Topological Cyclic Homology

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Assembly map (Loday)

$$K(A) \wedge BG \xrightarrow{\alpha} K(A[G])$$

A : ring , G : group

$K(A)$: non-connective algebraic K-theory spectrum

BG : classifying space

Homotopy groups RHS :

$$\pi_q(K(A[G])) =: K_q(A[G])$$

LHS : Atiyah-Hirzebruch sp. seq.

$$E^2_{s,t} = H_s(BG, K_t(A))$$

$$\Rightarrow \pi_q(K(A) \wedge BG)$$

Example For $G = \langle t \rangle$ infinite cyclic, get

$$K_q(A) \oplus K_{q-1}(A) \xrightarrow{\alpha} K_q(A[t, t^{-1}]).$$

An isomorphism for A regular, but not in general.

Farrell-Jones conjecture:

If A is regular and G is torsion-free, then

$$\pi_q(K(A) \wedge BG) \xrightarrow{\alpha} K_q(A[G])$$

is an isom. for all integers q.

Remark For $q < 0$, LHS $= 0$.

Theorem (Bartels - Lück - Reich)

True if G is a

word - hyperbolic group

Example $G = C_p$ cyclic, prime order

$$K_0(\mathbb{Z}[C_p]) \xrightarrow{\sim} K_0(\mathbb{Z}(\zeta_p))$$

$$\cong$$

Thm. of Rim ✓ $\quad \mathbb{Z} \oplus \mathrm{Pic}(\mathbb{Z}(\zeta_p))$

so assembly map not isom.

K-theoretic Novikov conjecture

For every group G and integer q, the map

$$\pi_q\left(K(\mathbb{Z}) \wedge BG\right) \otimes \mathbb{Q} \xrightarrow{\alpha} K_q(\mathbb{Z}[G]) \otimes \mathbb{Q}$$

is injective.

Theorem (Bökstedt-Hsiang-Madsen)

True, if the homology groups

$$H_q(G,\mathbb{Z}) = H_q(BG,\mathbb{Z})$$

are f.g. for all q.

Proof The Hurewicz map

$$\mathbb{S} \longrightarrow \mathbb{Z}$$

is a rational equivalence by Serre's thesis.

By Waldhausen, can consider

$$\begin{array}{ccc} K(\mathbb{Z}) \wedge BG & \xrightarrow{\alpha} & K(\mathbb{Z}[G]) \\ \uparrow \sim_{\mathbb{Q}} & & \uparrow \sim_{\mathbb{Q}} \\ K(\mathbb{S}) \wedge BG & \xrightarrow{\alpha} & K(\mathbb{S}[G]) \end{array}$$

Cyclotomic trace

$$\begin{array}{ccc} K(\mathbb{S}) \propto BG & \xrightarrow{\alpha} & K(\mathbb{S}[G]) \\ \Big\downarrow {\scriptstyle tr \wedge id} & & \Big\downarrow {\scriptstyle tr} \\ TC(\mathbb{S};p) \propto BG & \underset{\longleftarrow}{\xrightarrow{\alpha}} & TC(\mathbb{S}[G];p) \end{array}$$

rationally injective, if p regular

show retract exists (in htpy. cat.)

Advantage of replacing

$\mathbb{Z}$ by $\mathbb{S}$

is that one can

calculate . . .

$$
\begin{array}{ccc}
TC(\mathbb{S}[G];p) & \longrightarrow & \underset{n}{\operatorname{holim}}\, \mathbb{H}_{\cdot}(C_{p^n}, \mathbb{S}\otimes \mathcal{L}BG) \\
\big\downarrow & \text{cart} & \big\downarrow \, pr_0 \\
\mathbb{S}\otimes \mathcal{L}BG & \xrightarrow{\ \mathrm{id}-\Delta_p\ } & \mathbb{S}\otimes \mathcal{L}BG
\end{array}
$$

free loop space

$$\mathcal{L}X = \operatorname{Map}(\mathbb{T}, X)$$

$$E^2_{s,t} = H_s(C_{p^n}, \pi_t(\mathbb{S} \propto \mathcal{L}X))$$

$$\Rightarrow \mathbb{H}_{s+t}(C_{p^n}, \mathbb{S} \propto \mathcal{L}X)$$

In particular,

$$
\begin{array}{ccc}
TC(\mathbb{S};p) & \longrightarrow & \operatorname{holim}_{n} \mathbb{H}_{\cdot}(C_{p^n}, \mathbb{S}) \\
\Big\downarrow & \text{cart} & \Big\downarrow {\scriptstyle pr_0} \\
\mathbb{S} & \xrightarrow{\quad 0 \quad} & \mathbb{S}
\end{array}
$$

Assumption that $H_q(BG, \mathbb{Z})$ is f.g. for all q implies that

$$\left(\operatorname{holim}_n \mathbb{H}_\cdot(C_{p^n}, \mathbb{S})\right) \propto BG$$

$$\xrightarrow{\sim} \operatorname{holim}_n \mathbb{H}_\cdot(C_{p^n}, \mathbb{S} \propto BG)$$

Hence, also

$$\begin{array}{ccc}
TC(\mathbb{S};p) \ltimes BG & \longrightarrow & \operatorname{holim}_n \mathbb{H}_\cdot(C_{p^n}, \mathbb{S} \ltimes BG) \\
\downarrow & \textcolor{red}{\text{cart}} & \downarrow \textcolor{blue}{pr_0} \\
\mathbb{S} \ltimes BG & \xrightarrow{\textcolor{blue}{0}} & \mathbb{S} \ltimes BG
\end{array}$$

and

$$TC(\mathbb{S};p) \propto BG \xrightarrow{\;\propto\;} TC(\mathbb{S}[G];p)$$

is induced by ...

… the inclusion

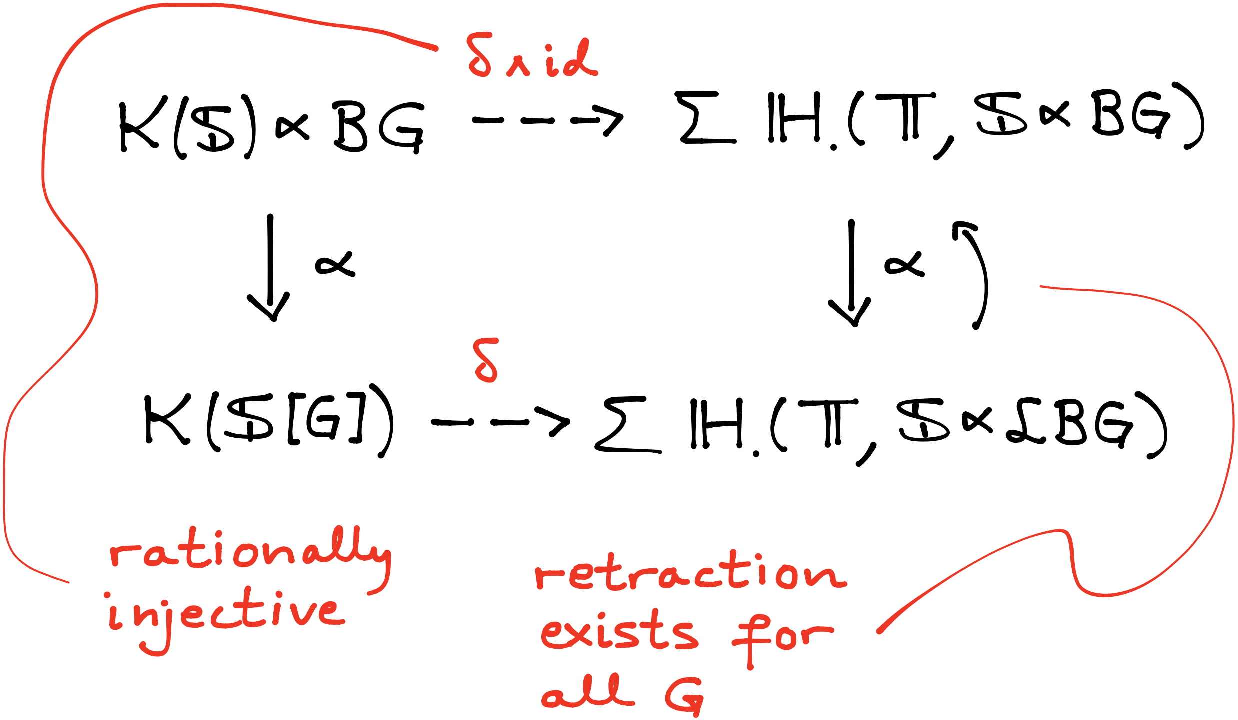
$$\begin{array}{ccc} BG & \xrightarrow{\ \alpha\ } & \mathcal{L}BG \\ \downarrow \wr & & \Vert \\ \mathcal{L}_{[1]}BG & \xrightarrow{\ in_{[1]}\ } & \coprod_{[g]} \mathcal{L}_{[g]}BG \end{array}$$

Get retraction by mapping summands $[g] \neq [1]$ to zero. $\lrcorner$

Suppose there exists a lifting δ ...

$$
\begin{array}{ccc}
K(\mathbb{S}[G]) & \overset{\delta}{\dashrightarrow} & \Sigma\, \mathbb{H}_{\bullet}(\mathbb{T}, \mathbb{S} \rtimes \mathcal{L}BG) \\
\downarrow & & \downarrow {\scriptstyle \mathrm{trf}} \\
TC(\mathbb{S}[G];p) & \longrightarrow & \operatorname{holim}_{n}\, \mathbb{H}_{\bullet}(C_{p^n}, \mathbb{S} \rtimes \mathcal{L}BG) \\
\downarrow & \mathrm{cart} & \downarrow {\scriptstyle \mathrm{pr}_0} \\
\mathbb{S} \rtimes \mathcal{L}BG & \xrightarrow[\mathrm{id}-\Delta_p]{} & \mathbb{S} \rtimes \mathcal{L}BG
\end{array}
$$

Then ...


δ ∧ id
K(𝕊) ∝ BG
Σ ℍ.(𝕋, 𝕊 ∝ BG)
α
α
δ
K(𝕊[G])
Σ ℍ.(𝕋, 𝕊 ∝ ℒBG)
rationally injective
retraction exists for all G

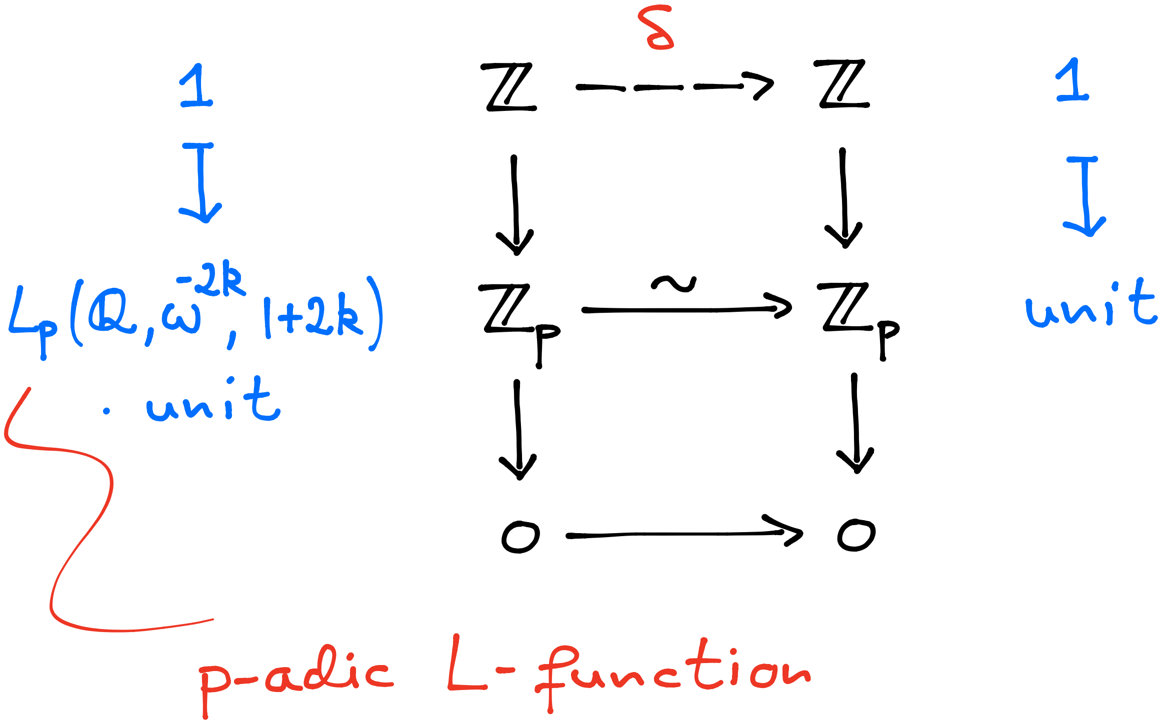
... so K-theoretic Novikov conjecture for all groups G would follow

Number-theoretic consequences :

Up to finite torsion . . .

$$
\pi_{4k+1}\left(\begin{array}{ccc}
K(\mathbb{S}) & \overset{\delta}{\dashrightarrow} & \Sigma \mathbb{H}_{\cdot}(\mathbb{T}, \mathbb{S}) \\
\downarrow & & \downarrow \\
TC(\mathbb{S};p) & \longrightarrow & \operatorname{holim}_{n} \mathbb{H}_{\cdot}(C_{p^n}, \mathbb{S}) \\
\downarrow & & \downarrow \\
\mathbb{S} & \xrightarrow{\ 0\ } & \mathbb{S}
\end{array}\right)
$$

$$= \ \cdots$$


δ
ℤ
ℤ
1
1
~
ℤ_p
ℤ_p
L_p(ℚ, ω^{-2k}, 1+2k)
· unit
unit
0
0
p-adic L-function

So existence of δ implies:

"For every $k \geq 1$, there are only finitely many p s.t.

$$p \mid L_p(\mathbb{Q}, \omega^{-2k}, 1+2k)$$"

Equivalently,

"For every $k \geq 1$, there are only finitely many p s.t.

$$p \mid B_{p-(1+2k)}$$"

For $k=1$, the only $p < 10^9$ s.t.

$$p \mid B_{p-(1+2k)}$$

are

$$p = 16{,}843 \quad \text{and} \quad p = 2{,}124{,}679$$

Thank you