otimes_Y \mathcal{G} & \xrightarrow[\varphi']{} & \mathcal{G} \end{array}$$

commutes. In general it may be rather difficult to determine whether such a ψ exists, and consequently, whether two seemingly different A-module structures are actually isomorphic. The problem becomes increasingly difficult as the size of $\mathrm{Aut}_Y \mathcal{G}$ increases. Luckily, in our case, this issue is easily manageable.

Example 4.7. We can illustrate of the above phenomenon with two (related) examples. Recall from Theorem 3.12 that an A-line bundle had two possible $\mathcal{O}_Y$-module structures: either it was $\mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1-1)$ or $A \otimes_Y \mathcal{O}_Y(-1,0) \overset{Y}{\simeq} \mathcal{O}_Y(-1,0) \oplus \mathcal{O}_Y(-1,-2)$. The former, as we later saw, had infinitely many non-isomorphic A-module structures whilst the latter, only had one. The fact that that $\mathcal{O}_Y(-1,0) \oplus \mathcal{O}_Y(-1,-2)$ has only one A-module structure is only clear, when it is written as $A \otimes_Y \mathcal{O}_Y(-1,0)$ which clearly only has one A-module structure. Hence, if one does not realise that $\mathcal{O}_Y(-1,0) \oplus \mathcal{O}_Y(-1,-2) \overset{Y}{\simeq} A \otimes_Y \otimes \mathcal{O}_Y(-1,0)$ then determining the fact that all possible A-module structures are isomorphic may be very hard indeed.

A similar phenomenon occurs for quotients of A. Let $Q := A \otimes_Y \mathcal{O}_F$ and forget the natural A-module structure, and ask: how many (non-isomorphic) A-module structures can Q have? If one does not realise that at least as an $\mathcal{O}_Y$-module $Q \simeq A \otimes_Y \mathcal{O}_F$ it will be difficult to prove that all the potentially different A-module structures are in fact isomorphic. Furthermore, as we are about to see, for most σ-invariant $(1,1)$-divisors C, $\mathcal{O}_C$ will have several, but finitely many, A-module structures.

The reason for the difference in the number of A-module structures is partly due to the size of the endomorphism ring of the modules. In the first example, $\dim_k \mathrm{End}_Y(\mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)) = 4$ whilst $\dim_k \mathrm{End}_Y(A \otimes_Y \mathcal{O}_Y(-1,0)) = 5$. A larger automorphism group means it is "easier" for two A-modules structures to be isomorphic.

We now study the number of A-module structures that $\mathcal{O}_C$ may possess. For any $p \in (\mathbb{P}^2)^\vee$ we will denote by l_p the corresponding line in $\mathbb{P}^2$ and we let $C_p := \pi^* l_p$ which is a σ-invariant $(1,1)$-divisor.

As we saw, for every $p \in (\mathbb{P}^2)^\vee$, giving $\mathcal{O}_{C_p}$ an A-module structure amounts to giving a left $\mathcal{O}_Y$-module map $A \otimes_Y \mathcal{O}_{C_p} \to \mathcal{O}_{C_p}$ satisfying the necessary associativity condition. In order to better understand this we first introduce some notation: we let $\bar{L} := L \otimes_Y \mathcal{O}_{C_p} = L|_{C_p}$ and $\bar{D} := D \cap C_p$. Then, since $A \otimes_Y \mathcal{O}_{C_p} = A|_{C_p}$ and because $L_\sigma^{\otimes 2} \simeq \mathcal{O}_Y(-D)$ this condition is equivalent to giving a map $m \colon \bar{L} \to \mathcal{O}_{C_p}$ such that

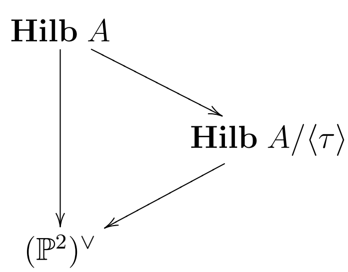
$$\mathcal{O}_{C_p}(-\bar{D}) \simeq \bar{L}_\sigma \otimes_{C_p} \bar{L}_\sigma \xrightarrow{1 \otimes m} \bar{L}_\sigma \xrightarrow{m} \mathcal{O}_{C_p}(-\bar{D})$$

is the identity. Note that given such a map m, the map $-m$ gives a different, non isomorphic A-module structure to $\mathcal{O}_{C_p}$. This observation gives us the following:

Proposition 4.8. *There exist an involution $\tau \colon \mathbf{Hilb}\ A \to \mathbf{Hilb}\ A$ sending an A-module structure given by m to the one given by $-m$. The fixed points are those which corresponds to quotients of A that are not quotients of $\mathcal{O}_Y$.*

Proof. If τ sends the A-module structure given by m to the one given by $-m$ then if the module is also a quotient of $\mathcal{O}_Y$ then as we just saw, these two A-module structures are not isomorphic. If the module is not a quotient of $\mathcal{O}_Y$ then by Corollary 4.2 it must be isomorphic to $A \otimes_Y \mathcal{O}_F$ which can only be given one A-module structure. □

Corollary 4.9. *The map $\Psi \colon \mathbf{Hilb}\ A \to (\mathbb{P}^2)^\vee$ factors through $\mathbf{Hilb}\ A/\langle\tau\rangle$. I.e. we have the following commutative diagram*



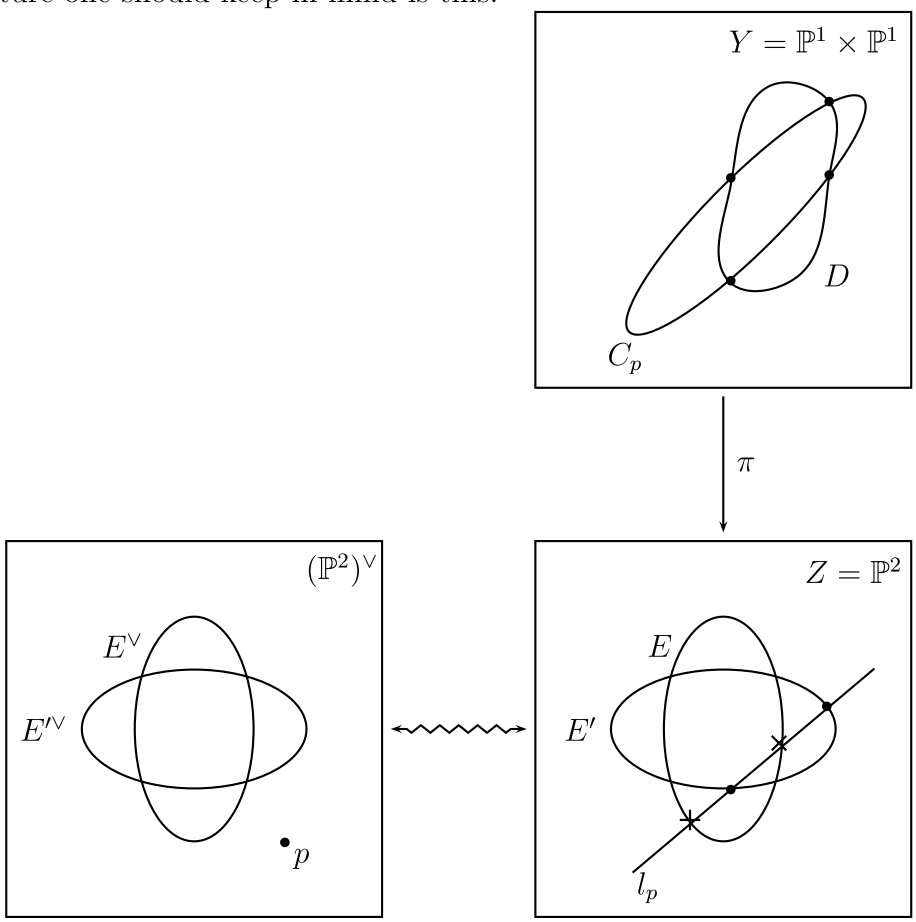
Proof. Clear from the above proposition and Theorem 4.6. □

We can view m as an element of $H^0(C_p, \bar{L}^{-1})$ and, up to multiplication by ± 1, the associativity condition then simply says that we need div $m +$ div $\sigma^* m = \bar{D}$, where each such m gives rise to two A-module structures. Since $\bar{D}$ is a finite number of points we have proved the following lemma, which also finishes off the proof of the Theorem 4.6:

Lemma 4.10. *The map Ψ is finite.*

This way of thinking, allows us to view the problem of giving $\mathcal{O}_{C_p}$ an A-module structure geometrically. As we are about to see, the number of A-module structures that $\mathcal{O}_{C_p}$ can be given depends primarily how many points l_p intersects with E and E'.

Note also that the dual of a smooth conic in $\mathbb{P}^2$ is another smooth conic in $(\mathbb{P}^2)^\vee$. We denote the duals of E and E' by $E^\vee$ and $E'^\vee$ respectively. The picture one should keep in mind is this:



We mark where l_p intersects E with a "×" and where l_p intersects E' with a "•". The problem of giving $\mathcal{O}_{C_p}$ an A-module structure breaks up into two cases:

1. l_p is not tangential to E. In this case we get $C_p \to l_p$ is a $2:1$ cover ramified at two points an hence $C_p \simeq \mathbb{P}^1$, in particular it is smooth. We analyse this case first, in Section 4.2.1.

2. l_p is tangential to E. In this case $C_p \to l_p$ is ramified at only one point

and hence C_p is the union of two $\mathbb{P}^1$'s, in particular it is singular. We analyse this case second, in Section 4.2.2.

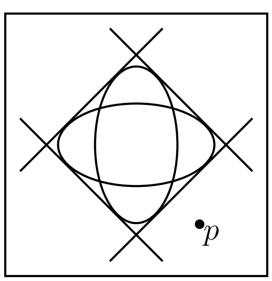
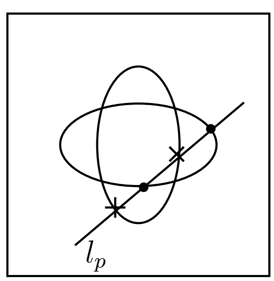
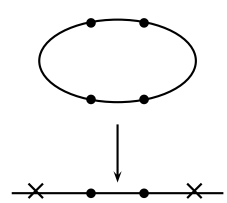
From now on, in any subsequent diagrams, any vertical conic on Z will be E, any horizontal one will be E' and similarly with $E^\vee$ and $E'^\vee$ on $(\mathbb{P}^2)^\vee$ and hence will not longer be labelled.

4.2.1 If C is smooth

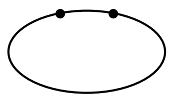
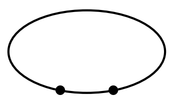
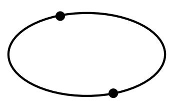
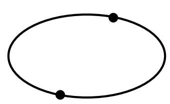
As mentioned earlier, we begin by studying the first of the two cases mentioned above. Recall that C_p is smooth, in fact $C_p \simeq \mathbb{P}^1$, precisely when l_p is not tangential to E or, equivalently, when p doesn't lie on $E^\vee$. In this case, from Corollary 4.3 we know that all quotients of A with this support have their underlying $\mathcal{O}_Y$-module structure isomorphic to $\mathcal{O}_C$.

This happens when l_p is not a tangent to E which is equivalent to p not lying on $E^\vee$. In this case, since Pic $C_p \simeq \mathbb{Z}$ we have $H^0(C_p, \bar{L}^{-1}) = H^0(C_p, \mathcal{O}_{C_p}(2))$ and so to give $\mathcal{O}_{C_p}$ an A-module structure corresponds to choosing two points $\bar{D}' \subseteq \bar{D} := C_p \cap D$ such that $\bar{D}' + \sigma^* \bar{D}' = \bar{D}$. As mentioned earlier, any such choice gives rise to precisely two A-module structures. There are several cases that need to be considered depending on precisely where p lies.

Case 1: p does not lie on either $E^\vee$ or $E'^\vee$ nor on any of the four bitangents to them and so we see that this is the generic case. In summary we have:

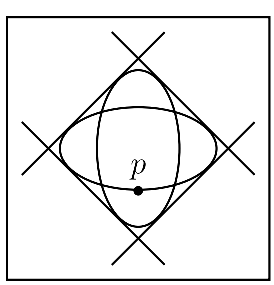
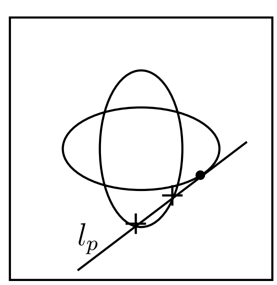
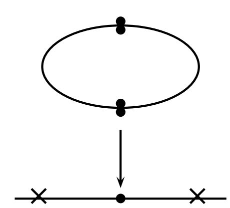
position of $p \in (\mathbb{P}^2)^\vee$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

Thus there are 4 choices for $\bar{D}'$ which results in 8 different A-module structures on $\mathcal{O}_{C_p}$. In order for us to later study the ramification of Ψ we also include the column which shows which branch corresponds to which module structure.

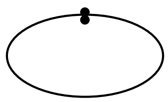
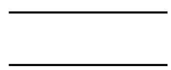
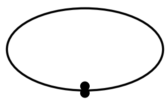
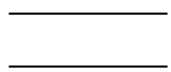
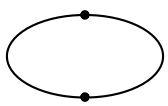
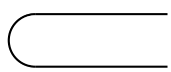
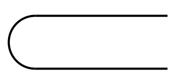
$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
	$1a$ $1b$	
	$2a$ $2b$	
	$3a$ $3b$	8
	$4a$ $4b$	

We may thus conclude that Ψ is an $8:1$ cover of $(\mathbb{P}^2)^\vee$. The other cases are used to study the ramification of this map.

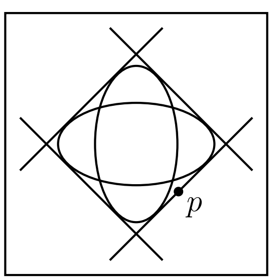
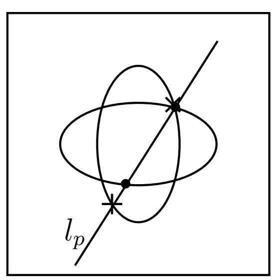
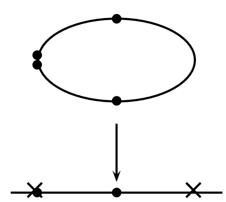
Case 2: p lies on $E'^\vee$ but not on $E^\vee$ nor on any of the four bitangents.

position of $p \in (\mathbb{P}^2)^\vee$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

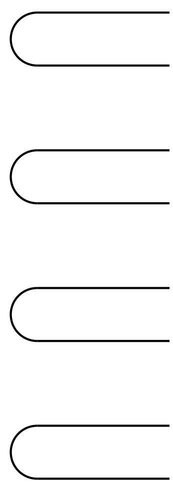
There are now only 3 choices for $\bar{D}'$ as we see in the table below.

$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
		
		6
		

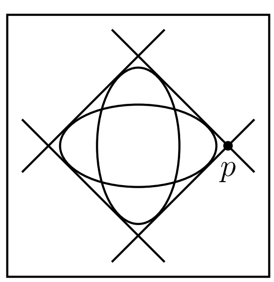
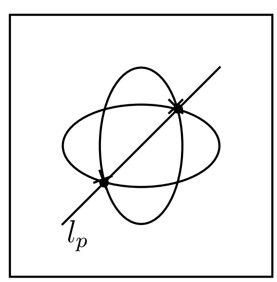
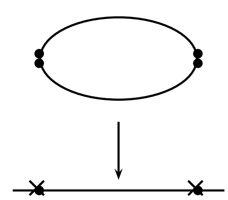
Case 3: p lies on one exactly one of the four bitangents but not where they meet the conics.

position of $p \in (\mathbb{P}^2)^\vee$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

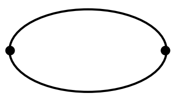
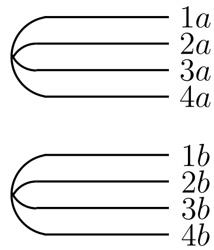
There are now only 2 choices for $\bar{D}'$ as we explain in the table below.

$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
		4

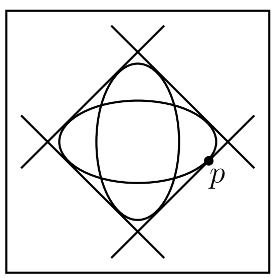
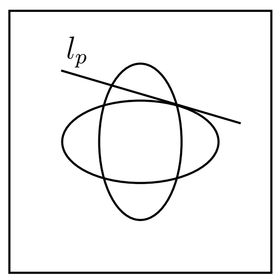
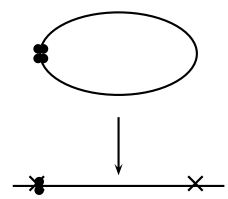
Case 4: p is chosen to be the point of intersection of two bitangents to $E^\vee$ and $E'^\vee$.

position of $p \in (\mathbb{P}^2)^\vee$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

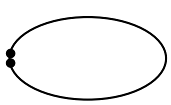
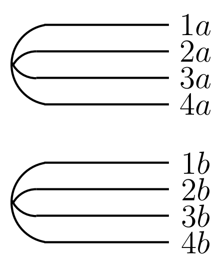
There is now only 1 choice for $\bar{D}'$ as we explain in the table below.

$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
		2

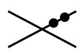
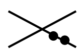
Case 5: p lies on the intersection of one of the bitangents and $E'^{\vee}$.

position of $p \in (\mathbb{P}^2)^{\vee}$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

There is now only 1 choice for $\bar{D}'$ as we explain in the table below.

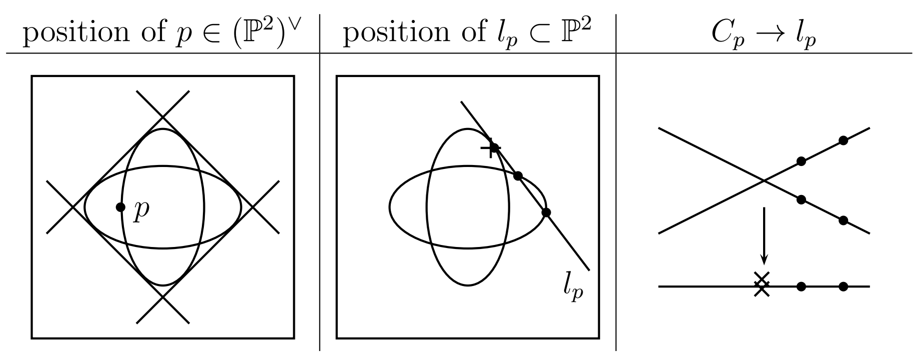
$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
		2

4.2.2 If C is singular.

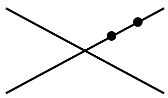
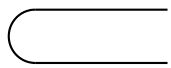
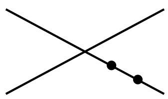
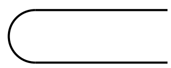
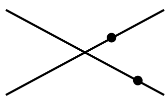
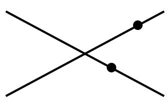
We now analyse the second case mentioned on page 21. Here C_p is singular, in fact it is the union of two $\mathbb{P}^1$'s crossing at one point. This occurs precisely when l_p is tangential to E or, equivalently, when p lies on $E^\vee$. Let $C_p = F_p + F'_p$ where F_p is a $(1,0)$-divisor and $F'_p = \sigma^* F_p$ which is a $(0,1)$-divisor. In this case Pic $C_p = \mathbb{Z} \oplus \mathbb{Z}$. Thus $H^0(C_p, \bar{L}^{-1}) = H^0(C_p, \mathcal{O}_{C_p}(1,1))$ and so to give $\mathcal{O}_{C_p}$ an A-module structure corresponds to choosing two points $\bar{D}' \subseteq \bar{D} := C_p \cap D$ one lying on F_p the other on F'_p such that $\bar{D}' + \sigma^* \bar{D}' = \bar{D}$. As before, any such choice gives rise to precisely two A-module structures. Since we must choose one point from F_p and the other from F'_p (and can not choose both points to lie on F_p nor on F'_p) implies that we have "lost" some quotients of A corresponding to p. From a geometric view point, this means that $\bar{D}' =$  and $\bar{D}' =$  do not correspond to A-module structures on $\mathcal{O}_{C_p}$.

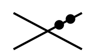
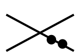
However we are now in the case where Corollary 4.3 no longer applies, and so not all quotients of A have their underlying $\mathcal{O}_Y$-module structure equal to $\mathcal{O}_C$ for some $(1,1)$-divisor C. In fact from Corollary 4.2 we know that for every p lying on $E'^\vee$ there are two additional quotients of A (in the sense that they have no analogue in Cases 1-5 because they are not quotients of $\mathcal{O}_Y$) with support C_p and they are $A \otimes_Y \mathcal{O}_{F_p}$ and $A \otimes_Y \mathcal{O}_{F'_p}$. It is thus natural to think of the above two choices of $\bar{D}'$ as giving rise to these two quotients of A and so we make this association in our future analysis of Ψ.

Case 6: p lies on $E^\vee$ but not on $E'^\vee$ nor on any of the four bitangents.

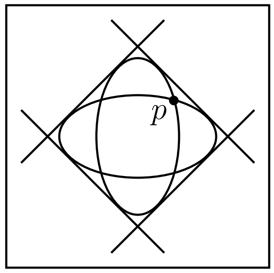
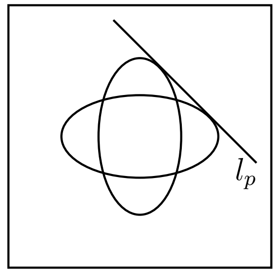
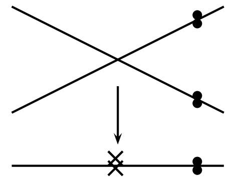


There are now the full 4 choices for $\bar{D}'$, however they only gives rise to six quotients of A as we explain below.

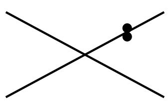
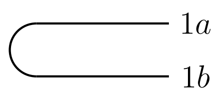
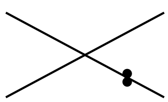
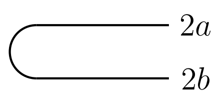
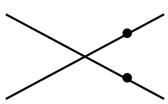
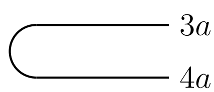
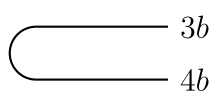
$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
	 $1a$, $1b$	6
	 $2a$, $2b$	
	$3a$, $3b$	
	$4a$, $4b$	

Let us explain further why branches $1a$ and $1b$ come together here and why this case is different to Case 1. Recall that to picking $\bar{D}' =$  and  we associate not a total of four A-module structure on $\mathcal{O}_{C_p}$ but the two quotients of A that are not quotients of $\mathcal{O}_Y$ with support C_p, namely $A \otimes_Y \mathcal{O}_F$ and $A \otimes_Y \mathcal{O}_{F'}$. We also saw that the involution τ from Proposition 4.8 fixes points of **Hilb** A corresponding to $A \otimes_Y \mathcal{O}_F$ and that by Corollary 4.9 the map Ψ factors through τ. Hence the branches $1a$ and $1b$ must intersect at precisely points corresponding to $A \otimes_Y \mathcal{O}_F$. The same argument applies to explain why the branches $2a$ and $2b$ also merge.

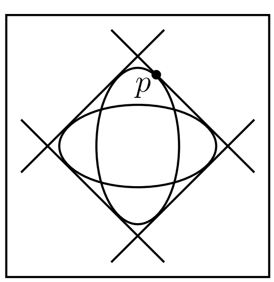
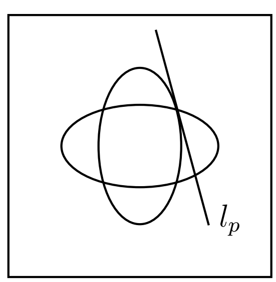
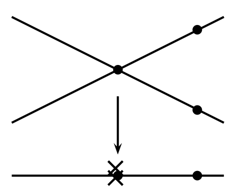
Case 7: p lies on the intersection of $E^\vee$ and $E'^\vee$.

position of $p \in (\mathbb{P}^2)^\vee$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

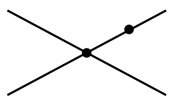
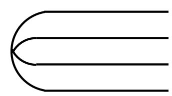
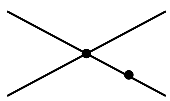
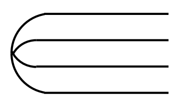
There are now only 4 choices for $\bar{D}'$ as we explain in the table below.

$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
		
		4
		
		

Case 8: p is one of the points of intersection of the bitangents with $E^\vee$.

position of $p \in (\mathbb{P}^2)^\vee$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

There are now only 2 choices for $\bar{D}'$ as we explain in the table below.

$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
		2
		

Note that the two A-module structures with support $C_p = F + F'$ are $A \otimes_Y \mathcal{O}_F$ and $A \otimes_Y \mathcal{O}_{F'}$.

By carefully following which branch connects to which branch we can see that **Hilb** A is in fact connected and thus we may conclude that **Hilb** A is in fact a smooth projective surface.

4.3 Possible second Chern classes of A-line bundles

In this section we tie up one loose end that we have left from Section 3.1 and prove the existence of lines bundles with all possible combinations of Chern classes, provided they satisfy our Bogomolov-type inequality. We continue with the same notation as before.

Theorem 4.11. *Let $c_1 \in \operatorname{Pic} Y$ and $c_2 \in \mathbb{Z}$ such that $4c_2 - c_1^2 \geq -2$. Then there exists an $M \in \operatorname{Pic} A$ with these Chern classes.*

Before we begin the proof, we need the following lemma:

Lemma 4.12. *Let C be a smooth, σ-invariant $(1,1)$-divisor on Y and $N \in \operatorname{Pic} C$. Endow $\mathcal{O}_C$ with an A-module structure, which we saw is always possible from Cases 1-5 previously. Then N inherits an A-module structure from $\mathcal{O}_C$.*

Proof. We need give an $\mathcal{O}_Y$-module morphism $A \otimes_C N \to N$ satisfying the required associativity condition. Suppose $\psi \colon A \otimes_C \mathcal{O}_C \to \mathcal{O}_C$ is the morphism which gives $\mathcal{O}_C$ its A-module structure. Then $A \otimes_C N \to A \otimes_Y \mathcal{O}_C \otimes_C N \xrightarrow{\psi \otimes 1} \mathcal{O}_C \otimes_C N \to N$ is the required morphism. $\square$

Proof of theorem. The discriminant of any rank two vector bundle M, defined to be the integer $4c_2(M) - c_1(M)^2$, is unchanged by tensoring with a line bundle (see Chapter 12.1 of in [LP97]) and so as we saw before we can thus assume $c_1 = \mathcal{O}_Y(-1,-1)$ or $c_1 = \mathcal{O}_Y$. We deal with these two cases separately although the proofs will be very similar. Fix for the remainder of the proof a smooth σ-invariant $(1,1)$-divisor C and an A-module structure on $\mathcal{O}_C$.

We will now construct an A-line bundle with $c_1 = \mathcal{O}_Y$ and $c_2 = n$ for an arbitrary $n \geq 0$. Using Lemma 4.12 endow $\mathcal{O}_C(n+2)$ with an A-module structure. Note that

$$\begin{aligned}\mathrm{Hom}_A\left(A \otimes_Y \mathcal{O}_Y(1,1), \mathcal{O}_C(n+2)\right) &= \mathrm{Hom}_Y\left(\mathcal{O}_Y(1,1), \mathcal{O}_C(n+2)\right) \\ &= \mathrm{Hom}_C\left(\mathcal{O}_C, \mathcal{O}_C(n)\right) \\ &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) \neq 0.\end{aligned}$$

We claim that there is at least one morphism $\varphi\colon A \otimes_Y \mathcal{O}_Y(1,1) \to \mathcal{O}_C(n+2)$ which is surjective. From the above computation, we see that any A-module morphism $A \otimes_Y \mathcal{O}_Y(1,1) \to \mathcal{O}_C(n+2)$ arises from an $\mathcal{O}_Y$-module morphism $\phi\colon \mathcal{O}_C \to \mathcal{O}_C(n)$. Choose ϕ in such a away that coker $\phi = \oplus k_{p_i}$, where k_{p_i} is the skyscraper sheaf at p_i, with the p_i lying in the Azumaya locus of A. Then, since $A|_p = M_2(k)$, when we extend ϕ to a morphism $\varphi\colon A \otimes_Y \mathcal{O}_Y(1,1) \to \mathcal{O}_C(n+2)$ we must have coker $\varphi = 0$ for the simple representations of $M_2(k)$ are all two dimensional. Letting $M :=$ ker φ we have

$$0 \longrightarrow M \longrightarrow A \otimes_Y \mathcal{O}_Y(1,1) \longrightarrow \mathcal{O}_C(n+2) \longrightarrow 0. \qquad (*)$$

It is easy to check that $M \in$ Pic A with $c_1(M) = \mathcal{O}_Y$ and $c_2(M) = n$.

Constructing an A-line bundle with $c_1 = \mathcal{O}_Y(-1,-1)$ and $c_2 = n$ for an arbitrary $n \geq 0$ is an almost identical process where one finds a surjective morphism $\varphi : A \to \mathcal{O}_C(n)$ in the same manner as before and then proves that the kernel must be a line bundle. A simple computation shows that this kernel has the desired Chern classes. □

5 The Link

In this section we establish a link between the moduli space of A-line bundles with $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$, which as before we denote by **Pic** A, and the Hilbert scheme of A, which parameterises quotients of A with $c_1 = \mathcal{O}_Y(1,1)$ and $c_2 = 2$, which as before is denoted by **Hilb** A. In particular we will show that **Hilb** A is a ruled surface over **Pic** A. Thus by using the map

Ψ from the previous section, we will calculate $(K_{\mathbf{Hilb}\ A})^2$, which will allow us to determine the genus of **Pic** A.

We have already seen this link between line bundles on A and quotients of A. It is summarised with the following exact sequence

$$0 \longrightarrow M \longrightarrow A \longrightarrow Q \longrightarrow 0$$

where:

- $Q \simeq \mathcal{O}_C$ as an $\mathcal{O}_Y$-module, which occurs precisely when $M \in \text{Pic } A$ with $M \simeq \mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$ as an $\mathcal{O}_Y$-module, or
- $Q \simeq A \otimes_Y \mathcal{O}_F$, where F is a $(1,0)$ (respectively $(0,1)$) divisor, which occurs precisely when $M \simeq A \otimes_Y \mathcal{O}_Y(-1,0)$ (respectively $A \otimes_Y \mathcal{O}_Y(0,-1)$).

Furthermore, we saw in Proposition 3.14 that in both cases $\hom_A(M, A) = 2$ which suggests there is a $\mathbb{P}:1$ map **Hilb** $A \to$ **Pic** A. We prove this now.

Theorem 5.1. **Hilb** *A is a ruled surface over* **Pic** *A.*

Proof. Let $\mathcal{F}$ be the universal family on **Hilb** A. From Proposition 4.1 $\ker(A_{\mathbf{Hilb}\ A} \to \mathcal{F})$ is a flat family of A-line bundles on **Hilb** A and so we get a map $\Phi\colon$ **Hilb** $A \to$ **Pic** A. **M** being smooth and together with Proposition 3.13 implies one of its components is a curve. However, from the previous section we know that **Hilb** A is a smooth projective surface and thus **Pic** A must in fact be connected and hence must be a smooth projective curve. It thus suffice to show that every fibre of Φ is isomorphic to $\mathbb{P}^1$ which is clear from Proposition 3.14 □

Since **Hilb** A is a ruled surface over **Pic** A we can determine the genus of **Pic** A using Corollary 2.11 in Chapter 5 of [Har77] which states that

$$(K_{\mathbf{Hilb}\ A})^2 = 8(1 - g(\mathbf{Pic}\ A)).$$

Furthermore, we can determine $(K_{\mathbf{Hilb}\ A})^2$ using the map Ψ.

Theorem 5.2. *The moduli space parameterising A-line bundles with $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$ is a smooth projective curve of genus 2.*

Proof. As discussed above, all that we need to do is compute $(K_{\mathbf{Hilb}\ A})^2$. Recall from before that we have an $8:1$ map $\Psi :$ **Hilb** $A \to (\mathbb{P}^2)^\vee$. Thus using Formula 19 of Section 16 in Chapter 1 of [BPVdV84] we have:

$$K_{\mathbf{Hilb}\ A} = \Psi^* K_{(\mathbb{P}^2)^\vee} + R$$

where R is the ramification divisor on **Hilb** A.

Let us describe R. Looking at Case 2 of Section 4.2.1 we define R_1 and U_1 to be the divisors such that $\Psi^* E'^{\vee} = 2R_1 + U_1$. Similarly looking at Case 6 in Section 4.2.2 we define R_2 and U_2 to be such that $\Psi^* E^{\vee} = 2R_2 + U_2$. Denote by $L_3, \cdots, L_6$ the four bitangents to $E^{\vee} \cup E'^{\vee}$. Looking at Case 3 of Section 4.2.1 we see that $\Psi^* L_i$ is two divisible and we let R_i be such that $\Psi^* L_i = 2R_i$. Thus $R = R_1 + R_2 + \cdots + R_6$.

We now compute $(K_{\mathbf{Hilb}\ A})^2 = (\Psi^*(K_{(\mathbb{P}^2)^{\vee}}) + R_1 + \cdots + R_6)^2$. Throughout this calculation K denotes $K_{(\mathbb{P}^2)^{\vee}}$.

- $(\Psi^* K)^2 = 8 \cdot (-3)^2 = 72$
- $(\Psi^* K).R_1 = K.(\Psi_* R_1) = 2K.E' = 2 \cdot (-6) = -12$. Similarly,
- $(\Psi^* K).R_2 = -12$.
- $(\Psi^* K).R_i = K.(\Psi_* R_i) = 4K.L_i = 4 \cdot (-3) = -12$ for $i = 3, \cdots, 6$.
- $R_1.R_2 = 0$ from Case 7 on page 29.
- $R_1.R_i = \frac{1}{2} R_1.(\Psi^* L_i) = \frac{1}{2}(\Psi_* R_1).L_i = \frac{1}{2} 2E'.L_i = 2$ for $i = 3, \cdots, 6$. Similarly,
- $R_2.R_i = 2$ for $i = 3, \cdots, 6$.
- $R_i.R_j = \frac{1}{2}(\Psi^* L_i).R_j = \frac{1}{2} L_i.(\Psi_* R_j) = \frac{1}{2} L_i.4L_j = 2L_i.L_j = 2$ for $i, j = 3, \cdots, 6$.

What remains is to compute R_1^2 and R_2^2.

We can see from Case 2, 5 and 7 that $\Psi|_{R_1} : R_1 \to E'$ is an étale double cover of E'. Thus $R_1 = R_1' + R_1''$ where both R_1' and R_1'' have genus zero. We now use the adjunction formula (Proposition 1.5 in Chapter V of [Har77]) to compute $R_1'^2$. We have

$$\begin{aligned} -2 &= R_1'.(2R_1' + \Psi^* K + R_2 + R_3 + \cdots + R_6) \\ &= 2R_1'^2 + E'.K + 0 + 4 \cdot R_1'.\frac{1}{2}(\Psi^* L_3) \\ &= 2R_1'^2 - 6 + 4 \cdot 1 = 2R_1'^2 - 2. \end{aligned}$$

Thus $R_1'^2 = 0$ and a similar computation shows $R_1''^2 = 0$. Thus $R_1^2 = 0$.

The same argument shows $R_2^2 = 0$ since $R_2 \to E$ is also an étale double cover.

Thus

$$\begin{aligned}(K_{\mathbf{Hilb}\ A})^2 &= (\Psi^*K)^2 + R_1^2 + \cdots + R^6 + \\ &+ 2\left((\Psi^*K).R_1 + \cdots (\Psi^*K).R_6 + \sum R_i.R_j\right) \\ &= 72 + 0 + 0 + 4\cdot 2 + 2\,(6\cdot(-12)4\cdot 2 + 4\cdot 2 + 6\cdot 2) \\ &= -8\end{aligned}$$

and so $g(\mathbf{Pic}\ A) = 2$. □

Note that at no stage did we use the fact that **Hilb** A is ruled in order to calculate $(K_{\mathbf{Hilb}\ A})^2$. In particular, we didn't use the fact that we knew in advance that $(K_{\mathbf{Hilb}\ A})^2$ is a multiple of eight. We could have simplified the computation above if we had done so, but it seemed nice to spend the extra work and get an independent confirmation that fact.

As we saw in the above proof R_2 is the union of two $\mathbb{P}^1$'s. These $\mathbb{P}^1$'s are fibres of $\Phi\colon \mathbf{Hilb}\ A \to \mathbf{Pic}\ A$ above the two very special points on **Pic** A corresponding to the A-line bundles $A \otimes_Y \mathcal{O}_Y(-1,0)$ and $A \otimes_Y \mathcal{O}_Y(0,-1)$. Since R_1 is also a union of two $\mathbb{P}^1$'s it would have been nice to find the two A-line bundles which they are fibres of, but unfortunately, we were unable to do so.

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