

# On the Whitehead Spectrum of the Circle

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**Abstract** The seminal work of Waldhausen, Farrell and Jones, Igusa, and Weiss and Williams shows that the homotopy groups in low degrees of the space of homeomorphisms of a closed Riemannian manifold of negative sectional curvature can be expressed as a functor of the fundamental group of the manifold. To determine this functor, however, it remains to determine the homotopy groups of the topological Whitehead spectrum of the circle. The cyclotomic trace of Bökstedt, Hsiang, and Madsen and a theorem of Dundas, in turn, lead to an expression for these homotopy groups in terms of the equivariant homotopy groups of the homotopy fiber of the map from the topological Hochschild  $\mathbb{T}$ -spectrum of the sphere spectrum to that of the ring of integers induced by the Hurewicz map. We evaluate the latter homotopy groups, and hence, the homotopy groups of the topological Whitehead spectrum of the circle in low degrees. The result extends earlier work by Anderson and Hsiang and by Igusa and complements recent work by Grunewald, Klein, and Macko.

## Introduction

Let  $M$  be a closed smooth manifold of dimension  $m \geq 5$ . Then, the stability theorem of Igusa [22] and a theorem of Weiss and Williams [35, Theorem A] show that, for all integers  $q$  less both  $(m - 4)/3$  and  $(m - 7)/2$ , there is a long-exact sequence

$$\cdots \rightarrow \mathbb{H}_{q+2}(C_2, \tau_{\geq 2} \text{Wh}^{\text{Top}}(M)) \rightarrow \pi_q(\text{Homeo}(M)) \rightarrow \pi_q(\widetilde{\text{Homeo}}(M)) \rightarrow \cdots$$

where the middle group is the  $q$ th homotopy group of the space of homeomorphisms of  $M$ . In particular, the group  $\pi_0(\text{Homeo}(M))$  is the mapping class group of  $M$ . The right-hand term is the  $q$ th homotopy group of the space of block homeomorphisms

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of  $M$  and is the subject of surgery theory. The left-hand term is the  $(q + 2)$ th homotopy group of the Borel quotient of the 2-connective cover of the topological Whitehead spectrum of  $M$  by the canonical involution. It is one of the great past achievements that the left-hand term can be expressed by Waldhausen’s algebraic  $K$ -theory of spaces [32–34].

Suppose, in addition, that  $M$  carries a Riemannian metric of negative, but not necessarily constant, sectional curvature. Another great achievement is the topological rigidity theorems [11, Remark 1.10, Theorem 2.6] of Farrell and Jones which, in this case, give considerable simplifications of the left and right-hand terms in the above sequence. For the right-hand term, there are canonical isomorphisms

$$\pi_q(\widetilde{\text{Homeo}}(M)) \xrightarrow{\sim} \pi_q(\widetilde{\text{HoAut}}(M)) \xleftarrow{\sim} \pi_q(\text{HoAut}(M)),$$

where  $\text{HoAut}(M)$  and  $\widetilde{\text{HoAut}}(M)$  are the spaces of self-homotopy equivalences and block self-homotopy equivalences of  $M$ , respectively. We note that, as  $M$  is aspherical with  $\pi_1(M)$  centerless [27, Theorems 22, 24], it follows from [13, Theorem III.2] that the canonical map from  $\text{HoAut}(M)$  to the discrete group  $\text{Out}(\pi_1(M))$  is a weak equivalence. For the left-hand term, there is a canonical isomorphism

$$\bigoplus_{(C)} \text{Wh}_q^{\text{Top}}(S^1) \xrightarrow{\sim} \text{Wh}_q^{\text{Top}}(M),$$

where the sum ranges over the set of conjugacy classes of maximal cyclic subgroups of the torsion-free group  $\pi_1(M)$ ; see also [24, Theorem 139]. Hence, in order to evaluate the groups  $\pi_q(\text{Homeo}(M))$ , it remains to evaluate

$$\text{Wh}_q^{\text{Top}}(S^1) = \pi_q(\text{Wh}^{\text{Top}}(S^1))$$

and the canonical involution on these groups. We prove the following result.

**Theorem 1.** *The groups  $\text{Wh}_0^{\text{Top}}(S^1)$  and  $\text{Wh}_1^{\text{Top}}(S^1)$  are zero. Moreover, there are canonical isomorphisms*

$$\begin{aligned} \text{Wh}_2^{\text{Top}}(S^1) &\xrightarrow{\sim} \bigoplus_{r \geq 1} \bigoplus_{j \in \mathbb{Z} \setminus 2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \\ \text{Wh}_3^{\text{Top}}(S^1) &\xrightarrow{\sim} \bigoplus_{r \geq 0} \bigoplus_{j \in \mathbb{Z} \setminus 2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{r \geq 1} \bigoplus_{j \in \mathbb{Z} \setminus 2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

The statement for  $q = 0$  and  $q = 1$  was proved earlier by Anderson and Hsiang [1] by different methods. It was also known by work of Igusa [21] that the two sides of the statement for  $q = 2$  are abstractly isomorphic. The statement for  $q = 3$  is new. We also note that in recent work, Grunewald et al. [15] have proved that for  $p$  an odd prime and  $q \leq 4p - 7$ , the  $p$ -primary torsion subgroup of  $\text{Wh}_q^{\text{Top}}(S^1)$  is a countably dimensional  $\mathbb{F}_p$ -vector space, if  $q = 2p - 2$  or  $2p - 1$ , and zero, otherwise. Hence, we will here focus the attention on the 2-primary torsion subgroup.

We briefly outline the proof of Theorem 1. The seminal work of Waldhausen establishes a cofibration sequence of spectra

$$S_+^1 \wedge K(\mathbb{S}) \xrightarrow{\alpha} K(\mathbb{S}[x^{\pm 1}]) \rightarrow \text{Wh}^{\text{Top}}(S^1) \xrightarrow{\partial} \Sigma S_+^1 \wedge K(\mathbb{S}),$$

which identifies the topological Whitehead spectrum of the circle as the mapping cone of the assembly map in algebraic  $K$ -theory [33, Theorem 3.3.3], [34, Theorem 0.1]. Here  $\mathbb{S}$  is the sphere spectrum and  $\mathbb{S}[x^{\pm 1}]$  is the Laurent polynomial extension. If we replace the sphere spectrum by the ring of integers, the assembly map

$$\alpha: S_+^1 \wedge K(\mathbb{Z}) \rightarrow K(\mathbb{Z}[x^{\pm 1}])$$

becomes a weak equivalence by the fundamental theorem of algebraic  $K$ -theory [28, Theorem 8, Corollary]. Hence, we obtain a cofibration sequence of spectra

$$S_+^1 \wedge K(\mathbb{S}, I) \xrightarrow{\alpha} K(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]) \rightarrow \text{Wh}^{\text{Top}}(S^1) \xrightarrow{\partial} \Sigma S_+^1 \wedge K(\mathbb{S}, I),$$

where the spectra  $K(\mathbb{S}, I)$  and  $K(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}])$  are defined to be the homotopy fibers of the maps of  $K$ -theory spectra induced by the Hurewicz maps  $\ell: \mathbb{S} \rightarrow \mathbb{Z}$  and  $\ell: \mathbb{S}[x^{\pm 1}] \rightarrow \mathbb{Z}[x^{\pm 1}]$ , respectively. The Hurewicz maps are rational equivalences, as was proved by Serre. This implies that  $K(\mathbb{S}, I)$  and  $K(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}])$  are rationally trivial spectra. It follows that, for all integers  $q$ ,

$$\text{Wh}_q^{\text{Top}}(S^1) \otimes \mathbb{Q} = 0.$$

Therefore, it suffices to evaluate, for every prime number  $p$ , the homotopy groups with  $p$ -adic coefficients,

$$\text{Wh}_q^{\text{Top}}(S^1; \mathbb{Z}_p) = \pi_q(\text{Wh}^{\text{Top}}(S^1)_p),$$

that are defined to be the homotopy groups of the  $p$ -completion [6].

The cyclotomic trace map of Bökstedt et al. [4] induces a map

$$\text{tr}: K(\mathbb{S}, I) \rightarrow \text{TC}(\mathbb{S}, I; p)$$

from the relative  $K$ -theory spectrum to the relative topological cyclic homology spectrum. It was proved by Dundas [8] that this map becomes a weak equivalence after  $p$ -completion. The same is true for the Laurent polynomial extension. Hence, we have a cofibration sequence of implicitly  $p$ -completed spectra

$$S_+^1 \wedge \text{TC}(\mathbb{S}, I; p) \xrightarrow{\alpha} \text{TC}(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; p) \rightarrow \text{Wh}^{\text{Top}}(S^1) \xrightarrow{\partial} \Sigma S_+^1 \wedge \text{TC}(\mathbb{S}, I; p).$$

There is also a “fundamental theorem” for topological cyclic homology which was proved by Madsen and the author in [19, Theorem C]. If  $A$  is a symmetric ring

spectrum whose homotopy groups are  $\mathbb{Z}_{(p)}$ -modules, this theorem expresses, up to an extension, the topological cyclic homology groups  $\mathrm{TC}_q(A[x^{\pm 1}]; p)$  of the Laurent polynomial extension in terms of the equivariant homotopy groups

$$\mathrm{TR}_q^n(A; p) = [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(A)]_{\mathbb{T}}$$

of the topological Hochschild  $\mathbb{T}$ -spectrum  $T(A)$  and the maps

$$\begin{aligned} R: \mathrm{TR}_q^n(A; p) &\rightarrow \mathrm{TR}_q^{n-1}(A; p) && \text{(restriction)} \\ F: \mathrm{TR}_q^n(A; p) &\rightarrow \mathrm{TR}_q^{n-1}(A; p) && \text{(Frobenius)} \\ V: \mathrm{TR}_q^{n-1}(A; p) &\rightarrow \mathrm{TR}_q^n(A; p) && \text{(Verschiebung)} \\ d: \mathrm{TR}_q^n(A; p) &\rightarrow \mathrm{TR}_{q+1}^n(A; p) && \text{(Connes' operator)} \end{aligned}$$

which relate these groups. Here  $\mathbb{T}$  is the multiplicative group of complex numbers of modulus 1, and  $C_{p^{n-1}} \subset \mathbb{T}$  is the subgroup of the indicated order. We recall the groups  $\mathrm{TR}_q^n(A; p)$  in Section 1 and give a detailed discussion of the fundamental theorem in Section 2. In the following Sections 3 and 4, we briefly recall the cyclotomic trace map and the skeleton spectral sequence which we use extensively in later sections. A minor novelty here is Proposition 4 which generalizes of the fundamental long-exact sequence [17, Theorem 2.2] to a long-exact sequence

$$\cdots \rightarrow \mathbb{H}_q(C_{p^m}, \mathrm{TR}^n(A; p)) \rightarrow \mathrm{TR}_q^{m+n}(A; p) \xrightarrow{R^n} \mathrm{TR}_q^m(A; p) \rightarrow \cdots$$

valid for all positive integers  $m$  and  $n$ .

The problem to evaluate  $\mathrm{Wh}_q^{\mathrm{Top}}(S^1)$  is thus reduced to the homotopy theoretical problem of evaluating the equivariant homotopy groups  $\mathrm{TR}_q^n(\mathbb{S}, I; p)$  along with the maps listed above. In the paper [15] mentioned earlier, the authors approximate the Hurewicz map  $\ell: \mathbb{S} \rightarrow \mathbb{Z}$  by a map of suspension spectra  $\theta: \mathbb{S}[SG] \rightarrow \mathbb{S}$  and use the Segal-tom Dieck splitting to essentially evaluate the groups  $\mathrm{TR}_q^n(\mathbb{S}, I; p)$ , for  $p$  odd and  $q \leq 4p - 7$ . However, this approach is not available, for  $q > 4p - 7$ , where a genuine understanding of the domain and target of the map

$$\mathrm{TR}_q^n(\mathbb{S}; p) \rightarrow \mathrm{TR}_q^n(\mathbb{Z}; p)$$

appears necessary. We evaluate  $\mathrm{TR}_q^n(\mathbb{S}, I; 2)$ , for  $q \leq 3$ , and we partly evaluate the four maps listed above. The result, which is Theorem 25, is the main result of the paper, and the proof occupies Sects. 5–7. The homotopy theoretical methods we employ here are perhaps somewhat simple-minded and more sophisticated methods will certainly make it possible to evaluate the groups  $\mathrm{TR}_q^n(\mathbb{S}, I; p)$  in a wider range of degrees. In particular, it would be very interesting to understand the corresponding homology groups. However, to evaluate the groups  $\mathrm{TR}_q^n(\mathbb{S}, I; p)$  is at least as difficult as to evaluate the stable homotopy groups of spheres. In the Section 8, we apply the fundamental theorem to the result of Theorem 25 and prove Theorem 1.

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## 1 The Groups $\text{TR}_q^n(A; p)$

Let  $A$  be a symmetric ring spectrum [20, Section 5.3]. The topological Hochschild  $\mathbb{T}$ -spectrum  $T(A)$  is a cyclotomic spectrum in the sense of [17, Definition 2.2]. In particular, it is an object of the  $\mathbb{T}$ -stable homotopy category. Let  $C_r \subset \mathbb{T}$  be the subgroup of order  $r$ , and let  $(\mathbb{T}/C_r)_+$  be the suspension  $\mathbb{T}$ -spectrum of the union of  $\mathbb{T}/C_r$  and a disjoint basepoint. One defines the equivariant homotopy group

$$\text{TR}_q^n(A; p) = [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(A)]_{\mathbb{T}}.$$

to be the abelian group of maps in the  $\mathbb{T}$ -stable homotopy category between the indicated  $\mathbb{T}$ -spectra. The Frobenius map, Verschiebung map, and Connes' operator, which we mentioned in the Introduction, are induced by maps

$$\begin{aligned} f: (\mathbb{T}/C_{p^{n-2}})_+ &\rightarrow (\mathbb{T}/C_{p^{n-1}})_+ \\ v: (\mathbb{T}/C_{p^{n-1}})_+ &\rightarrow (\mathbb{T}/C_{p^{n-2}})_+ \\ \delta: \Sigma(\mathbb{T}/C_{p^{n-1}})_+ &\rightarrow (\mathbb{T}/C_{p^{n-1}})_+ \end{aligned}$$

in the  $\mathbb{T}$ -stable homotopy category defined as follows. The map  $f$  is the map of suspension  $\mathbb{T}$ -spectra induced by the canonical projection  $\text{pr}: \mathbb{T}/C_{p^{n-2}} \rightarrow \mathbb{T}/C_{p^{n-1}}$ , and the map  $v$  is the corresponding transfer map. To define the latter, we choose an embedding  $\iota: \mathbb{T}/C_{p^{n-2}} \hookrightarrow \lambda$  into a finite dimensional orthogonal  $\mathbb{T}$ -presentation. The product embedding  $(\iota, \text{pr}): \mathbb{T}/C_{p^{n-2}} \rightarrow \lambda \times \mathbb{T}/C_{p^{n-1}}$  has trivial normal bundle, and the linear structure of  $\lambda$  determines a preferred trivialization. Hence, the Pontryagin–Thom construction gives a map of pointed  $\mathbb{T}$ -spaces

$$S^\lambda \wedge (\mathbb{T}/C_{p^{n-1}})_+ \rightarrow S^\lambda \wedge (\mathbb{T}/C_{p^{n-2}})_+$$

and  $v$  is the induced map of suspension  $\mathbb{T}$ -spectra. Finally, there is a unique homotopy class of maps of pointed spaces  $\delta'': S^1 \rightarrow (\mathbb{T}/C_{p^{n-1}})_+$  such that image by the Hurewicz map is the fundamental class  $[\mathbb{T}/C_{p^{n-1}}]$  corresponding to the counter-clockwise orientation of the circle  $\mathbb{T} \subset \mathbb{C}$  and such that the composite of  $\delta''$  and the map  $(\mathbb{T}/C_{p^{n-1}})_+ \rightarrow S^0$  that collapses  $\mathbb{T}/C_{p^{n-1}}$  to the non-base point of  $S^0$  is the null-map. The map  $\delta''$  induces the map of suspension  $\mathbb{T}/C_{p^{n-1}}$ -spectra

$$\delta': \Sigma(\mathbb{T}/C_{p^{n-1}})_+ \rightarrow (\mathbb{T}/C_{p^{n-1}})_+$$

which, in turn, induces the map  $\delta$ .

The definition of the restriction map is more delicate. We let  $E$  be the unit sphere in  $\mathbb{C}^\infty$  and consider the cofibration sequence of pointed  $\mathbb{T}$ -spaces

$$E_+ \rightarrow S^0 \rightarrow \tilde{E} \rightarrow \Sigma E_+$$

where the left-hand map collapses  $E$  onto the non-base point of  $S^0$ ; the  $\mathbb{T}$ -space  $\tilde{E}$  is canonically homeomorphic to the one-point compactification of  $\mathbb{C}^\infty$ . It induces a cofibration sequence of  $\mathbb{T}$ -spectra

$$E_+ \wedge T(A) \rightarrow T(A) \rightarrow \tilde{E} \wedge T(A) \rightarrow \Sigma E_+ \wedge T(A),$$

and hence, a long-exact sequence of equivariant homotopy groups. By [17, Theorem 2.2], the latter sequence is canonically isomorphic to the sequence

$$\cdots \rightarrow \mathbb{H}_q(C_{p^{n-1}}, T(A)) \xrightarrow{N} \mathrm{TR}_q^n(A; p) \xrightarrow{R} \mathrm{TR}_q^{n-1}(A; p) \rightarrow \cdots$$

which is called the fundamental long-exact sequence. The left-hand term is the group homology of  $C_{p^{n-1}}$  with coefficients in the underlying  $C_{p^{n-1}}$ -spectrum of  $T(A)$  and is defined to be the equivariant homotopy group

$$\mathbb{H}_q(C_{p^{n-1}}, T(A)) = [S^q, E_+ \wedge T(A)]_{C_{p^{n-1}}}.$$

The isomorphism of the left-hand terms in the two sequences is given by the canonical change-of-groups isomorphism

$$[S^q, E_+ \wedge T(A)]_{C_{p^{n-1}}} \xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, E_+ \wedge T(A)]_{\mathbb{T}}$$

and the resulting map  $N$  in the fundamental long-exact sequence is called the norm map. The isomorphism of the right-hand terms in the two sequences involves the cyclotomic structure of the spectrum  $T(A)$  as we now explain. The  $C_p$ -fixed points of the  $\mathbb{T}$ -spectrum  $T(A)$  is a  $\mathbb{T}/C_p$ -spectrum  $T(A)^{C_p}$ . Moreover, the isomorphism

$$\rho_p: \mathbb{T} \rightarrow \mathbb{T}/C_p$$

given by the  $p$ th root induces an equivalence of categories that to the  $\mathbb{T}/C_p$ -spectrum  $D$  associates the  $\mathbb{T}$ -spectrum  $\rho_p^* D$ . Then the additional cyclotomic structure of the topological Hochschild  $\mathbb{T}$ -spectrum  $T(A)$  consists of a map of  $\mathbb{T}$ -spectra

$$r: \rho_p^*((\tilde{E} \wedge T(A))^{C_p}) \rightarrow T(A)$$

with the property that the induced map of equivariant homotopy groups

$$[S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, \rho_p^*((\tilde{E} \wedge T(A))^{C_p})]_{\mathbb{T}} \rightarrow [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(A)]_{\mathbb{T}}$$

is an isomorphism for all positive integers  $n$ . The right-hand sides of the two sequences above are now identified by the composition

$$\begin{aligned}
 [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, \tilde{E} \wedge T(A)]_{\mathbb{T}} &\xleftarrow{\sim} [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, (\tilde{E} \wedge T(A))^{C_p}]_{\mathbb{T}/C_p} \\
 &\xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_{p^{n-2}})_+, \rho_p^*((\tilde{E} \wedge T(A))^{C_p})]_{\mathbb{T}} \xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_{p^{n-2}})_+, T(A)]_{\mathbb{T}}
 \end{aligned}$$

of the canonical isomorphism, the isomorphism  $\rho_p^*$ , and the isomorphism induced by the map  $r$ . By definition, the restriction map is the resulting map  $R$  in the fundamental long-exact sequence. Since  $r$  is a map of  $\mathbb{T}$ -spectra, the restriction map commutes with the Frobenius map, the Verschiebung map, and Connes' operator.

We mention that, if the symmetric ring spectrum  $A$  is commutative, then  $T(A)$  has the structure of a commutative ring  $\mathbb{T}$ -spectrum which, in turn, gives the graded abelian group  $\mathrm{TR}_*^n(A; p)$  the structure of an anti-symmetric graded ring, for all  $n \geq 1$ . The restriction and Frobenius maps are both ring homomorphisms, the Frobenius and Verschiebung maps satisfy the projection formula

$$xV(y) = V(F(x)y),$$

and Connes' operator is a derivation with respect to the product.

In general, the restriction map does not admit a section. However, if  $A = \mathbb{S}$  is the sphere spectrum, there exists a map

$$s: T(\mathbb{S}) \rightarrow \rho_p^*(T(\mathbb{S})^{C_p})$$

in the  $\mathbb{T}$ -stable homotopy category such that the composition

$$T(\mathbb{S}) \xrightarrow{s} \rho_p^*(T(\mathbb{S})^{C_p}) \rightarrow \rho_p^*((\tilde{E} \wedge T(\mathbb{S}))^{C_p}) \xrightarrow{r} T(\mathbb{S})$$

is the identity map [25, Corollary 4.4.8]. The map  $s$  induces a section

$$S = S_n: \mathrm{TR}_q^{n-1}(\mathbb{S}; p) \rightarrow \mathrm{TR}_q^n(\mathbb{S}; p) \quad (\text{Segal-tom Dieck splitting})$$

of the restriction map. The section  $S$  is a ring homomorphism and commutes with the Verschiebung map and Connes' operator. The composition  $FS_n$  is equal to  $S_{n-1}F$ , for  $n \geq 3$ , and to the identity map, for  $n = 2$ . It follows that, for every symmetric ring spectrum  $A$ , the graded abelian group  $\mathrm{TR}_*^n(A; p)$  is a graded module over the graded ring  $\mathrm{TR}_*^1(\mathbb{S}; p)$  which is canonically isomorphic to the graded ring given by the stable homotopy groups of spheres. It is proved in [16, Section 1] that Connes' operator satisfies the following additional relations

$$\begin{aligned}
 FdV &= d + (p - 1)\eta, \\
 dd &= d\eta = \eta d,
 \end{aligned}$$

where  $\eta$  indicates multiplication by the Hopf class  $\eta \in \mathrm{TR}_1^1(\mathbb{S}; p)$ . It follows from the above that  $FV = p$ ,  $dF = pFd$ , and  $Vd = pdV$ .

The zeroth space  $A_0$  of the symmetric spectrum  $A$  is a pointed monoid which is commutative if  $A$  is commutative. There is a canonical map

$$[-]_n: \pi_0(A_0) \rightarrow \mathrm{TR}_0^n(A; p) \quad (\text{Teichmüller map}),$$

which satisfies  $R([a]_n) = [a]_{n-1}$  and  $F([a]_n) = [a^p]_{n-1}$ ; see [19, Section 2.5]. If  $A$  is commutative, the Teichmüller map is multiplicative and satisfies

$$Fd([a]_n) = [a]_{n-1}^{p-1} d([a]_{n-1}).$$

## 2 The Fundamental Theorem

Let  $A$  be a symmetric ring spectrum, and let  $\Gamma$  be the free group on a generator  $x$ . We define the symmetric ring spectrum  $A[x^{\pm 1}]$  to be the symmetric spectrum

$$A[x^{\pm 1}] = A \wedge \Gamma_+$$

with the multiplication map given by the composition of the canonical isomorphism from  $A \wedge \Gamma_+ \wedge A \wedge \Gamma_+$  to  $A \wedge A \wedge \Gamma_+ \wedge \Gamma_+$  that permutes the second and third smash factors and the smash product  $\mu_A \wedge \mu_\Gamma$  of the multiplication maps of  $A$  and  $\Gamma$  and with the unit map given by the composition of the canonical isomorphism from  $\mathbb{S}$  to  $\mathbb{S} \wedge S^0$  and the smash product  $e_A \wedge e_\Gamma$  of the unit maps of  $A$  and  $\Gamma$ . There is a natural map of symmetric ring spectra  $f: A \rightarrow A[x^{\pm 1}]$  defined to be the composition of the canonical isomorphism from  $A$  to  $A \wedge S^0$  and the smash product  $\mathrm{id}_A \wedge e_\Gamma$  of the identity map of  $A$  and the unit map of  $\Gamma$ . It induces a natural map

$$f_*: \mathrm{TR}_q^n(A; p) \rightarrow \mathrm{TR}_q^n(A[x^{\pm 1}]; p).$$

Moreover, there is a map of symmetric ring spectra  $g: \mathbb{S}[x^{\pm 1}] \rightarrow A[x^{\pm 1}]$  defined to be the smash product  $e_A \wedge \mathrm{id}_\Gamma$  of the unit map of  $A$  and the identity map of  $\Gamma$ . The map  $g$  makes  $A[x^{\pm 1}]$  into an algebra spectrum over the commutative symmetric ring spectrum  $\mathbb{S}[x^{\pm 1}]$ . It follows that there is a natural pairing

$$v: \mathrm{TR}_q^n(A[x^{\pm 1}]; p) \otimes \mathrm{TR}_{q'}^n(\mathbb{S}[x^{\pm 1}]; p) \rightarrow \mathrm{TR}_{q+q'}^n(A[x^{\pm 1}]; p)$$

which makes the graded abelian group  $\mathrm{TR}_*^n(A[x^{\pm 1}]; p)$  a graded module over the anti-symmetric graded ring  $\mathrm{TR}_*^n(\mathbb{S}[x^{\pm 1}]; p)$ . The element  $[x]_n \in \mathrm{TR}_0^n(\mathbb{S}[x^{\pm 1}]; p)$  is a unit with inverse  $[x]_n^{-1} = [x^{-1}]_n$  and we define

$$d \log[x]_n = [x]_n^{-1} d[x]_n \in \mathrm{TR}_1^n(\mathbb{S}[x^{\pm 1}]; p).$$



It follows from the general relations that

$$F(d \log[x]_n) = R(d \log[x]_n) = d \log[x]_{n-1}.$$

Now, given an integer  $j$  and element  $a \in \text{TR}_q^n(A; p)$ , we define

$$\begin{aligned} a[x]_n^j &= v(f_*(a) \otimes [x]_n^j) \in \text{TR}_q^n(A[x^{\pm 1}]; p) \\ a[x]_n^j d \log[x]_n &= v(f_*(a) \otimes [x]_n^j d \log[x]_n) \in \text{TR}_{q+1}^n(A[x^{\pm 1}]; p). \end{aligned}$$

The following theorem, which is similar to the fundamental theorem of algebraic  $K$ -theory, was proved by Ib Madsen and the author in [19, Theorem C]. The assumption in loc. cit. that the prime  $p$  be odd is unnecessary; the same proof works for  $p = 2$ . However, the formulas for  $F$ ,  $V$ , and  $d$  given in loc. cit. are valid for odd primes only. Below, we give a formula for the Frobenius which holds for all primes  $p$ .

**Theorem 2.** *Let  $p$  be a prime number, and let  $A$  be a symmetric ring spectrum whose homotopy groups are  $\mathbb{Z}_{(p)}$ -modules. Then every element  $\omega \in \text{TR}_q^n(A[x^{\pm 1}]; p)$  can be written uniquely as a (finite) sum*

$$\sum_{j \in \mathbb{Z}} (a_{0,j} [x]_n^j + b_{0,j} [x]_n^j d \log[x]_n) + \sum_{\substack{1 \leq s < n \\ j \in \mathbb{Z} \setminus p\mathbb{Z}}} (V^s(a_{s,j} [x]_{n-s}^j) + dV^s(b_{s,j} [x]_{n-s}^j))$$

with  $a_{s,j} = a_{s,j}(\omega) \in \text{TR}_q^{n-s}(A; p)$  and  $b_{s,j} = b_{s,j}(\omega) \in \text{TR}_{q-1}^{n-s}(A; p)$ . The corresponding statement for the equivariant homotopy groups with  $\mathbb{Z}_p$ -coefficients is valid for every symmetric ring spectrum  $A$ .

It is perhaps helpful to point out that the formula in the statement of Theorem 2 defines a canonical map from the direct sum

$$\bigoplus_{j \in \mathbb{Z}} (\text{TR}_q^n(A; p) \oplus \text{TR}_{q-1}^n(A; p)) \oplus \bigoplus_{\substack{1 \leq s < n \\ j \in \mathbb{Z} \setminus p\mathbb{Z}}} (\text{TR}_q^{n-s}(A; p) \oplus \text{TR}_{q-1}^{n-s}(A; p))$$

to the group  $\text{TR}_q^n(A[x^{\pm 1}]; p)$  and that the theorem states that this map is an isomorphism. We also remark that the assembly map

$$\alpha: \text{TR}_q^n(A; p) \oplus \text{TR}_{q-1}^n(A; p) \rightarrow \text{TR}_q^n(A[x^{\pm 1}]; p)$$

is given by the formula

$$\alpha(a, b) = a[x]_n^0 + b[x]_n^0 d \log[x]_n,$$

where  $[x]_n^0 = [1]_n \in \text{TR}_0^n(\mathbb{S}[x^{\pm 1}]; p)$  is the multiplicative unit element.

The value of the restriction and Frobenius maps on  $\text{TR}_q^n(A[x^{\pm 1}]; p)$  are readily derived from the general relations. Indeed, if  $\omega \in \text{TR}_q^n(A[x^{\pm 1}]; p)$  is equal to the sum in the statement of Theorem 2, then

$$\begin{aligned}
R(\omega) &= \sum_{j \in \mathbb{Z}} (R(a_{0,j})[x]_{n-1}^j + R(b_{0,j})[x]_{n-1}^j d \log[x]_{n-1}) \\
&\quad + \sum_{\substack{1 \leq s < n-1 \\ j \in \mathbb{Z} \setminus p\mathbb{Z}}} (V^s(R(a_{s,j})[x]_{n-1-s}^j) + dV^s(R(b_{s,j})[x]_{n-1-s}^j))
\end{aligned}$$

$$\begin{aligned}
F(\omega) &= \sum_{j \in p\mathbb{Z}} (F(a_{0,j/p})[x]_{n-1}^j + F(b_{0,j/p})[x]_{n-1}^j d \log[x]_{n-1}) \\
&\quad + \sum_{j \in \mathbb{Z} \setminus p\mathbb{Z}} ((pa_{1,j} + db_{1,j} + (p-1)\eta b_{1,j})[x]_{n-1}^j \\
&\quad\quad\quad + (-1)^{q-1} j b_{1,j} [x]_{n-1}^j d \log[x]_{n-1}) \\
&\quad + \sum_{\substack{1 \leq s < n-1 \\ j \in \mathbb{Z} \setminus p\mathbb{Z}}} (V^s((pa_{s+1,j} + (p-1)\eta b_{s+1,j})[x]_{n-1-s}^j) \\
&\quad\quad\quad + dV^s(b_{s+1,j}[x]_{n-1-s}^j)).
\end{aligned}$$

We leave it to the reader to derive the corresponding formulas for the Verschiebung map and Connes' operator. The following result is an immediate consequence.

We recall that the limit system  $\{M_n\}$  satisfies the Mittag-Leffler condition if, for every  $n$ , there exists  $m \geq n$  such that, for all  $k \geq m$ , the image of  $M_k \rightarrow M_n$  is equal to the image of  $M_m \rightarrow M_n$ . This implies that the derived limit  $R^1 \lim_n M_n$  vanishes.

**Corollary 3.** *Let  $p$  be a prime number, let  $A$  be a symmetric ring spectrum whose homotopy groups are  $\mathbb{Z}_{(p)}$ -modules, and let  $q$  be an integer. If both of the limit systems  $\{\mathrm{TR}_q^n(A; p)\}$  and  $\{\mathrm{TR}_{q-1}^n(A; p)\}$  satisfy the Mittag-Leffler condition, then so does the limit system  $\{\mathrm{TR}_q^n(A[x^{\pm 1}]; p)\}$ . Moreover, the element*

$$\omega = (\omega^{(n)}) \in \lim_R \mathrm{TR}_q^n(A[x^{\pm 1}]; p)$$

lies in the kernel of the map  $1 - F$  if and only if the coefficients

$$\begin{aligned}
a_{s,j}^{(n)} &= a_{s,j}(\omega^{(n)}) \in \mathrm{TR}_q^{n-s}(A; p) \\
b_{s,j}^{(n)} &= b_{s,j}(\omega^{(n)}) \in \mathrm{TR}_{q-1}^{n-s}(A; p)
\end{aligned}$$

satisfy the equations

$$a_{s,j}^{(n-1)} = \begin{cases} F(a_{0,j/p}^{(n)}) & (s = 0 \text{ and } j \in p\mathbb{Z}) \\ pa_{1,j}^{(n)} + db_{1,j}^{(n)} + (p-1)\eta b_{1,j}^{(n)} & (s = 0 \text{ and } j \in \mathbb{Z} \setminus p\mathbb{Z}) \\ pa_{s+1,j}^{(n)} + (p-1)\eta b_{s+1,j}^{(n)} & (1 \leq s < n-1 \text{ and } j \in \mathbb{Z} \setminus p\mathbb{Z}) \end{cases}$$

$$b_{s,j}^{(n-1)} = \begin{cases} F(b_{0,j/p}^{(n)}) & (s = 0 \text{ and } j \in p\mathbb{Z}) \\ (-1)^{q-1} j b_{1,j}^{(n)} & (s = 0 \text{ and } j \in \mathbb{Z} \setminus p\mathbb{Z}) \\ b_{s+1,j}^{(n)} & (1 \leq s < n-1 \text{ and } j \in \mathbb{Z} \setminus p\mathbb{Z}) \end{cases}$$

for all  $n \geq 1$ . The corresponding statements for the equivariant homotopy groups with  $\mathbb{Z}_p$ -coefficients is valid for every symmetric ring spectrum  $A$ .

We do not have a good description of the cokernel of  $1 - F$ . In particular, it is generally not easy to decide whether or not this map is surjective.

### 3 Topological Cyclic Homology

Let  $A$  be a symmetric ring spectrum. We recall the definition of the topological cyclic homology groups  $\mathrm{TC}_q(A; p)$  and refer to [17, 18] for a full discussion.

We consider the  $\mathbb{T}$ -fixed point spectrum

$$\mathrm{TR}^n(A; p) = F((\mathbb{T}/C_{p^{n-1}})_+, T(A))^{\mathbb{T}}$$

of the function  $\mathbb{T}$ -spectrum  $F((\mathbb{T}/C_{p^{n-1}})_+, T(A))$ . There is a canonical isomorphism

$$\iota: \pi_q(\mathrm{TR}^n(A; p)) \xrightarrow{\sim} \mathrm{TR}_q^n(A; p)$$

and maps of spectra

$$R^!, F^!: \mathrm{TR}^n(A; p) \rightarrow \mathrm{TR}^{n-1}(A; p)$$

such that the following diagrams commute

$$\begin{array}{ccc} \pi_q(\mathrm{TR}^n(A; p)) & \xrightarrow{\iota} & \mathrm{TR}_q^n(A; p) & & \pi_q(\mathrm{TR}^n(A; p)) & \xrightarrow{\iota} & \mathrm{TR}_q^n(A; p) \\ \downarrow R_*^! & & \downarrow R & & \downarrow F_*^! & & \downarrow F \\ \pi_q(\mathrm{TR}^{n-1}(A; p)) & \xrightarrow{\iota} & \mathrm{TR}_q^{n-1}(A; p) & & \pi_q(\mathrm{TR}^{n-1}(A; p)) & \xrightarrow{\iota} & \mathrm{TR}_q^{n-1}(A; p). \end{array}$$

The map  $F^!$  is induced by the map of  $\mathbb{T}$ -spectra  $f: (\mathbb{T}/C_{p^{n-2}})_+ \rightarrow (\mathbb{T}/C_{p^{n-1}})_+$  and the map  $R^!$  is defined to be the composition of the map

$$F((\mathbb{T}/C_{p^{n-1}})_+, T(A))^{\mathbb{T}} \rightarrow F((\mathbb{T}/C_{p^{n-1}})_+, \tilde{E} \wedge T(A))^{\mathbb{T}}$$

induced by the canonical inclusion of  $S^0$  in  $\tilde{E}$  and the weak equivalence

$$\begin{aligned} F((\mathbb{T}/C_{p^{n-1}})_+, \tilde{E} \wedge T(A))^{\mathbb{T}} &\xleftarrow{\sim} F((\mathbb{T}/C_{p^{n-1}})_+, (\tilde{E} \wedge T(A))^{C_p})^{\mathbb{T}/C_p} \\ &\xrightarrow{\sim} F((\mathbb{T}/C_{p^{n-2}})_+, \rho_p^*((\tilde{E} \wedge T(A))^{C_p}))^{\mathbb{T}} \xrightarrow{\sim} F((\mathbb{T}/C_{p^{n-2}})_+, T(A))^{\mathbb{T}} \end{aligned}$$

defined by the composition of the canonical isomorphism, the isomorphism  $\rho_p^*$ , and the map induced by the map  $r$  which we recalled in Section 1. We then define  $\mathrm{TC}^n(A; p)$  to be the homotopy equalizer of the maps  $R^l$  and  $F^l$  and

$$\mathrm{TC}(A; p) = \mathrm{holim}_n \mathrm{TC}^n(A; p)$$

to be the homotopy limit with respect to the maps  $R^l$ . We also define

$$\mathrm{TR}(A; p) = \mathrm{holim}_n \mathrm{TR}^n(A; p)$$

to be the homotopy limit with respect to the maps  $R^l$  such that we have a long-exact sequence of homotopy groups

$$\cdots \rightarrow \mathrm{TC}_q(A; p) \rightarrow \mathrm{TR}_q(A; p) \xrightarrow{1-F} \mathrm{TR}_q(A; p) \xrightarrow{\partial} \mathrm{TC}_{q-1}(A; p) \rightarrow \cdots$$

We recall Milnor’s short-exact sequence

$$0 \rightarrow R^1 \lim_n \mathrm{TR}_{q+1}^n(A; p) \rightarrow \mathrm{TR}_q(A; p) \rightarrow \lim_n \mathrm{TR}_q^n(A; p) \rightarrow 0.$$

In the cases we consider below, the derived limit on the left-hand side vanishes.

The cyclotomic trace map of Bökstedt–Hsiang–Madsen [4] is a map of spectra

$$\mathrm{tr}: K(A) \rightarrow \mathrm{TC}(A; p).$$

A technically better definition of the cyclotomic trace map was given by Dundas–McCarthy [10, Section 2.0] and [9]. From the latter definition it is clear that every class  $x$  in the image of the composite map

$$K_q(A) \xrightarrow{\mathrm{tr}} \mathrm{TC}_q(A; p) \rightarrow \mathrm{TC}_q^n(A; p) \rightarrow \mathrm{TR}_q^n(A; p)$$

is annihilated by Connes’ operator. It is also not difficult to show that, for  $A$  commutative, the cyclotomic trace is multiplicative; see [12, Appendix].

The spectrum  $\mathrm{TR}^n(A; p)$  considered here is canonically isomorphic to the underlying non-equivariant spectrum associated with the  $\mathbb{T}$ -spectrum

$$\mathrm{TR}^n(A; p) = \rho_{p^{n-1}}^*(T(A)^{C_{p^{n-1}}}).$$

Moreover, the fundamental long-exact sequence of [17, Theorem 2.2] has the following generalization which is used in the proof of Lemma 26.

**Proposition 4.** *Let  $A$  be a symmetric ring spectrum, and let  $m$  and  $n$  be positive integers. Then there is a natural long-exact sequence*

$$\cdots \rightarrow \mathbb{H}_q(C_{p^m}, \mathrm{TR}^n(A; p)) \xrightarrow{N_n} \mathrm{TR}_q^{m+n}(A; p) \xrightarrow{R^n} \mathrm{TR}_q^m(A; p) \rightarrow \cdots$$

where the left-hand term is the group homology of  $C_{p^m}$  with coefficients in the underlying  $C_{p^m}$ -spectrum of  $TR^n(A; p)$ .

*Proof.* A map of  $\mathbb{T}$ -spectra  $f: T \rightarrow T'$  is defined to be an  $\mathcal{F}_p$ -equivalence if it induces an isomorphism of equivariant homotopy groups

$$f_*: [S^q \wedge (\mathbb{T}/C_{p^v})_+, T]_{\mathbb{T}} \rightarrow [S^q \wedge (\mathbb{T}/C_{p^v})_+, T']_{\mathbb{T}}$$

for all integers  $q$  and  $v \geq 0$ . The cofibration sequence of pointed  $\mathbb{T}$ -spaces

$$E_+ \xrightarrow{\pi} S^0 \xrightarrow{\iota} \tilde{E} \xrightarrow{\partial} \Sigma E_+,$$

which we considered in Section 1, induces a cofibration sequence of  $\mathbb{T}$ -spectra

$$E_+ \wedge \rho_{p^s}^*(T(A)^{C_{p^s}}) \rightarrow \rho_{p^s}^*(T(A)^{C_{p^s}}) \rightarrow \tilde{E} \wedge \rho_{p^s}^*(T(A)^{C_{p^s}}) \rightarrow \Sigma E_+ \wedge \rho_{p^s}^*(T(A)^{C_{p^s}}).$$

We show that with  $s = n - 1$ , the induced long-exact sequence of equivariant homotopy groups is isomorphic to the sequence of the statement. The isomorphism of the left-hand terms in the two sequences is defined as in Section 1. To define the isomorphism of the right-hand terms in the two sequences, we first show that the cyclotomic structure map  $r$  gives rise to an  $\mathcal{F}_p$ -equivalence

$$r': \tilde{E} \wedge \rho_{p^{n-1}}^*(T(A)^{C_{p^{n-1}}}) \xrightarrow{\sim} \tilde{E} \wedge T(A).$$

Since the map  $\pi: E_+ \rightarrow S^0$  induces a weak equivalence

$$E_+ \wedge \rho_{p^s}^*((E_+ \wedge T(A))^{C_{p^s}}) \xrightarrow{\sim} \rho_{p^s}^*((E_+ \wedge T(A))^{C_{p^s}}),$$

a diagram chase shows that the map  $\iota: S^0 \rightarrow \tilde{E}$  induces a weak equivalence

$$\tilde{E} \wedge \rho_{p^s}^*(T(A)^{C_{p^s}}) \xrightarrow{\sim} \tilde{E} \wedge \rho_{p^s}^*(\tilde{E} \wedge T(A)^{C_{p^s}}).$$

The cyclotomic structure map  $r$  induces an  $\mathcal{F}_p$ -equivalence

$$\tilde{E} \wedge \rho_{p^s}^*(\tilde{E} \wedge T(A)^{C_{p^s}}) \xrightarrow{\sim} \tilde{E} \wedge \rho_{p^{s-1}}^*(T(A)^{C_{p^{s-1}}}),$$

which, composed with the former equivalence, defines an  $\mathcal{F}_p$ -equivalence

$$\tilde{E} \wedge \rho_{p^s}^*(T(A)^{C_{p^s}}) \xrightarrow{\sim} \tilde{E} \wedge \rho_{p^{s-1}}^*(T(A)^{C_{p^{s-1}}}).$$

The composition of these  $\mathcal{F}_p$ -equivalence as  $s$  varies from  $n - 1$  to 1 gives the desired  $\mathcal{F}_p$ -equivalence  $r'$ . The isomorphism of the right-hand terms in the two sequences is now given by the composition of the isomorphism

$$[S^q \wedge (\mathbb{T}/C_{p^m})_+, \tilde{E} \wedge \rho_{p^{n-1}}^*(T(A)^{C_{p^{n-1}}})]_{\mathbb{T}} \xrightarrow{\sim} [S^q \wedge (\mathbb{T}/C_{p^m})_+, \tilde{E} \wedge T(A)]_{\mathbb{T}}$$

induced by the map  $r'$  and the isomorphism

$$[S^q \wedge (\mathbb{T}/C_{p^m})_+, \tilde{E} \wedge T(A)]_{\mathbb{T}} \xrightarrow{\sim} [S^q \wedge (T/C_{p^{m-1}})_+, T(A)]_{\mathbb{T}}$$

defined in Section 1. □

### 4 The Skeleton Spectral Sequence

The left-hand groups in Proposition 4 are the abutment of the strongly convergent skeleton spectral sequence which we now discuss in some detail. Let  $G$  be a finite group, and let  $T$  be a  $G$ -spectrum. Then we define

$$\mathbb{H}_q(G, T) = [S^q, E_+ \wedge T]_G,$$

where  $E$  is a free contractible  $G$ -CW-complex. The group  $\mathbb{H}_q(G, T)$  is well-defined up to canonical isomorphism. Indeed, if also  $E'$  is a free contractible  $G$ -CW-complex, then there is a unique homotopy class of  $G$ -maps  $u: E \rightarrow E'$ , and the induced map  $u_*: [S^q, E_+ \wedge T]_G \rightarrow [S^q, E'_+ \wedge T]_G$  is the canonical isomorphism. The skeleton filtration of the  $G$ -CW-complex  $E$  gives rise to a spectral sequence

$$E_{s,t}^2 = H_s(G; \pi_t(T)) \Rightarrow \mathbb{H}_{s+t}(G, T)$$

from the homology of the group  $G$  with coefficients in the  $G$ -module  $\pi_t(T)$ . We will need the precise identification of the  $E^2$ -term below. The augmented cellular complex of  $E$  is the augmented chain complex  $\epsilon: P \rightarrow \mathbb{Z}$  defined by

$$P_s = \tilde{H}_s(E_s/E_{s-1}; \mathbb{Z})$$

with the differential  $d$  induced by the map  $\partial$  in the cofibration sequence

$$E_{s-1}/E_{s-2} \rightarrow E_s/E_{s-2} \rightarrow E_s/E_{s-1} \xrightarrow{\partial} \Sigma E_{s-1}/E_{s-2}.$$

and with the augmentation given by  $\epsilon(x) = 1$ , for all  $x \in E_0$ . It is a resolution of the trivial  $G$ -module  $\mathbb{Z}$  by free  $\mathbb{Z}[G]$ -modules. We define

$$H_s(G, \pi_t(T)) = H_s((P \otimes \pi_t(T))^G, d \otimes \text{id}).$$

The  $E^1$ -term of the spectral sequence is defined by

$$E_{s,t}^1 = [S^{s+t}, (E_s/E_{s-1}) \wedge T]_G$$

with the  $d^1$ -differential induced by the boundary map  $\partial$  in the cofibration sequence above. The quotient  $E_s/E_{s-1}$  is homeomorphic to a wedge of  $s$ -spheres indexed by a set on which the groups  $G$  acts freely. Therefore, the Hurewicz homomorphism

$$\pi_s(E_s/E_{s-1}) \rightarrow \tilde{H}(E_s/E_{s-1}; \mathbb{Z}),$$

the exterior product map

$$\pi_s(E_s/E_{s-1}) \otimes \pi_t(T) \rightarrow \pi_{s+t}((E_s/E_{s-1}) \wedge T),$$

and the canonical map

$$[S^{s+t}, (E_s/E_{s-1}) \wedge T]_G \rightarrow (\pi_{s+t}((E_s/E_{s-1}) \wedge T))^G$$

are all isomorphisms. These isomorphisms gives rise to a canonical isomorphism

$$h: (P_s \otimes \pi_t(T))^G \xrightarrow{\sim} E_{s,t}^1,$$

which satisfies  $h \circ (d \otimes \text{id}) = d^1 \circ h$ . The induced isomorphism of homology groups is then the desired identification of the  $E^2$ -term.

We consider the skeleton spectral sequence with  $G = C_{p^{n-1}}$  and  $T = TR^v(A; p)$ . Since the action by  $C_{p^{n-1}}$  on  $TR^v(A; p)$  is the restriction of an action by the circle group  $\mathbb{T}$ , the induced action on the homotopy groups  $TR_t^v(A; p)$  is trivial. Moreover, it follows from [16, Lemma 1.4.2] that the  $d^2$ -differential of the spectral sequence is related to Connes' operator  $d$  in the following way.

**Lemma 5.** *Let  $A$  be a symmetric ring spectrum. Then, in the spectral sequence*

$$E_{s,t}^2 = H_s(C_{p^{n-1}}, TR_t^v(A; p)) \Rightarrow \mathbb{H}_{s+t}(C_{p^{n-1}}, TR^v(A; p)),$$

*the  $d^2$ -differential  $d^2: E_{s,t}^2 \rightarrow E_{s-2,t+1}^2$  is equal to the map of group homology groups induced by  $d + \eta$ , if  $s$  is congruent to 0 or 1 modulo 4, and the map induced by  $d$ , if  $s$  is congruent to 2 or 3 modulo 4.*

The Frobenius and Verschiebung maps

$$\begin{aligned} F: \mathbb{H}_q(C_{p^{n-1}}, TR^v(A; p)) &\rightarrow \mathbb{H}_q(C_{p^{n-2}}, TR^v(A; p)) \\ V: \mathbb{H}_q(C_{p^{n-2}}, TR^v(A; p)) &\rightarrow \mathbb{H}_q(C_{p^{n-1}}, TR^v(A; p)) \end{aligned}$$

induce maps of spectral sequences which on the  $E^2$ -terms of the corresponding skeleton spectral sequence are given by the transfer and corestriction maps in group homology corresponding to the inclusion of  $C_{p^{n-2}}$  in  $C_{p^{n-1}}$ .

Let  $g \in C_{p^{n-1}}$  be the generator  $g = \exp(2\pi i/p^{n-1})$ , and let  $\epsilon: W \rightarrow \mathbb{Z}$  be the standard resolution which in degree  $s$  is a free  $\mathbb{Z}[C_{p^{n-1}}]$ -module of rank one on a

generator  $x_s$  with differential  $dx_s = Nx_{s-1}$ , for  $s$  even, and  $dx_s = (g-1)x_{s-1}$ , for  $s$  odd, and with augmentation  $\epsilon(x_0) = 1$ .

**Lemma 6.** *Let  $r$  and  $n$  be positive integers, and let  $p$  be a prime number.*

(1) *If  $r \leq n-1$ , then*

$$H_s(C_{p^{n-1}}, \mathbb{Z}/p^r\mathbb{Z}) = \mathbb{Z}/p^r\mathbb{Z} \cdot z_s,$$

where  $z_s = z_s(p, n, r)$  is the class of  $Nx_s \otimes 1$ .

(2) *If  $r \geq n-1$ , then*

$$H_s(C_{p^{n-1}}, \mathbb{Z}/p^r\mathbb{Z}) = \begin{cases} \mathbb{Z}/p^r\mathbb{Z} \cdot z_0 & (s = 0) \\ \mathbb{Z}/p^{n-1}\mathbb{Z} \cdot z_s & (s \text{ odd}) \\ \mathbb{Z}/p^{n-1}\mathbb{Z} \cdot p^{r-(n-1)}z_s & (s > 0 \text{ and even}) \end{cases}$$

where  $z_s = z_s(p, n, r)$  and  $p^{r-(n-1)}z_s = p^{r-(n-1)}z_s(p, n, r)$  are the classes of  $Nx_s \otimes 1$  and  $p^{r-(n-1)}Nx_s \otimes 1$ , respectively.

(3) *The transfer map*

$$F: H_s(C_{p^{n-1}}, \mathbb{Z}/p^r\mathbb{Z}) \rightarrow H_s(C_{p^{n-2}}, \mathbb{Z}/p^r\mathbb{Z})$$

maps  $z_s$  to  $z_s$ , if  $s$  is odd, maps  $z_s$  to  $pz_s$ , if  $s = 0$  or if  $s > 0$  is even and  $r \leq n-1$ , and maps  $p^{r-(n-1)}z_s$  to  $p^{r-(n-2)}z_s$ , if  $s > 0$  is even and  $r \geq n-1$ .

(4) *The corestriction map*

$$V: H_s(C_{p^{n-2}}, \mathbb{Z}/p^r\mathbb{Z}) \rightarrow H_s(C_{p^{n-1}}, \mathbb{Z}/p^r\mathbb{Z})$$

maps  $z_s$  to  $pz_s$ , if  $s$  is odd, maps  $z_s$  to  $z_s$ , if  $s = 0$  or if  $s > 0$  is even and  $r \leq n-1$ , and maps  $p^{r-(n-1)}z_s$  to  $p^{r-(n-1)}z_s$ , if  $s > 0$  is even and  $r \geq n-1$ .

*Proof.* The statements (1) and (2) are readily verified. To prove (3) and (4), we write  $\epsilon: W \rightarrow \mathbb{Z}$  and  $\epsilon': W' \rightarrow \mathbb{Z}$  for the standard resolutions for the groups  $C_{p^{n-1}}$  and  $C_{p^{n-2}}$ , respectively. Then  $\epsilon: W \rightarrow \mathbb{Z}$  is a resolution of  $\mathbb{Z}$  by free  $C_{p^{n-2}}$ -modules. The map  $h: W \rightarrow W'$  defined by

$$h(g^{dp+r}x_s) = \begin{cases} g'^r x'_s & (s \text{ even}) \\ \delta_{r,p-1} g'^d x'_s & (s \text{ odd}), \end{cases}$$

where  $0 \leq r < p$  and  $0 \leq d < n-2$ , is a  $C_{p^{n-2}}$ -linear chain map that lifts the identity map of  $\mathbb{Z}$ , and the  $C_{p^{n-2}}$ -linear map  $k: W' \rightarrow W$  defined by

$$k(x'_s) = \begin{cases} x_s & (s \text{ even}) \\ (1 + g + \dots + g^{p-1})x_s & (s \text{ odd}) \end{cases}$$



is a chain map and lifts the identity of  $\mathbb{Z}$ . Now the transfer map  $F$  is the map of homology groups induced by the composite chain map

$$(W \otimes \mathbb{Z}/p^r\mathbb{Z})^{C_{p^{n-1}}} \hookrightarrow (W \otimes \mathbb{Z}/p^r\mathbb{Z})^{C_{p^{n-2}}} \xrightarrow{h \otimes 1} (W' \otimes \mathbb{Z}/p^r\mathbb{Z})^{C_{p^{n-2}}},$$

where the left-hand map is the canonical inclusion. One verifies readily that this map takes  $Nx_{2i} \otimes 1$  to  $pN'x'_{2i} \otimes 1$  and  $Nx_{2i-1} \otimes 1$  to  $N'x'_{2i-1} \otimes 1$ . Similarly, the corestriction map  $V$  is the map of homology groups induced by the composite chain map

$$(W' \otimes \mathbb{Z}/p^r\mathbb{Z})^{C_{p^{n-2}}} \xrightarrow{k \otimes 1} (W \otimes \mathbb{Z}/p^r\mathbb{Z})^{C_{p^{n-2}}} \xrightarrow{N/N'} (W \otimes \mathbb{Z}/p^r\mathbb{Z})^{C_{p^{n-1}}},$$

where the right-hand map is multiplication by  $1 + g + \dots + g^{p-1}$ . This map takes  $N'x'_{2i} \otimes 1$  to  $Nx_{2i} \otimes 1$  and  $N'x'_{2i-1} \otimes 1$  to  $pNx_{2i-1} \otimes 1$ .  $\square$

**Lemma 7.** *Let  $n$  be a positive integer and let  $p$  be a prime number. Then*

$$H_s(C_{p^{n-1}}, \mathbb{Z}) = \begin{cases} \mathbb{Z} \cdot z_0 & (s = 0) \\ \mathbb{Z}/p^{n-1}\mathbb{Z} \cdot z_s & (s \text{ odd}) \\ 0 & (s > 0 \text{ even}) \end{cases},$$

where  $z_s = z_s(p, n)$  is the class of  $Nx_s \otimes 1$ . The transfer map

$$F: H_s(C_{p^{n-1}}, \mathbb{Z}) \rightarrow H_s(C_{p^{n-2}}, \mathbb{Z})$$

maps  $z_0$  to  $pz_0$  and  $z_s$  to  $z_s$ , for  $s > 0$ , and the corestriction map

$$V: H_s(C_{p^{n-2}}, \mathbb{Z}) \rightarrow H_s(C_{p^{n-1}}, \mathbb{Z})$$

maps  $z_0$  to  $z_0$  and  $z_s$  to  $pz_s$ , for  $s > 0$ .

*Proof.* The proof is similar to the proof of Lemma 6.  $\square$

## 5 The Groups $\text{TR}_q^n(\mathbb{S}; 2)$

In this section, we implicitly consider homotopy groups with  $\mathbb{Z}_2$ -coefficients. The groups  $\text{TR}_*^1(\mathbb{S}; 2)$  are the stable homotopy groups of spheres. The group  $\text{TR}_0^1(\mathbb{S}; 2)$  is isomorphic to  $\mathbb{Z}_2$  generated by the multiplicative unit element  $\iota = [1]_1$ ; the group  $\text{TR}_1^1(\mathbb{S}; 2)$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  generated by the Hopf class  $\eta$ ; the group  $\text{TR}_2^1(\mathbb{S}; 2)$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  generated by  $\eta^2$ ; the group  $\text{TR}_3^1(\mathbb{S}; 2)$  is isomorphic to  $\mathbb{Z}/8\mathbb{Z}$  generated by the Hopf class  $\nu$  and  $\eta^3 = 4\nu$ ; the groups  $\text{TR}_4^1(\mathbb{S}; 2)$  and  $\text{TR}_5^1(\mathbb{S}; 2)$  are zero; the group  $\text{TR}_6^1(\mathbb{S}; 2)$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  generated by  $\nu^2$ , and the group  $\text{TR}_7^1(\mathbb{S}; 2)$  is isomorphic to  $\mathbb{Z}/16\mathbb{Z}$  generated the Hopf class  $\sigma$ .

We consider the skeleton spectral sequence

$$E_{s,t}^2 = H_s(C_{2^n-1}, \text{TR}_t^1(\mathbb{S}; 2)) \Rightarrow \mathbb{H}_{s+t}(C_{2^n-1}, T(\mathbb{S})).$$

This sequence may be identified with the Atiyah–Hirzebruch spectral sequence that converges to the homotopy groups of the suspension spectrum of the pointed space  $(BC_{2^n-1})_+$  [14, Proposition 2.4]. Therefore, the edge-homomorphism onto the line  $s = 0$  has a retraction, and hence, the differentials  $d^r: E_{r,t}^r \rightarrow E_{0,t+r-1}^r$  are all zero.

Suppose first that  $n = 2$ . Then the  $E^2$ -term for  $s + t \leq 7$  is takes the form

$$\begin{array}{cccccccc} \mathbb{Z}/16\mathbb{Z} & & & & & & & \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\ 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & & \\ \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 4\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 4\mathbb{Z}/8\mathbb{Z} & & & \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \\ \mathbb{Z}_2 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z}, \end{array}$$

where  $s$  is the horizontal coordinate and  $t$  the vertical coordinate. The group  $E_{s,0}^2$  is generated by the class  $\iota z_s$ , the group  $E_{s,1}^2$  by the class  $\eta z_s$ , the group  $E_{s,2}^2$  by the class  $\eta^2 z_s$ , the group  $E_{s,3}^2$  with  $s = 0$  or  $s$  an odd positive integer by the class  $\nu z_s$ , the group  $E_{s,3}^2$  with  $s$  an even positive integer by the class  $4\nu z_s$ , the group  $E_{s,6}^2$  by the class  $\nu^2 z_s$ , the group  $E_{s,7}^2$  with  $s = 0$  or  $s$  an odd positive integer by the class  $\sigma z_s$ , and the group  $E_{s,7}^2$  with  $s$  an even positive integer by the class  $8\sigma z_s$ , where  $z_s$  are the classes defined in Lemmas 6 and 7. We recall from Lemma 5 that the  $d^2$ -differential is given by Connes’ operator and by multiplication by  $\eta$ . Since Connes’ operator on  $\text{TR}_*^1(\mathbb{S}; 2)$  is zero, we find that the  $E^3$ -term begins

$$\begin{array}{cccccccc} \mathbb{Z}/16\mathbb{Z} & & & & & & & \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\ 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & & \\ \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 4\mathbb{Z}/8\mathbb{Z} & & & \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & & \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & \\ \mathbb{Z}_2 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z}. \end{array}$$

Since the differential  $d^3: E_{3,0}^3 \rightarrow E_{0,2}^3$  is zero, the  $E^3$ -term is also the  $E^4$ -term. The following result is a consequence of Mosher [26, Proposition 5.2].

**Lemma 8.** *Let  $n$  be a positive integer. Then, in the spectral sequence*

$$E_{s,t}^2 = H_s(C_{2^n-1}, \text{TR}_t^1(\mathbb{S}; 2)) \Rightarrow \mathbb{H}_{s+t}(C_{2^n-1}, T(\mathbb{S})),$$

*the  $d^4$ -differential  $d^4: E_{s,t}^4 \rightarrow E_{s-4,t+3}^4$  is equal to the map of sub-quotients induced from the map of group homology groups induced from multiplication by  $\nu$ , if  $s$  is*

*congruent to 0, 1, 2, 3, 8, 9, 10, or 11 modulo 16, by  $2\nu$ , if  $s$  is congruent to 6, 7, 12, or 13 modulo 16, and by 0, if  $s$  is congruent to 4, 5, 14, or 15 modulo 16.*  $\square$

In the case at hand, we find that the  $d^4$ -differential is zero, for  $s + t \leq 7$ . For degree reasons, the only possible higher non-zero differential all have target on the fiber line  $s = 0$ . However, we argued above that these differentials are zero. Therefore, for  $s + t \leq 7$ , the  $E^3$ -term is also the  $E^\infty$ -term.

The  $E^2$ -term of the skeleton spectral sequence for  $\mathbb{H}_q(C_4, T(\mathbb{S}))$  for  $s + t \leq 7$  is

$$\begin{array}{cccccccc}
 \mathbb{Z}/16\mathbb{Z} & & & & & & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
 0 & 0 & 0 & & & & & \\
 0 & 0 & 0 & 0 & & & & \\
 \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/8\mathbb{Z} & & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \\
 \mathbb{Z}_2 & \mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z}.
 \end{array}$$

The generators of the groups  $E_{s,t}^2$  are as before with exception that the groups  $E_{s,3}^3$  and  $E_{s,7}^2$  with  $s$  an even positive integer are generated by  $2\nu z_s$  and  $4\sigma z_s$ , respectively. We find as before that the  $E^3$ -term for  $s + t \leq 7$  takes the form

$$\begin{array}{cccccccc}
 \mathbb{Z}/16\mathbb{Z} & & & & & & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
 0 & 0 & 0 & & & & & \\
 0 & 0 & 0 & 0 & & & & \\
 \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/8\mathbb{Z} & & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & \\
 \mathbb{Z}_2 & \mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z} & 0 & 2\mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z}.
 \end{array}$$

The only possible non-zero  $d^3$ -differential for  $s + t \leq 7$  is  $d^3: E_{6,1}^3 \rightarrow E_{3,3}^3$ . Since the corresponding differential in the previous spectral sequence is zero, a comparison by using the Verschiebung map shows that also this differential is zero. The  $d^4$ -differentials are given by Lemma 8. Hence, the  $E^5$ -term begins

$$\begin{array}{cccccccc}
 \mathbb{Z}/16\mathbb{Z} & & & & & & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
 0 & 0 & 0 & & & & & \\
 0 & 0 & 0 & 0 & & & & \\
 \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 2\mathbb{Z}/8\mathbb{Z} & & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & \\
 \mathbb{Z}_2 & \mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z} & 0 & 2\mathbb{Z}/4\mathbb{Z} & 0 & 2\mathbb{Z}/4\mathbb{Z}.
 \end{array}$$

We see as before that the  $E^5$ -term is also the  $E^\infty$ -term.

Finally, we consider the skeleton spectral sequence for  $\mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{S}))$ , where  $n \geq 4$ . The  $E^2$ -term for  $s + t \leq 7$ , takes the form

$$\begin{array}{cccccccc}
 \mathbb{Z}/16\mathbb{Z} & & & & & & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
 0 & 0 & 0 & & & & & \\
 0 & 0 & 0 & 0 & & & & \\
 \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \\
 \mathbb{Z}_2 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z}.
 \end{array}$$

The generators of the groups  $E_{s,t}^2$  are the same as in the skeleton spectral sequence for  $\mathbb{H}_q(C_2, T(\mathbb{S}))$  with the exception that the groups  $E_{s,3}^2$  and  $E_{s,7}^2$  are generated by the classes  $\nu z_s$  and  $\sigma z_s$ , respectively, for all  $s \geq 0$ . The  $d^2$ -differential is given by Lemma 5. We find that the  $E^3$ -term for  $s + t \leq 7$  becomes

$$\begin{array}{cccccccc}
 \mathbb{Z}/16\mathbb{Z} & & & & & & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
 0 & 0 & 0 & & & & & \\
 0 & 0 & 0 & 0 & & & & \\
 \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & 0 & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & \\
 \mathbb{Z}_2 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & 2\mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z}.
 \end{array}$$

A comparison with the previous spectral sequence by using the Verschiebung map shows that the  $d^3$ -differential is zero. The  $d^4$ -differential is given by Lemma 8. Hence, the  $E^5$ -term for  $s + t \leq 7$  becomes

$$\begin{array}{cccccccc}
 \mathbb{Z}/16\mathbb{Z} & & & & & & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & & & & \\
 0 & 0 & 0 & & & & & \\
 0 & 0 & 0 & 0 & & & & \\
 \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} & & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & 0 & & \\
 \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & \\
 \mathbb{Z}_2 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & 2\mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & 2\mathbb{Z}/2^{n-1}\mathbb{Z}
 \end{array}$$

and, for  $s + t \leq 7$ , this is also the  $E^\infty$ -term.

**Lemma 9.** (1) *There exists unique homotopy classes*

$$\xi_{1,n-1} \in \mathbb{H}_1(C_{2^{n-1}}, T(\mathbb{S})) \quad (n \geq 1)$$

such that  $\xi_{1,n-1}$  represents  $\iota z_1 \in E_{1,0}^\infty$ ,  $F(\xi_{1,n-1}) = \xi_{1,n-2}$ , and  $\xi_{1,0} = \eta$ .

(2) *There exists unique homotopy classes*

$$\xi_{3,n-1} \in \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S})) \quad (n \geq 1)$$

such that  $\xi_{3,n-1}$  represents  $\iota z_3 \in E_{3,0}^\infty$ ,  $F(\xi_{3,n-1}) = \xi_{3,n-2}$ , and  $\xi_{3,0} = \nu$ .



Moreover, if  $n \geq 3$ , then  $F^{n-1}: \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S})) \rightarrow \text{TR}_3^1(\mathbb{S}; 2)$  induces a map

$$\overline{F^{n-1}}: H_3(C_{2^{n-1}}, \text{TR}_0^1(\mathbb{S}; 2)) \rightarrow \text{TR}_3^1(\mathbb{S}; 2)/4 \text{TR}_3^1(\mathbb{S}; 2).$$

Indeed,  $F^{n-1}V^{n-1}(v) = 2^{n-1}v$  and  $F^{n-1}dV^{n-1}(\eta^2) = \eta^3 = 4v$ . The map  $\overline{F^{n-1}}$  is surjective by [36, Table 4]. One readily verifies that it maps the generator  $\iota_{\mathbb{Z}_3}$  to the modulo 4 reduction  $\bar{v}$  of the Hopf class  $v$ . Hence,  $F^{n-1}$  maps one of the classes  $\xi_{3,n-1}$  and  $\xi'_{3,n-1}$  to  $v$  and the other class to  $5v$ . The statement (2) follows.

Finally, the statement (3) follows immediately from the inverse limit of the spectral sequences displayed above.  $\square$

The group  $\mathbb{H}_5(C_8, T(\mathbb{S}))$  is equal to the direct sum of the subgroup generated by the class  $\xi_{5,3}$  and a cyclic group. We choose a generator  $\rho$  this cyclic group.

**Proposition 10.** *The groups  $\mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{S}))$  with  $q \leq 5$  are given by*

$$\begin{aligned} \mathbb{H}_0(C_{2^{n-1}}, T(\mathbb{S})) &= \mathbb{Z}_2 \cdot V^{n-1}(\iota), \\ \mathbb{H}_1(C_{2^{n-1}}, T(\mathbb{S})) &= \begin{cases} \mathbb{Z}/2\mathbb{Z} \cdot \eta & (n = 1) \\ \mathbb{Z}/2^{n-1}\mathbb{Z} \cdot \xi_{1,n-1} \oplus \mathbb{Z}/2\mathbb{Z} \cdot V^{n-1}(\eta) & (n \geq 2) \end{cases}, \\ \mathbb{H}_2(C_{2^{n-1}}, T(\mathbb{S})) &= \begin{cases} \mathbb{Z}/2\mathbb{Z} \cdot \eta^2 & (n = 1) \\ \mathbb{Z}/2\mathbb{Z} \cdot \eta\xi_{1,n-1} \oplus \mathbb{Z}/2\mathbb{Z} \cdot V^{n-1}(\eta^2) & (n \geq 2) \end{cases}, \\ \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S})) &= \begin{cases} \mathbb{Z}/8\mathbb{Z} \cdot v & (n = 1) \\ \mathbb{Z}/8\mathbb{Z} \cdot \xi_{3,1} \oplus \mathbb{Z}/8\mathbb{Z} \cdot V(v) & (n = 2) \\ \mathbb{Z}/2^n\mathbb{Z} \cdot \xi_{3,n-1} \oplus \mathbb{Z}/2\mathbb{Z} \cdot \eta^2\xi_{1,n-1} & (n \geq 3) \\ \oplus \mathbb{Z}/8\mathbb{Z} \cdot V^{n-1}(v) & \end{cases}, \\ \mathbb{H}_4(C_{2^{n-1}}, T(\mathbb{S})) &= \begin{cases} \mathbb{Z}/2^{n-1}\mathbb{Z} \cdot v\xi_{1,n-1} & (n \leq 3) \\ \mathbb{Z}/8\mathbb{Z} \cdot v\xi_{1,n-1} & (n \geq 4) \end{cases}, \\ \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S})) &= \begin{cases} 0 & (n \leq 2) \\ \mathbb{Z}/4\mathbb{Z} \cdot \xi_{5,2} & (n = 3) \\ \mathbb{Z}/2^{n-1}\mathbb{Z} \cdot \xi_{5,n-1} \oplus \mathbb{Z}/2\mathbb{Z} \cdot V^{n-4}(\rho) & (n \geq 4) \end{cases}. \end{aligned}$$

In addition,  $F(\xi_{q,n-1}) = \xi_{q,n-2}$ , where  $\xi_{1,0} = \eta$  and  $\xi_{3,0} = v$ , and  $F(\rho) = 0$ .

*Proof.* We have already evaluated the  $E^\infty$ -term of the spectral sequence

$$E_{s,t}^2 = H_s(C_{2^{n-1}}, \text{TR}_t^1(\mathbb{S}; 2)) \Rightarrow \mathbb{H}_{s+t}(C_{2^{n-1}}, T(\mathbb{S})),$$

for  $s + t \leq 7$ . We have also defined all the homotopy classes that appear in the statement. Hence, it remains only to prove that these homotopy classes have the indicated order. First, the edge homomorphism of the spectral sequence is the map

$$V^{n-1}: \text{TR}_t^1(\mathbb{S}; 2) \rightarrow \mathbb{H}_t(C_{2^{n-1}}, T(\mathbb{S})).$$

Since this map has a retraction, the classes  $V^{n-1}(\eta)$  and  $V^{n-1}(\eta^2)$  both generate a direct summand  $\mathbb{Z}/2\mathbb{Z}$  and the class  $V^{n-1}(v)$  generates a direct summand  $\mathbb{Z}/8\mathbb{Z}$  as stated. This completes the proof for  $q \leq 2$ . Next, the Frobenius map

$$F: \mathbb{H}_3(C_2, T(\mathbb{S})) \rightarrow \text{TR}_3^1(\mathbb{S}; 2)$$

is surjective by [36, Table 4]. This implies that the class  $\xi_{3,1}$  has order 8 and that the group  $\mathbb{H}_3(C_2, T(\mathbb{S}))$  is as stated. We note that  $4\xi_{3,1}$  is congruent to  $dV(\eta^2)$  modulo the image of the edge homomorphism.

Next, we show by induction on  $n \geq 3$  that the class  $\xi_{3,n-1}$  has order  $2^n$ . The class  $\xi_{3,2}$  has order either 8 or 16, because  $F(\xi_{3,2}) = \xi_{3,1}$  has order 8. If  $\xi_{3,2}$  has order 16, then the quotient of  $\mathbb{H}_3(C_4, T(\mathbb{S}))$  by the image of the edge homomorphism is equal to  $\mathbb{Z}/16\mathbb{Z}$  generated by the image of  $\xi_{3,2}$ . But then  $V(\xi_{3,1})$  has order 8 which contradicts that, modulo the image of the edge homomorphism,

$$4V(\xi_{3,1}) = V(4\xi_{3,1}) \equiv VdV(\eta^2) = 2dV^2(\eta^2) = 0.$$

Hence,  $\xi_{3,2}$  has order 8, and the group  $\mathbb{H}_3(C_4, T(\mathbb{S}))$  is as stated. So we let  $n \geq 4$  and assume, inductively, that  $\xi_{3,n-2}$  has order  $2^{n-1}$ . The class  $2^{n-2}\xi_{3,n-2}$  is represented in the spectral sequence by  $\eta_{z_2}$ . Now, by Lemma 6(4), we have  $V(\eta_{z_2}) = \eta_{z_2}$ , which shows that the class  $2^{n-2}V(\xi_{3,n-2}) = V(2^{n-2}\xi_{3,n-2})$  is non-zero and represented by  $\eta_{z_2}$ . This implies that  $2^{n-1}\xi_{3,n-1}$  is non-zero, and hence,  $\xi_{3,n-1}$  has order  $2^n$  as stated.

Next, we show that, for  $n \geq 3$ , the class  $\xi_{5,n-1}$  has order  $2^{n-1}$ . If  $n \geq 4$ , the spectral sequence shows that there is an extension

$$0 \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S})) \rightarrow \mathbb{Z}/2^{n-2}\mathbb{Z} \rightarrow 0.$$

The Verschiebung map induces a map of extensions from the extension for  $n$  to the extension for  $n + 1$ , and Lemma 6 shows that the resulting extension of colimits with respect to the Verschiebung maps is an extension

$$0 \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \text{colim}_V \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S})) \rightarrow \mathbb{Q}_2/\mathbb{Z}_2 \rightarrow 0.$$

It follows from [25, Lemma 4.4.9] that there is a canonical isomorphism

$$\text{Ext}(\mathbb{Q}_2/\mathbb{Z}_2, \text{colim}_V \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S}))) \xrightarrow{\sim} \lim_F \mathbb{H}_6(C_{2^{n-1}}, T(\mathbb{S}))$$

and, by the proof of Lemma 9, the right-hand group is cyclic of order 2. This implies that the extension for  $n \geq 4$  is equivalent to the extension

$$0 \rightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{(1, -2^{n-3})} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^{n-1}\mathbb{Z} \xrightarrow{2^{n-3}+1} \mathbb{Z}/2^{n-2}\mathbb{Z} \rightarrow 0.$$

It follows that, for  $n \geq 4$ , the class  $\xi_{5,n-1}$  has order  $2^{n-1}$  as stated. It remains to prove that  $\xi_{5,2}$  has order 4. If this is not the case, the map of extensions induced by the Verschiebung map  $V: \mathbb{H}_5(C_4, T(\mathbb{S})) \rightarrow \mathbb{H}_5(C_8, T(\mathbb{S}))$  takes the form

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{(1,0)} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \xrightarrow{0+1} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow 2 & & \downarrow V & & \downarrow 2 & & \\
 0 & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{(1,-2)} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} & \xrightarrow{2+1} & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & 0,
 \end{array}$$

where the middle map  $V$  takes  $(1, 0)$  to  $(0, 4)$  and  $(0, 1)$  to either  $(1, 0)$  or  $(1, 4)$ . The class  $\xi_{5,2}$  corresponds to either  $(0, 1)$  or  $(1, 1)$  in the top middle group. In either case, we find that the class  $V(\xi_{5,2})$  has order 2 and reduces to a generator of the quotient of  $\mathbb{H}_5(C_8, T(\mathbb{S}))$  by the subgroup  $\mathbb{Z}/8\mathbb{Z} \cdot \xi_{5,3}$ . It follows that the class

$$V(\xi_{5,2}) - 2\xi_{5,3} \in \mathbb{H}_5(C_8, T(\mathbb{S}))$$

generates the kernel of the edge homomorphism onto  $\mathbb{Z}/4\mathbb{Z} \cdot 2\iota_{\mathbb{Z}_5}$ . Then, Lemma 6 shows that the class  $F(V(\xi_{5,2}) - 2\xi_{5,3})$  generates the kernel of the edge homomorphism from  $\mathbb{H}_5(C_4, T(\mathbb{S}))$  onto  $\mathbb{Z}/2\mathbb{Z} \cdot 2\iota_{\mathbb{Z}_5}$ . But  $F(V(\xi_{5,2}) - 2\xi_{5,3}) = 0$  which is a contradiction. We conclude that the group  $\mathbb{H}_5(C_4, T(\mathbb{S}))$  is cyclic as stated.

Finally, the Frobenius map  $F: \mathbb{H}_5(C_8, T(\mathbb{S})) \rightarrow \mathbb{H}_5(C_4, T(\mathbb{S}))$  induces a map of extensions which takes the form

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{(1,-2)} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} & \xrightarrow{2+1} & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow 1 & & \downarrow 0+1 & & \downarrow 1 & & \\
 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{1} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0.
 \end{array}$$

The class  $\rho$  corresponds to one of the elements  $(1, 0)$  or  $(1, 4)$  of the top middle group both of which map to zero by the middle vertical map. It follows that  $F(\rho)$  is zero as stated. □

We define  $\xi_{q,s} \in \text{TR}_q^n(\mathbb{S}; 2)$  to be the image of  $\xi_{q,s} \in \mathbb{H}_q(C_{2^s}, T(\mathbb{S}))$  by the composition of the norm map and the iterated Segal-tom Dieck splitting

$$\mathbb{H}_q(C_{2^s}, T(\mathbb{S})) \rightarrow \text{TR}_q^s(\mathbb{S}; 2) \rightarrow \text{TR}_q^n(\mathbb{S}; 2).$$

Similarly, we define  $\rho \in \text{TR}_5^n(\mathbb{S}; 2)$  to be the image of  $\rho \in \mathbb{H}_5(C_8, T(\mathbb{S}))$  by the composition of the norm map and the iterated Segal-tom Dieck splitting

$$\mathbb{H}_5(C_8, T(\mathbb{S})) \rightarrow \text{TR}_5^4(\mathbb{S}; 2) \rightarrow \text{TR}_5^n(\mathbb{S}; 2).$$

Then we have the following result.



**Theorem 11.** *The groups  $\text{TR}_q^n(\mathbb{S}; 2)$  with  $q \leq 5$  are given by*

$$\begin{aligned} \text{TR}_0^n(\mathbb{S}; 2) &= \bigoplus_{0 \leq s < n} \mathbb{Z}_2 \cdot V^s(1) \\ \text{TR}_1^n(\mathbb{S}; 2) &= \bigoplus_{0 \leq s < n} \mathbb{Z}/2\mathbb{Z} \cdot V^s(\eta) \oplus \bigoplus_{1 \leq s < n} \mathbb{Z}/2^s\mathbb{Z} \cdot \xi_{1,s} \\ \text{TR}_2^n(\mathbb{S}; 2) &= \bigoplus_{0 \leq s < n} \mathbb{Z}/2\mathbb{Z} \cdot V^s(\eta^2) \oplus \bigoplus_{1 \leq s < n} \mathbb{Z}/2\mathbb{Z} \cdot \eta \xi_{1,s} \\ \text{TR}_3^n(\mathbb{S}; 2) &= \bigoplus_{0 \leq s < n} \mathbb{Z}/8\mathbb{Z} \cdot V^s(\nu) \oplus \bigoplus_{1 \leq s < n} \mathbb{Z}/2^u\mathbb{Z} \cdot \xi_{3,s} \oplus \bigoplus_{2 \leq s < n} \mathbb{Z}/2\mathbb{Z} \cdot \eta^2 \xi_{1,s} \\ \text{TR}_4^n(\mathbb{S}; 2) &= \bigoplus_{1 \leq s < n} \mathbb{Z}/2^v\mathbb{Z} \cdot \nu \xi_{1,s} \\ \text{TR}_5^n(\mathbb{S}; 2) &= \bigoplus_{2 \leq s < n} \mathbb{Z}/2^s\mathbb{Z} \cdot \xi_{5,s} \oplus \bigoplus_{3 \leq s < n} \mathbb{Z}/2\mathbb{Z} \cdot V^{s-3}(\rho) \end{aligned}$$

where  $u = u(s)$  is the larger of 3 and  $s + 1$ , and where  $v = v(s)$  is the smaller of 3 and  $s$ . The restriction map takes  $\xi_{q,s}$  to  $\xi_{s,q}$ , if  $s < n - 1$ , and to zero, if  $s = n - 1$ , and takes  $\rho$  to  $\rho$ , if  $n \geq 5$ , and to zero, if  $n = 4$ . The Frobenius map takes  $\xi_{q,s}$  to  $\xi_{q,s-1}$ , where  $\xi_{1,0} = \eta$  and  $\xi_{3,0} = \nu$ , and takes  $\rho$  to zero. Connes' operator takes  $V^s(1)$  to  $\xi_{1,s} + V^s(\eta)$ , and takes  $\xi_{1,s}$  to zero.

*Proof.* The Segal-tom Dieck splitting gives a section of the restriction map. Hence, the fundamental long-exact sequence

$$\cdots \rightarrow \mathbb{H}_q(C_{p^{n-1}}, T(\mathbb{S})) \xrightarrow{N} \text{TR}_q^n(\mathbb{S}; p) \xrightarrow{R} \text{TR}_q^{n-1}(\mathbb{S}; p) \xrightarrow{\partial} \mathbb{H}_{q-1}(C_{p^{n-1}}, T(\mathbb{S})) \rightarrow \cdots$$

of Proposition 4 breaks into split short-exact sequences and Proposition 10 then shows that the groups  $\text{TR}_q^n(\mathbb{S}; 2)$  are as stated. Since the Frobenius map and the Segal-tom Dieck splitting commute, the formula for the Frobenius also follows from Proposition 10. Finally, from the proof of Proposition 10, we have  $\xi_{1,s} = dV^s(1) + V^s(\eta)$ . This implies that

$$d\xi_{1,s} = ddV^s(1) + dV^s(\eta) = dV^s(\eta) + dV^s(\eta) = 0$$

as stated. □

*Remark 12.* We have not determined  $\eta \xi_{3,s}$  and  $d\xi_{3,s}$ .

## 6 The Groups $\text{TR}_q^n(\mathbb{Z}; 2)$

In this section, we again implicitly consider homotopy groups with  $\mathbb{Z}_2$ -coefficients. The groups  $\text{TR}_q^1(\mathbb{Z}; 2)$  were evaluated by Bökstedt [3]; see also [23, Theorem 1.1]. The group  $\text{TR}_0^1(\mathbb{Z}; 2)$  is equal to  $\mathbb{Z}_2$  generated by the multiplicative unit element

$\iota = [1]_1$ , and for positive integers  $q$ , the group  $\mathrm{TR}_q^1(\mathbb{Z}; 2)$  is finite cyclic of order

$$|\mathrm{TR}_q^1(\mathbb{Z}; 2)| = \begin{cases} 2^{v_2(i)} & (q = 2i - 1 \text{ odd}) \\ 1 & (q \text{ even}). \end{cases}$$

We choose a generator  $\lambda$  of  $K_3(\mathbb{Z})$  such that  $2\lambda = \nu$ . Then, by [5, Theorem 10.4], the image of  $\lambda$  by the cyclotomic trace map generates the group  $\mathrm{TR}_3^1(\mathbb{Z}; 2)$ . We also choose a generator  $\gamma$  of the group  $\mathrm{TR}_7^1(\mathbb{Z}; 2)$ . We first derive the following result from Rognes' paper [27].

**Proposition 13.** *The group  $\mathrm{TR}_q^n(\mathbb{Z}; 2)$  is zero, for every positive even integer  $q$  and every positive integer  $n$ .*

*Proof.* The group  $\mathrm{TR}_q^n(\mathbb{Z}; 2)$  is finite, for all positive integers  $q$  and  $n$ . Indeed, this is true, for  $n = 1$  by Bökstedt's result that we recalled above and follows, inductively, for  $n \geq 1$ , from the fundamental long-exact sequence of Proposition 4, the skeleton spectral sequence, and the fact that the boundary map

$$\partial: \mathrm{TR}_1^{n-1}(\mathbb{Z}; 2) \rightarrow \mathbb{H}_0(C_{2^{n-1}}, T(\mathbb{Z}))$$

in the fundamental long-exact sequence is zero [17, Proposition 3.3]. Moreover, the group  $\mathrm{TR}_0^n(\mathbb{Z}; 2)$  is a free  $\mathbb{Z}_2$ -module. It follows that, in the strongly convergent whole plane Bockstein spectral sequence

$$E_{s,t}^2 = \mathrm{TR}_{s+t}^n(\mathbb{Z}; 2, 2^{-s}\mathbb{Z}/2^{-(s-1)}\mathbb{Z}) \Rightarrow \mathrm{TR}_{s+t}^n(\mathbb{Z}; 2, \mathbb{Q}_2)$$

induced from the 2-adic filtration of  $\mathbb{Q}_2$ , all elements of total degree 0 survive to the  $E^\infty$ -term and all elements of positive total degree are annihilated by differentials. The differentials are periodic in the sense that the isomorphism  $2: \mathbb{Q}_2 \rightarrow \mathbb{Q}_2$  induces an isomorphism of spectral sequences

$$\underline{2}: E_{s,t}^r \xrightarrow{\sim} E_{s-1,t+1}^r.$$

We recall from [27, Lemma 9.4] that, for all positive integers  $n$  and  $i$ ,

$$\dim_{\mathbb{F}_2} \mathrm{TR}_{2i-1}^n(\mathbb{Z}; 2, \mathbb{F}_2) = \dim_{\mathbb{F}_2} \mathrm{TR}_{2i}^n(\mathbb{Z}; 2, \mathbb{F}_2).$$

Using this result, we show, by induction on  $i \geq 1$ , that every element of total degree  $2i - 1$  is an infinite cycle and that every non-zero element of total degree  $2i$  supports a non-zero differential. The proof of the case  $i = 1$  and of the induction step are similar. The statement that every element in total degree  $2i - 1$  is an infinite cycle follows, for  $i = 1$ , from the fact that every element of total degree 0 survives to the  $E^\infty$ -term, and for  $i > 1$ , from the inductive hypothesis that every non-zero element of total degree  $2i - 2$  supports a non-zero differential. Since no element of total degree  $2i - 1$  survives to the  $E^\infty$ -term, it is hit by a differential supported

on an element of total degree  $2i$ . Since the differentials are periodic and  $E_{s,2i-1-s}^2$  and  $E_{s,2i-s}^2$  have the same dimension, we find that non-zero every element of total degree  $2i$  supports a non-zero differential as stated.

Finally, we consider the strongly convergent left half-plane Bockstein spectral sequence induced from the 2-adic filtration of  $\mathbb{Z}_2$ ,

$$E_{s,t}^2 = \text{TR}_{s+t}^n(\mathbb{Z}; 2, 2^{-s}\mathbb{Z}/2^{-(s-1)}\mathbb{Z}) \Rightarrow \text{TR}_{s+t}^n(\mathbb{Z}; 2, \mathbb{Z}_2).$$

The differentials in this spectral sequence are obtained by restricting the differentials in the whole plan Bockstein spectral sequence above. It follows that in this spectral sequence, too, every non-zero element of positive even total degree supports a non-zero differential. This completes the proof.  $\square$

*Remark 14.* The same argument based on Bökstedt and Madsen’s paper [5], shows that, for an odd prime  $p$ , the groups  $\text{TR}_q^n(\mathbb{Z}; p)$  are zero, for every positive even integer  $q$  and every positive integer  $n$ .

We next consider the skeleton spectral sequence

$$E_{s,t}^2 = H_s(C_{2^{n-1}}, \text{TR}_t^1(\mathbb{Z}; 2)) \Rightarrow \mathbb{H}_{s+t}(C_{2^{n-1}}, T(\mathbb{Z})).$$

The  $E^2$ -term, for  $s + t \leq 7$ , takes the form

$$\begin{array}{cccccccc} \mathbb{Z}/4\mathbb{Z} & & & & & & & \\ 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & & \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \mathbb{Z}_2 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z}. \end{array}$$

The group  $E_{s,0}^2$  is generated by  $\iota_{z_s}$  and the group  $E_{s,3}^2$  is generated by  $\lambda_{z_s}$ . The group  $E_{s,7}^2$  is generated by  $\gamma_{z_s}$ , if  $s = 0$  or if  $s$  is odd of if  $n > 1$ , and is generated by  $2\gamma_{z_s}$ , if  $n = 1$  and  $s$  is positive and even. It follows from [27, Theorem 8.14] that the group  $\mathbb{H}_4(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2)$  is an  $\mathbb{F}_2$ -vector space of dimension 1. This implies that  $d^4(\iota_{z_5}) = \lambda_{z_1}$ . On the other hand,  $d^4(\iota_{z_7}) = 0$ , since  $\iota_{z_7}$  survives to the  $E^4$ -term of the skeleton spectral sequence for  $\mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{S}))$  and is a  $d^4$ -cycle. This shows that the  $E^5$ -term for  $s + t \leq 7$  is given by

$$\begin{array}{cccccccc} \mathbb{Z}/4\mathbb{Z} & & & & & & & \\ 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & & \\ \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \mathbb{Z}_2 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & 2\mathbb{Z}/2^{n-1}\mathbb{Z} & 0 & \mathbb{Z}/2^{n-1}\mathbb{Z}. \end{array}$$

We claim that the differential  $d^5: E_{5,3}^5 \rightarrow E_{0,7}^5$  is zero. Indeed, let

$$F^{m-n}: 'E_{s,t}^r \rightarrow E_{s,t}^r$$

be the map of spectral sequences induced by the iterated Frobenius map

$$F^{m-n}: \mathbb{H}_q(C_{2^{m-1}}, T(\mathbb{Z})) \rightarrow \mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{Z})).$$

It follows from Lemma 6 that the map  $F^{m-n}: 'E_{5,3}^5 \rightarrow E_{5,3}^5$  is an isomorphism and that, for  $m \geq n + 2$ , the map  $F^{m-n}: 'E_{5,3}^5 \rightarrow E_{5,3}^5$  is zero. Hence, the differential in question is zero as claimed. It follows that the  $E^5$ -term of the spectral sequence is also the  $E^\infty$ -term.

We choose a generator  $\kappa$  of the infinite cyclic group  $K_5(\mathbb{Z})$  and recall the generator  $\lambda$  of the group  $K_3(\mathbb{Z})$ . We continue to write  $\lambda$  and  $\kappa$  for the images of  $\lambda$  and  $\kappa$  in  $\text{TR}_3^n(\mathbb{Z}; 2)$  and  $\text{TR}_5^n(\mathbb{Z}; 2)$  by the cyclotomic trace map. The norm map

$$\mathbb{H}_5(C_2, T(\mathbb{Z})) \rightarrow \text{TR}_5^2(\mathbb{Z}; 2)$$

is an isomorphism, and we will also write  $\kappa$  for the unique class on the left-hand side whose image by the norm map is the class  $\kappa$  on the right-hand side. Finally, we continue to write  $\xi_{q,n} \in \mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{Z}))$  for the image by the map induced from the Hurewicz map  $\ell: \mathbb{S} \rightarrow \mathbb{Z}$  of the class  $\xi_{q,n} \in \mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{S}))$ .

**Proposition 15.** *The groups  $\mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{Z}))$  with  $q \leq 6$  are given by*

$$\begin{aligned} \mathbb{H}_0(C_{2^{n-1}}, T(\mathbb{Z})) &= \mathbb{Z}_2 \cdot V^{n-1}(t), \\ \mathbb{H}_1(C_{2^{n-1}}, T(\mathbb{Z})) &= \begin{cases} 0 & (n = 1) \\ \mathbb{Z}/2^{n-1}\mathbb{Z} \cdot \xi_{1,n-1} & (n \geq 2) \end{cases}, \\ \mathbb{H}_2(C_{2^{n-1}}, T(\mathbb{Z})) &= 0 \\ \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{Z})) &= \begin{cases} \mathbb{Z}/2\mathbb{Z} \cdot \lambda & (n = 1) \\ \mathbb{Z}/2^n\mathbb{Z} \cdot \xi_{3,n-1} & (n \geq 2) \end{cases}, \\ \mathbb{H}_4(C_{2^{n-1}}, T(\mathbb{Z})) &= 0 \\ \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z})) &= \begin{cases} 0 & (n = 1) \\ \mathbb{Z}/2\mathbb{Z} \cdot \kappa & (n = 2) \\ \mathbb{Z}/2^{n-2}\mathbb{Z} \cdot \xi_{5,n-1} \oplus \mathbb{Z}/2\mathbb{Z} \cdot V^{n-2}(\kappa) & (n \geq 3) \end{cases}, \\ \mathbb{H}_6(C_{2^{n-1}}, T(\mathbb{Z})) &= \begin{cases} 0 & (n = 1) \\ \mathbb{Z}/2\mathbb{Z} \cdot dV^{n-2}(\kappa) & (n \geq 2) \end{cases}. \end{aligned}$$

In addition,  $F(\xi_{q,n-1}) = \xi_{q,n-2}$ , where  $\xi_{1,0}$ ,  $\xi_{3,0}$ , and  $\xi_{5,0}$  are zero.

*Proof.* The cases  $q = 0$  and  $q = 1$  follow immediately from the spectral sequence above and from the fact that the map  $\text{TR}_0^1(\mathbb{S}; 2) \rightarrow \text{TR}_0^1(\mathbb{Z}; 2)$  induced by the Hurewicz map is an isomorphism. The cases  $q = 2$  and  $q = 4$  follow

directly from the spectral sequence above. It follows from [27, Theorem 8.14] that  $\mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2)$  is an  $\mathbb{F}_2$ -vector space of dimension 1, for all  $n \geq 1$ . The statement for  $q = 3$  follows. It also follows from loc. cit. that  $\mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2)$  is an  $\mathbb{F}_2$ -vector space of dimension 0, if  $n = 1$ , dimension 1, if  $n = 2$ , and dimension 2, if  $n \geq 3$ . Hence, to prove the statement for  $q = 5$ , it will suffice to show that the group  $\mathbb{H}_5(C_2, T(\mathbb{Z}))$  is generated by the class  $\kappa$ , or equivalently, that the composition

$$K_5(\mathbb{Z}) \rightarrow \mathrm{TC}_5^2(\mathbb{Z}; 2) \rightarrow \mathrm{TR}_5^2(\mathbb{Z}; 2) \xrightarrow{\sim} \mathrm{TR}_5^2(\mathbb{Z}; 2, \mathbb{F}_2)$$

of the cyclotomic trace map and the modulo 2 reduction map is surjective. But this is the statement that  $i_1(\kappa) = \xi_5(0)$  in [30, Proposition 4.2]. (Here  $\xi_5(0)$  is name given in loc. cit. to the generator of the right-hand group; it is unrelated to the class  $\xi_{5,0}$ .) Finally, the statement for  $q = 6$  follows from [18, Proposition 4.4.1].  $\square$

**Corollary 16.** *The cokernel of the map induced by the Hurewicz map*

$$\ell: \mathrm{TR}_3^n(\mathbb{S}; 2) \rightarrow \mathrm{TR}_3^n(\mathbb{Z}; 2)$$

is equal to  $\mathbb{Z}/2\mathbb{Z} \cdot \lambda$ .

*Proof.* The proof is by induction on  $n \geq 1$ . In the case  $n = 1$ , the Hurewicz map induces the zero map  $\mathrm{TR}_q^1(\mathbb{S}; 2) \rightarrow \mathrm{TR}_q^1(\mathbb{Z}; 2)$ , for all positive integers  $q$ . Indeed, the spectrum  $\mathrm{TR}^1(\mathbb{Z}; 2)$  is a module spectrum over the Eilenberg–MacLane spectrum for  $\mathbb{Z}$  and therefore is weakly equivalent to a product of Eilenberg–MacLane spectra. As we recalled above,  $\mathrm{TR}_3^1(\mathbb{Z}; 2) = \mathbb{Z}/2\mathbb{Z} \cdot \lambda$ , which proves the case  $n = 1$ . To prove the induction step, we use that the Hurewicz map induces a map of fundamental long-exact sequences which takes the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S})) & \longrightarrow & \mathrm{TR}_3^n(\mathbb{S}; 2) & \longrightarrow & \mathrm{TR}_3^{n-1}(\mathbb{S}; 2) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{Z})) & \longrightarrow & \mathrm{TR}_3^n(\mathbb{Z}; 2) & \longrightarrow & \mathrm{TR}_3^{n-1}(\mathbb{Z}; 2) \longrightarrow 0. \end{array}$$

The zero on the lower right-hand side follows from Proposition 15, and the zero on the lower left-hand side from Proposition 13. Since Propositions 10 and 15 show that the left-hand vertical map is surjective, the induction step follows.  $\square$

We owe the proof of the following result to Marcel Bökstedt.

**Lemma 17.** *The square of homotopy groups with  $\mathbb{Z}_2$ -coefficients*

$$\begin{array}{ccc} K_5(\mathbb{S}; \mathbb{Z}_2) & \longrightarrow & K_5(\mathbb{S}_2; \mathbb{Z}_2) \\ \downarrow & & \downarrow \\ K_5(\mathbb{Z}; \mathbb{Z}_2) & \longrightarrow & K_5(\mathbb{Z}_2; \mathbb{Z}_2), \end{array}$$

where the vertical maps are induced by the Hurewicz maps and the horizontal maps are induced by the completion maps, takes the form

$$\begin{array}{ccc}
 \mathbb{Z}_2 \cdot 8\kappa & \longrightarrow & \mathbb{Z}_2 \cdot (4\kappa + \tau) \\
 \downarrow & & \downarrow \\
 \mathbb{Z}_2 \cdot \kappa & \longrightarrow & \mathbb{Z}_2 \cdot \kappa \oplus \mathbb{Z}/2\mathbb{Z} \cdot \tau
 \end{array}$$

*Proof.* It was proved in [30, Proposition 4.2] that the group  $K_5(\mathbb{Z}_2; \mathbb{Z}_2)$  is the direct sum of a free  $\mathbb{Z}_2$ -module of rank one generated by  $\kappa$  and a torsion subgroup of order 2; the class  $\tau$  is the unique generator of the torsion subgroup. Moreover, [31, Theorem 5.8] shows that the group  $K_5(\mathbb{S}; \mathbb{Z}_2)$  is a free  $\mathbb{Z}_2$ -module of rank one, and [31, Theorem 2.11] and [4, Theorem 5.17] show that the group  $K_5(\mathbb{S}_2; \mathbb{Z}_2)$  is a free  $\mathbb{Z}_2$ -module of rank one. To complete the proof of the lemma, it remains to show that the left-hand vertical map in the diagram in the statement is equal to the inclusion of a subgroup of index 8. This is essentially proved in [2] as we now explain. In op. cit., Bökstedt constructs a homotopy commutative diagram of pointed spaces

$$\begin{array}{ccccc}
 G/O & \xrightarrow{e} & \text{Fib}(s) & \xleftarrow{g} & \text{Fib}(t) \\
 \downarrow & & \downarrow & & \downarrow \\
 BSO & \xlongequal{\quad} & BSO & \xlongequal{\quad} & BSO \\
 \downarrow & & \downarrow s & & \downarrow t \\
 BSG & \xrightarrow{\eta} & SG & \xleftarrow{\quad} & SJ
 \end{array}$$

in which the columns are fibration sequences. The induced diagram of fourth homotopy groups with  $\mathbb{Z}_2$ -coefficients is isomorphic to the diagram

$$\begin{array}{ccccc}
 \mathbb{Z}_2 & \xrightarrow{8} & \mathbb{Z}_2 & \xleftarrow{\text{id}} & \mathbb{Z}_2 \\
 \downarrow 8 & & \downarrow \text{id} & & \downarrow \text{id} \\
 \mathbb{Z}_2 & \xlongequal{\quad} & \mathbb{Z}_2 & \xlongequal{\quad} & \mathbb{Z}_2 \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbb{Z}/8\mathbb{Z} & \longrightarrow & 0 & \longleftarrow & 0.
 \end{array}$$

We compare this diagram to the following diagram constructed by Waldhausen.

$$\begin{array}{ccccccc}
 & & G/O & \xrightarrow{f} & \Omega \text{Wh}^{\text{Diff}}(*) & & \\
 & & \downarrow e & & \downarrow & & \\
 \text{Fib}(t) & \xrightarrow{g} & \text{Fib}(s) & \longrightarrow & \Omega K(\mathbb{Z}) & \longrightarrow & \Omega JK(\mathbb{Z}).
 \end{array}$$

It is proved in [2, p. 30] that the composition of the lower horizontal maps in this diagram becomes a weak equivalence after 2-completion. Moreover, it is proved in [31, Theorem 7.5] that the upper horizontal map induces an isomorphism of homotopy groups with  $\mathbb{Z}_2$ -coefficients in degrees less than or equal to 8. Hence, the induced diagram of fourth homotopy groups with  $\mathbb{Z}_2$ -coefficients is isomorphic to the diagram

$$\begin{array}{ccccc}
 & & \mathbb{Z}_2 & \xrightarrow{\text{id}} & \mathbb{Z}_2 \\
 & & \downarrow 8 & & \downarrow 8 \\
 \mathbb{Z}_2 & \xrightarrow{\text{id}} & \mathbb{Z}_2 & \xrightarrow{\text{id}} & \mathbb{Z}_2 & \xrightarrow{\text{id}} & \mathbb{Z}_2.
 \end{array}$$

The right-hand vertical map in this diagram is induced by the composition

$$\text{Wh}^{\text{Diff}}(*) \rightarrow K(\mathbb{S}) \rightarrow K(\mathbb{Z})$$

of the canonical section of the canonical map  $K(\mathbb{S}) \rightarrow \text{Wh}^{\text{Diff}}(*)$  and the map induced by the Hurewicz map. The left-hand map induces an isomorphism of fifth homotopy groups with  $\mathbb{Z}_2$ -coefficients because  $\pi_5(\mathbb{S}; \mathbb{Z}_2)$  is zero. This completes the proof that the map induced by the Hurewicz map  $K_5(\mathbb{S}; \mathbb{Z}_2) \rightarrow K_5(\mathbb{Z}; \mathbb{Z}_2)$  is the inclusion of an index eighth subgroup. The lemma follows.  $\square$

We define the class  $\xi_{q,s} \in \text{TR}_q^n(\mathbb{Z}; 2)$  to be the image of the class  $\xi_{q,s} \in \text{TR}_q^n(\mathbb{S}; 2)$  by the map induced by the Hurewicz map.

**Theorem 18.** *The groups  $\text{TR}_q^n(\mathbb{Z}; 2)$  with  $q \leq 6$  are given by*

$$\begin{aligned}
 \text{TR}_0^n(\mathbb{Z}; 2) &= \bigoplus_{0 \leq s < n} \mathbb{Z}_2 \cdot V^s(1), \\
 \text{TR}_1^n(\mathbb{Z}; 2) &= \bigoplus_{1 \leq s < n} \mathbb{Z}/2^s \mathbb{Z} \cdot \xi_{1,s}, \\
 \text{TR}_2^n(\mathbb{Z}; 2) &= 0, \\
 \text{TR}_3^n(\mathbb{Z}; 2) &= \begin{cases} \mathbb{Z}/2\mathbb{Z} \cdot \lambda & (n = 1) \\ \mathbb{Z}/8\mathbb{Z} \cdot \lambda \oplus \bigoplus_{2 \leq s < n} \mathbb{Z}/2^{s+1}\mathbb{Z} \cdot \xi_{3,s} & (n \geq 2) \end{cases}, \\
 \text{TR}_4^n(\mathbb{Z}; 2) &= 0, \\
 \text{TR}_5^n(\mathbb{Z}; 2) &= \mathbb{Z}/2^{n-1}\mathbb{Z} \cdot \kappa \oplus \bigoplus_{2 \leq s < n} \mathbb{Z}/2^{s-1}\mathbb{Z} \cdot (\xi_{5,s} + \dots + \xi_{5,n-1} + 4u\kappa), \\
 \text{TR}_6^n(\mathbb{Z}; 2) &= 0,
 \end{aligned}$$

where  $u \in \mathbb{Z}_2^*$  is a unit.

*Proof.* The map induced by the Hurewicz map is an isomorphism, for  $q = 0$ , so the statement for the group  $\text{TR}_0^n(\mathbb{Z}; 2)$  follows from Theorem 11. The statement for  $q = 1$  follows from Proposition 15 and from the fact that the generator  $\xi_{1,s}$  is annihilated by  $2^s$ . For  $q = 3$ , the case  $n = 1$  was recalled at the beginning of the section, so suppose that  $n \geq 2$ . We know from Proposition 15 that the two sides of the statement are groups of the same order. We also know that both groups are the direct sum of  $n - 1$  cyclic groups. Indeed, this is trivial, for the right-hand side, and is proved in [27, Lemma 9.4], for the left-hand side. Now, it follows from Theorem 11 that  $\xi_{3,s}$  is annihilated by  $2^{s+1}$ , so it suffices to show that  $\lambda$  is annihilated by 8. We have a commutative diagram

$$\begin{CD} K_3(\mathbb{Z}; \mathbb{Z}_2) @>>> \text{TR}_3^n(\mathbb{Z}; 2, \mathbb{Z}_2) \\ @VVV @VVV \\ K_3(\mathbb{Z}_2; \mathbb{Z}_2) @>>> \text{TR}_3^n(\mathbb{Z}_2; 2, \mathbb{Z}_2) \end{CD}$$

where the horizontal maps are the cyclotomic trace maps, where the vertical maps are induced by the completion maps, and where we have explicitly indicated that we are considering the homotopy groups with  $\mathbb{Z}_2$ -coefficients. The right-hand vertical map is an isomorphism by [17, Addendum 6.2]. Therefore, it suffices to show that the image of  $\lambda$  in  $K_3(\mathbb{Z}_2; \mathbb{Z}_2)$  has order 8. But this is proved in [30, Proposition 4.2].

It remains to prove the statement for  $q = 5$ . We first show that  $\text{TR}_5^n(\mathbb{Z}; 2)$  is generated by the classes  $\kappa, \xi_{5,2}, \dots, \xi_{5,n-1}$ , or equivalently, that the group

$$\text{TR}_5^n(\mathbb{Z}; 2) / 2 \text{TR}_5^n(\mathbb{Z}; 2) \xrightarrow{\sim} \text{TR}_5^n(\mathbb{Z}; 2, \mathbb{F}_2)$$

is generated by the images of the classes  $\kappa, \xi_{5,2}, \dots, \xi_{5,n-1}$ . We prove this by induction on  $n \geq 2$ . The case  $n = 2$  is true, so we assume the statement for  $n - 1$  and prove it for  $n$ . The fundamental long-exact sequence takes the form

$$\mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2) \xrightarrow{N} \text{TR}_5^n(\mathbb{Z}; 2, \mathbb{F}_2) \xrightarrow{R} \text{TR}_5^{n-1}(\mathbb{Z}; 2, \mathbb{F}_2) \rightarrow 0.$$

Inductively, the right-hand group is generated by the classes  $\kappa, \xi_{5,2}, \dots, \xi_{5,n-2}$ , which are the images by the restriction map of the classes  $\kappa, \xi_{5,2}, \dots, \xi_{5,n-2}$  in the middle group. Moreover, Proposition 15 shows that the left-hand group is generated by the classes  $V^{n-2}(\kappa)$  and  $\xi_{5,n-1}$ . Hence, it will suffice to show that, for  $n \geq 3$ , the image of the class  $V^{n-2}(\kappa)$  in  $\text{TR}_5^n(\mathbb{Z}; 2, \mathbb{F}_2)$  is zero. This follows from [27, Theorem 8.14] as we now explain. We have the commutative diagram with exact rows

$$\begin{CD} \text{TR}_{q+1}^{n-1}(\mathbb{Z}; 2, \mathbb{F}_2) @>\partial>> \mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2) @>N>> \text{TR}_q^n(\mathbb{Z}; 2, \mathbb{F}_2) \\ @VV\hat{\Gamma}V @VVV @VV\Gamma V \\ \hat{\mathbb{H}}^{-q-1}(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2) @>\partial^h>> \mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2) @>N^h>> \mathbb{H}^{-q}(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2) \end{CD}$$



considered first in [5, (6.1)]. It follows from [27, Theorems 0.2, 0.3] that the left-hand vertical map  $\hat{\Gamma}$  is an isomorphism, for all integers  $q + 1 \geq 0$  and  $n \geq 1$ . Hence, it suffices to show that the class  $V^{n-2}(\kappa)$  in the lower middle group is in the image of the lower left-hand horizontal map  $\partial^h$ . The lower left-hand group is the abutment of the strongly convergent, upper half-plan Tate spectral sequence

$$\hat{E}_{s,t}^2 = \hat{H}^{-s}(C_{2^{n-1}}, \text{TR}_t^1(\mathbb{Z}; 2, \mathbb{F}_2)) \Rightarrow \hat{\mathbb{H}}^{-s-t}(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2),$$

and the middle groups are the abutment of the strongly convergent, first quadrant skeleton spectral sequence

$$E_{s,t}^2 = H_s(C_{2^{n-1}}, \text{TR}_t^1(\mathbb{Z}; 2, \mathbb{F}_2)) \Rightarrow \mathbb{H}_{s+t}(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2).$$

Moreover, the map  $\partial^h$  induces a map of spectral sequences

$$\partial^{h,r}: \hat{E}_{s,t}^r \rightarrow E_{s-1,t}^r,$$

which is an isomorphism, for  $r = 2$  and  $s \geq 1$ . Suppose that the homotopy class  $\tilde{x} \in \mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2)$  is represented by the infinite cycle  $x \in E_{s,t}^2$ , and let  $y \in \hat{E}_{s+1,t}^2$  be the unique element with  $\partial^{h,2}(y) = x$ . Then, if  $y$  is an infinite cycle, there exists a homotopy class  $\tilde{y} \in \hat{\mathbb{H}}^{-q-1}(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2)$  represented by  $y$  such that  $\partial^h(\tilde{y}) = \tilde{x}$ ; compare [5, Theorem 2.5]. We now return to [27, Theorem 8.14]. The homotopy class  $V^{n-2}(\kappa)$  is represented by the unique generator of  $E_{2,3}^2$  which, in turn, is the image by the map  $\partial^{h,2}$  of the unique generator of  $\hat{E}_{3,3}^{h,2}$ . In loc. cit., the latter generator is given the name  $u_{n-1}t^{-2}e_3$  and proved to be an infinite cycle for  $n \geq 3$ . This shows that the image of the class  $V^{n-2}(\kappa)$  by the norm map

$$N: \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z}); \mathbb{F}_2) \rightarrow \text{TR}_5^n(\mathbb{Z}; 2, \mathbb{F}_2)$$

is zero as stated. We conclude that  $\kappa, \xi_{5,2}, \dots, \xi_{5,n-1}$  generate  $\text{TR}_5^n(\mathbb{Z}; 2)$ .

We know from Theorem 11 that  $\xi_{5,s}$  is annihilated by  $2^s$  and further claim that  $\kappa$  is annihilated by  $2^{n-1}$  and that, for some unit  $u \in \mathbb{Z}_2^*$ ,

$$2 \cdot (\xi_{5,2} + \dots + \xi_{5,n-1} + 4u\kappa) = 0.$$

This implies the statement of the theorem for  $q = 5$ . Indeed, the abelian group generated by  $\kappa, \xi_{5,2}, \dots, \xi_{5,n-1}$  and subject to the relations above is equal to

$$\mathbb{Z}/2^{n-1}\mathbb{Z} \cdot \kappa \oplus \bigoplus_{2 \leq s < n} \mathbb{Z}/2^{s-1}\mathbb{Z} \cdot (\xi_{5,s} + \dots + \xi_{5,n-1} + 4u\kappa)$$

and subjects onto  $\text{TR}_5^n(\mathbb{Z}; 2)$ . Hence, it suffices to show that the two groups have the same order. But this follows by an induction argument based on the exact sequence

$$0 \rightarrow \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z})) \rightarrow \mathrm{TR}_5^n(\mathbb{Z}; 2) \rightarrow \mathrm{TR}_5^{n-1}(\mathbb{Z}; 2) \rightarrow 0$$

and Proposition 15.

It remains to prove the claim. We first show that the class  $2^{n-1} \cdot \kappa$  is zero by induction on  $n \geq 2$ . The case  $n = 2$  is true, so we assume the statement for  $n - 1$  and prove it for  $n$ . We again use the exact sequence

$$0 \rightarrow \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z})) \rightarrow \mathrm{TR}_5^n(\mathbb{Z}; 2) \rightarrow \mathrm{TR}_5^{n-1}(\mathbb{Z}; 2) \rightarrow 0$$

and the calculation of the left-hand group in Proposition 15. The inductive hypothesis implies that the image of the class  $2^{n-2} \cdot \kappa$  by the right-hand map is zero, and hence, this class is in the image of the left-hand map. It follows that we can write

$$2^{n-2} \cdot \kappa = a \cdot \xi_{5,n-1} + b \cdot V^{n-2}(\kappa)$$

with  $a \in \mathbb{Z}/2^{n-2}\mathbb{Z}$  and  $b \in \mathbb{Z}/2\mathbb{Z}$ . We apply the Frobenius map

$$F: \mathrm{TR}_5^n(\mathbb{Z}; 2) \rightarrow \mathrm{TR}_5^{n-1}(\mathbb{Z}; 2)$$

to this equation. The image of the left-hand side is zero, by induction, and the image of the right-hand side is  $\bar{a} \cdot \xi_{5,n-2}$ , where  $\bar{a} \in \mathbb{Z}/2^{n-3}\mathbb{Z}$  is reduction of  $a$  modulo  $2^{n-3}$ . It follows that  $\bar{a}$  is zero, or equivalently, that  $a \in 2^{n-3}\mathbb{Z}/2^{n-2}\mathbb{Z}$ . This shows that  $2^{n-1} \cdot \kappa$  is zero as desired.

Finally, to prove the relation  $2 \cdot (\xi_{5,2} + \dots + \xi_{5,n-1} + 4u\kappa) = 0$ , we consider the following long-exact sequence

$$\dots \rightarrow \mathrm{TR}_6(\mathbb{S}; 2) \xrightarrow{1-F} \mathrm{TR}_6(\mathbb{S}; 2) \xrightarrow{\partial} K_5(\mathbb{S}_2; \mathbb{Z}_2) \xrightarrow{\mathrm{tr}} \mathrm{TR}_5(\mathbb{S}; 2) \xrightarrow{1-F} \mathrm{TR}_5(\mathbb{S}; 2) \rightarrow \dots$$

We know from Lemma 17 that the group  $K_5(\mathbb{S}_2; \mathbb{Z}_2)$  is a free  $\mathbb{Z}_2$ -module of rank one generated by the class  $4\kappa + \tau$ . Moreover, it follows from Theorem 11 that the left-hand map  $1 - F$  is surjective and that the kernel of the right-hand map  $1 - F$  is isomorphic to a free  $\mathbb{Z}_2$ -module of rank one generated by the element  $\Delta = (\Delta^{(n)})$  with  $\Delta^{(n)} = \xi_{5,2} + \dots + \xi_{5,n-1}$ . It follows that there exists a unit  $u \in \mathbb{Z}_2^*$  such that  $\Delta + u(4\kappa + \tau) = 0$  in  $\mathrm{TR}_5(\mathbb{S}; 2)$ . But then  $2(\Delta + 4u\kappa) = 0$ , since  $2(4\kappa + \tau) = 8\kappa$ . This completes the proof.  $\square$

**Corollary 19.** *The cokernel of the map induced by the Hurewicz map*

$$\ell: \mathrm{TR}_5^n(\mathbb{S}; 2) \rightarrow \mathrm{TR}_5^n(\mathbb{Z}; 2)$$

*is equal to  $\mathbb{Z}/2^v\mathbb{Z} \cdot \kappa$ , where  $v = v(n - 1)$  is the smaller of 3 and  $n - 1$ .*

*Proof.* It follows immediately from Theorem 18 that the cokernel of the map  $\ell$  is generated by the class of  $\kappa$ . Moreover, since the class

$$\xi_{5,1} + \dots + \xi_{5,n-1} + 4u\kappa$$

has order 2, it is also clear that the cokernel of the map  $\ell$  is annihilated by multiplication by 8. Hence, it will suffice to show that, for  $n = 4$ , the cokernel of the map  $\ell$  is not annihilated by 4, or equivalently, that the map  $\ell$  takes the class  $\rho \in \text{TR}_5^4(\mathbb{S}; 2)$  to zero. But this follows immediately from the structure of the spectral sequences that abuts  $\mathbb{H}_5(C_8, T(\mathbb{S}))$  and  $\mathbb{H}_5(C_8, T(\mathbb{Z}))$ .  $\square$

### 7 The Groups $\text{TR}_q^n(\mathbb{S}, I; 2)$

We again implicitly consider homotopy groups with  $\mathbb{Z}_2$ -coefficients. The Hurewicz map from the sphere spectrum  $\mathbb{S}$  to the Eilenberg MacLane spectrum  $\mathbb{Z}$  for the ring of integers induces a map of topological Hochschild  $\mathbb{T}$ -spectra

$$\ell: T(\mathbb{S}) \rightarrow T(\mathbb{Z}).$$

In [5, Appendix], Bökstedt and Madsen constructs a sequence of cyclotomic spectra

$$T(\mathbb{S}, I) \xrightarrow{i} T(\mathbb{S}) \xrightarrow{\ell} T(\mathbb{Z}) \xrightarrow{\partial} \Sigma T(\mathbb{S}, I)$$

such that the underlying sequence of  $\mathbb{T}$ -spectra is a cofibration sequence. As a consequence, the equivariant homotopy groups

$$\text{TR}_q^n(\mathbb{S}, I; p) = [S^q \wedge (\mathbb{T}/C_{p^{n-1}})_+, T(\mathbb{S}, I)]_{\mathbb{T}}$$

come equipped with maps

$$\begin{aligned} R: \text{TR}_q^n(\mathbb{S}, I; p) &\rightarrow \text{TR}_q^{n-1}(\mathbb{S}, I; p) && \text{(restriction)} \\ F: \text{TR}_q^n(\mathbb{S}, I; p) &\rightarrow \text{TR}_q^{n-1}(\mathbb{S}, I; p) && \text{(Frobenius)} \\ V: \text{TR}_q^{n-1}(\mathbb{S}, I; p) &\rightarrow \text{TR}_q^n(\mathbb{S}, I; p) && \text{(Verschiebung)} \\ d: \text{TR}_q^n(\mathbb{S}, I; p) &\rightarrow \text{TR}_{q+1}^n(\mathbb{S}, I; p) && \text{(Connes' operator)} \end{aligned}$$

and all maps in the long-exact sequence of equivariant homotopy groups induced by the cofibration sequence above,

$$\dots \rightarrow \text{TR}_q^n(\mathbb{S}, I; 2) \xrightarrow{i} \text{TR}_q^n(\mathbb{S}; 2) \xrightarrow{\ell} \text{TR}_q^n(\mathbb{Z}; 2) \xrightarrow{\partial} \text{TR}_{q-1}^n(\mathbb{S}, I; 2) \rightarrow \dots,$$

are compatible with restriction maps, Frobenius maps, Verschiebung maps, and Connes' operator. Moreover, this is a sequence of graded modules over the graded ring  $\text{TR}_*^n(\mathbb{S}; p)$ .

**Lemma 20.** *The following sequence is exact, for all  $n \geq 1$ .*



map is zero. Now,  $V(\bar{\lambda}) = V(\partial(\lambda)) = \partial(V(\lambda))$ , and by Proposition 18, the class  $V(\lambda) \in \text{TR}_3^2(\mathbb{Z}; 2)$  is either zero or equal to  $4\lambda$ . But  $\partial(4\lambda) = 4\partial(\lambda) = 4\bar{\lambda}$  which is zero, by Corollary 16. This proves the claim. The  $d^2$ -differential is now given by Lemma 5. We find that the  $E^3$ -term for  $s + t \leq 5$  takes the form

$$\begin{array}{cccccc}
 0 & & & & & \\
 0 & & 0 & & & \\
 \mathbb{Z}/8\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & & 0 & \\
 \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} \\
 \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & & 0 & & 0 & & 0 \\
 0 & & 0 & & 0 & & 0 & & 0 & & 0
 \end{array}$$

and, for degree reasons, this is also the  $E^\infty$ -term. The group  $E_{s,2}^3$  is generated by the class of  $\eta\tilde{\eta}z_s$ . The class of  $\bar{\lambda}z_s$  in  $E_{s,2}^3$  is equal to zero, if  $s$  is congruent to 0 or 1 modulo 4, and is equal to the class of  $\eta\tilde{\eta}z_s$ , if  $s$  is congruent to 2 or 3 modulo 4.

The spectral sequences for  $n \geq 3$  are similar with the only difference being the groups  $E_{s,3}^r$  with  $s > 0$ . In the case  $n = 3$ , the  $E^\infty$ -term for  $s + t \leq 5$  takes the form

$$\begin{array}{cccccc}
 0 & & & & & \\
 0 & & 0 & & & \\
 \mathbb{Z}/8\mathbb{Z} & & \mathbb{Z}/4\mathbb{Z} & & 2\mathbb{Z}/4\mathbb{Z} & \\
 \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} \\
 \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & & 0 & & 0 & & 0 \\
 0 & & 0 & & 0 & & 0 & & 0 & & 0
 \end{array}$$

and in the case  $n \geq 4$ , it takes the form

$$\begin{array}{cccccc}
 0 & & & & & \\
 0 & & 0 & & & \\
 \mathbb{Z}/8\mathbb{Z} & & \mathbb{Z}/8\mathbb{Z} & & \mathbb{Z}/4\mathbb{Z} & \\
 \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} \\
 \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & & 0 & & 0 & & 0 \\
 0 & & 0 & & 0 & & 0 & & 0 & & 0.
 \end{array}$$

We define  $\epsilon \in \mathbb{H}_5(C_4, T(\mathbb{S}, I))$  and  $\tilde{\rho} \in \mathbb{H}_5(C_8, T(\mathbb{S}, I))$  to be the unique homotopy classes that represent  $2\tilde{v}z_2$  and  $\tilde{v}z_2$ , respectively. We note that  $V(\epsilon) = 2\tilde{\rho}$ . We further define  $\bar{\kappa} = \partial(\kappa) \in \mathbb{H}_4(C_2, T(\mathbb{S}, I))$ .

**Proposition 22.** *The groups  $\mathbb{H}_q(C_{2^{n-1}}, T(\mathbb{S}, I))$  with  $q \leq 5$  are given by*

$$\begin{aligned}
 \mathbb{H}_0(C_{2^{n-1}}, T(\mathbb{S}, I)) &= 0, \\
 \mathbb{H}_1(C_{2^{n-1}}, T(\mathbb{S}, I)) &= \mathbb{Z}/2\mathbb{Z} \cdot V^{n-1}(\tilde{\eta}), \\
 \mathbb{H}_2(C_{2^{n-1}}, T(\mathbb{S}, I)) &= \mathbb{Z}/2\mathbb{Z} \cdot dV^{n-1}(\tilde{\eta}) \oplus \mathbb{Z}/2\mathbb{Z} \cdot V^{n-1}(\eta\tilde{\eta}), \\
 \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S}, I)) &= \mathbb{Z}/2\mathbb{Z} \cdot dV^{n-1}(\eta\tilde{\eta}) \oplus \mathbb{Z}/8\mathbb{Z} \cdot V^{n-1}(\tilde{v}), \\
 \mathbb{H}_4(C_{2^{n-1}}, T(\mathbb{S}, I)) &= \begin{cases} 0 & (n = 1) \\ \mathbb{Z}/2^v\mathbb{Z} \cdot dV^{n-1}(\tilde{v}) \oplus \mathbb{Z}/2\mathbb{Z} \cdot V^{n-2}(\bar{\kappa}) & (n \geq 2), \end{cases}
 \end{aligned}$$

$$\mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S}, I)) = \begin{cases} 0 & (n = 1) \\ \mathbb{Z}/2\mathbb{Z} \cdot d\bar{\kappa} & (n = 2) \\ \mathbb{Z}/2\mathbb{Z} \cdot dV(\bar{\kappa}) \oplus \mathbb{Z}/2\mathbb{Z} \cdot \epsilon & (n = 3) \\ \mathbb{Z}/2\mathbb{Z} \cdot dV^{n-2}(\bar{\kappa}) \oplus \mathbb{Z}/4\mathbb{Z} \cdot V^{n-4}(\bar{\rho}) & (n \geq 4), \end{cases}$$

where  $v = v(n - 1)$  is the smaller of 3 and  $n - 1$ .

*Proof.* The statement for  $q \leq 3$  follows immediately from the spectral sequence above since the generators given in the statement have the indicated orders. To prove the statement for  $q = 4$ , we first note that  $dV^{n-1}(\tilde{v})$  has order  $v(n - 1)$ . Indeed, the class  $\tilde{v}$  has order 8 and  $d\tilde{v} = 0$ . Moreover, the image of the map

$$\ell: \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S})) \rightarrow \mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{Z}))$$

does not contain the class  $V^{n-2}(\kappa)$ . Indeed, this follows immediately from the induced map of spectral sequences. It follows that  $V^{n-2}(\bar{\kappa})$  is a non-zero class of order 2 which is represented by the element  $\eta\tilde{\eta}_{z_2} = \bar{\lambda}_{z_2}$  in the  $E^\infty$ -term of the spectral sequence above. This proves the statement for  $q = 4$ . It remains to prove the statement for  $q = 5$ . It follows from [18, Proposition 4.4.1] that the element  $\eta\tilde{\eta}_{z_3} = \bar{\lambda}_{z_3}$  in the  $E^\infty$ -term of the spectral sequence above represents the class  $dV^{n-2}(\bar{\kappa})$ . Hence, this class is non-zero and has order 2. Moreover, the spectral sequence shows that the subgroup of  $\mathbb{H}_5(C_{2^{n-1}}, T(\mathbb{S}, I))$  generated by  $dV^{n-2}(\bar{\kappa})$  is a direct summand. This completes the proof.  $\square$

The following result was proved by Costeanu in [7, Proposition 2.6].

**Lemma 23.** *The map*

$$\ell: \text{TR}_1^n(\mathbb{S}; 2) \rightarrow \text{TR}_1^n(\mathbb{Z}; 2)$$

*takes the class  $\eta = \eta \cdot [1]_n$  to the class  $\xi_{1,1}$ .*

*Proof.* We temporarily write  $[1_{\mathbb{S}}]_n$  and  $[1_{\mathbb{Z}}]_n$  for the multiplicative unit elements of the graded rings  $\text{TR}_*^n(\mathbb{S}; 2)$  and  $\text{TR}_*^n(\mathbb{Z}; 2)$ , respectively. By [17, Proposition 2.7.1], the map  $\ell$  is a map of graded algebras over the graded ring given by the stable homotopy groups of spheres. Hence, it takes the class  $\eta \cdot [1_{\mathbb{S}}]_n$  to the class  $\eta \cdot [1_{\mathbb{Z}}]_n$ . Similarly, it is proved in [12, Corollary 6.4.1] that the cyclotomic trace map

$$\text{tr}: K_*(\mathbb{Z}) \rightarrow \text{TR}_*^n(\mathbb{Z}; 2)$$

is a map of graded algebras over the graded ring given by the stable homotopy groups of spheres. Hence, the class  $\eta \cdot [1_{\mathbb{Z}}]_n$  is equal to the image by the cyclotomic trace map of the class  $\eta \cdot 1_{\mathbb{Z}} \in K_1(\mathbb{Z})$ . The latter class is known to be equal to the generator  $\{-1\} \in K_1(\mathbb{Z})$ . It is proved in [18, Lemma 2.3.3] that the image by the cyclotomic trace map of the generator  $\{-1\}$  is equal to the class

$$d \log[-1]_n \in \text{TR}_1^n(\mathbb{Z}; 2).$$

To evaluate this class, we recall from [17, Theorem F] that the ring  $\text{TR}_0^n(\mathbb{Z}; 2)$  is canonically isomorphic to the ring of Witt vectors  $W_n(\mathbb{Z})$ . One readily verifies that

$$[-1]_n = -[1]_n + V([1]_{n-1})$$

by evaluating the ghost coordinates. It follows that  $d[-1]_n = dV([1]_{n-1})$ , and since the class  $[-1]_n$  is a square root of 1, we find

$$\begin{aligned} d \log[-1]_n &= [-1]_n d[-1]_n = (-[1]_n + V([1]_{n-1})) \cdot dV([1]_{n-1}) \\ &= dV([1]_{n-1}) + V(FdV([1]_{n-1})) = dV([1]_{n-1}) + V(\eta \cdot [1]_{n-1}). \end{aligned}$$

But by Theorem 11, this is the class  $\xi_{1,1}$  as stated. □

*Remark 24.* It follows from Lemma 23 that  $\ell(V^s(\eta)) = \sum_{t \geq s} 2^{t-1} \xi_{1,t}$ , if  $s \geq 2$ .

At present, we do not know the precise value of the map

$$\ell: \text{TR}_q^n(\mathbb{S}; 2) \rightarrow \text{TR}_q^n(\mathbb{Z}; 2)$$

for  $q \geq 3$ . However, we have the following result. We define  $\tilde{\eta} \in \text{TR}_1^n(\mathbb{S}, I; 2)$  to be the unique class such that  $i(\tilde{\eta}) = \eta - \xi_{1,1}$ . The class  $\tilde{v}$  that appears in the statement will be defined in the course of the proof. It would be desirable to better understand this class. In particular, we do not know the values of  $\eta^2 \tilde{\eta}$  or  $Fd\tilde{v}$ .

**Theorem 25.** *The groups  $\text{TR}_q^n(\mathbb{S}, I; 2)$  with  $q \leq 3$  are given by*

$$\begin{aligned} \text{TR}_0^n(\mathbb{S}, I; 2) &= 0, \\ \text{TR}_1^n(\mathbb{S}, I; 2) &= \bigoplus_{0 \leq s < n} \mathbb{Z}/2\mathbb{Z} \cdot V^s(\tilde{\eta}), \\ \text{TR}_2^n(\mathbb{S}, I; 2) &= \bigoplus_{0 \leq s < n} (\mathbb{Z}/2\mathbb{Z} \cdot V^s(\eta\tilde{\eta}) \oplus \mathbb{Z}/2\mathbb{Z} \cdot dV^s(\tilde{\eta})), \\ \text{TR}_3^n(\mathbb{S}, I; 2) &= \bigoplus_{0 \leq s < n} \mathbb{Z}/8\mathbb{Z} \cdot V^s(\tilde{v}) \oplus \bigoplus_{1 \leq s < n} \mathbb{Z}/2\mathbb{Z} \cdot dV^s(\eta\tilde{\eta}), \end{aligned}$$

and the group  $\text{TR}_4^n(\mathbb{S}, I; 2)$  is generated by  $dV^s(\tilde{v})$  with  $0 \leq s < n$ . Moreover, the restriction map takes  $\tilde{\eta}$  to  $\tilde{\eta}$  and  $\tilde{v}$  to  $\tilde{v}$ , and the Frobenius map takes both  $\tilde{\eta}$  and  $\tilde{v}$  to zero and takes  $d\tilde{\eta}$  to  $d\tilde{\eta}$ . The class  $d(\eta\tilde{\eta}) = \eta d(\tilde{\eta})$  is zero.

*Proof.* The statement for  $q = 0$  follows immediately from Theorem 11 and Proposition 18. For  $q = 1$ , Lemma 13 shows that the map  $i: \text{TR}_1^n(\mathbb{S}, I; 2) \rightarrow \text{TR}_1^n(\mathbb{S}; 2)$  is injective, and Lemma 23 shows that the class  $\eta - \xi_{1,1}$  is in the image. As said above, we define  $\tilde{\eta} \in \text{TR}_1^n(\mathbb{S}, I; 2)$  to be the unique class with  $i(\tilde{\eta}) = \eta - \xi_{1,1}$ . The statement for  $q = 1$  now follows immediately from Theorem 11 and Proposition 18. For  $q = 2$ , a similar argument shows that the group  $\text{TR}_2^n(\mathbb{S}, I; 2)$  contains the subgroup

$$\mathrm{TR}_2^n(\mathbb{S}, I; 2)' = \bigoplus_{0 \leq s < n} \mathbb{Z}/2\mathbb{Z} \cdot V^s(\eta\tilde{\eta}) \oplus \bigoplus_{1 \leq s < n} \mathbb{Z}/2\mathbb{Z} \cdot dV^s(\tilde{\eta}),$$

which maps isomorphically onto the image of  $i: \mathrm{TR}_2^n(\mathbb{S}, I; 2) \rightarrow \mathrm{TR}_2^n(\mathbb{S}; 2)$ , and Lemma 16 shows that the kernel of the latter map is  $\mathbb{Z}/2\mathbb{Z} \cdot \bar{\lambda}$ . Therefore, to prove the statement for  $q = 2$ , it remains to prove that  $d\tilde{\eta} = \bar{\lambda}$ . We have already proved this equality, for  $n = 1$ , in the discussion preceding Proposition 22. It follows that the iterated restriction map  $R^{n-1}: \mathrm{TR}_2^n(\mathbb{S}, I; 2) \rightarrow \mathrm{TR}_2^1(\mathbb{S}, I; 2)$  takes the class  $d\tilde{\eta}$  to the class  $\bar{\lambda}$ . Since the kernel of this map is equal to the subgroup  $\mathrm{TR}_2^n(\mathbb{S}, I; 2)'$ , it suffices to show that the class  $i(d\tilde{\eta} - \bar{\lambda}) \in \mathrm{TR}_2^n(\mathbb{S}; 2)$  is zero. We have

$$i(d\tilde{\eta} - \bar{\lambda}) = i(d\tilde{\eta}) = d(i(\tilde{\eta})) = d\eta - d\xi_{1,1}.$$

The class  $d\eta$  is zero, since  $\eta$  is in the image of the cyclotomic trace map, and we proved in Theorem 11 that  $d\xi_{1,1}$  is zero. The statement for  $q = 2$  follows. It also follows that  $F(d\tilde{\eta}) = d\tilde{\eta}$ , since  $d\tilde{\eta} = \partial(\lambda)$  and  $\lambda$  is in the image of the cyclotomic trace map.

We next prove the statement for  $q = 3$ . By Lemma 20, the sequences

$$0 \rightarrow \mathbb{H}_3(C_{2^{n-1}}, T(\mathbb{S}, I)) \xrightarrow{N} \mathrm{TR}_3^n(\mathbb{S}, I; 2) \xrightarrow{R} \mathrm{TR}_3^{n-1}(\mathbb{S}, I; 2) \rightarrow 0$$

are exact. The left-hand group was evaluated in Proposition 22. To complete the proof, we inductively construct classes

$$\tilde{v} = \tilde{v}_n \in \mathrm{TR}_3^n(\mathbb{S}, I; 2) \quad (n \geq 1)$$

such that  $R(\tilde{v}_n) = \tilde{v}_{n-1}$  and  $F(\tilde{v}_n) = 0$ , and such that  $\tilde{v}_1$  is the class  $\tilde{v}$  already defined. By Proposition 13 and Corollary 16, we have a short-exact sequence

$$0 \rightarrow \mathrm{TR}_3^n(\mathbb{S}, I; 2) \xrightarrow{i} \mathrm{TR}_3^n(\mathbb{S}; 2) \xrightarrow{\ell} \mathrm{TR}_3^n(\mathbb{Z}; 2)' \rightarrow 0,$$

where the right-hand group is the index two subgroup of  $\mathrm{TR}_3^n(\mathbb{Z}; 2)$  defined by

$$\mathrm{TR}_3^n(\mathbb{Z}; 2)' = \bigoplus_{1 \leq s < n} \mathbb{Z}/2^{s+1}\mathbb{Z} \cdot \xi_{3,s}.$$

To define the class  $\tilde{v}_2$ , we first note that  $\ell(v) = a_1\xi_{3,1}$ , where  $a_1 \in (\mathbb{Z}/4\mathbb{Z})^*$  is a unit, and choose a unit  $\tilde{a}_1 \in (\mathbb{Z}/8\mathbb{Z})^*$  whose reduction modulo 4 is  $a_1$ . Then, we have  $\ell(v - \tilde{a}_1\xi_{3,1}) = 0$  and  $F(v - \tilde{a}_1\xi_{3,1}) = (1 - \tilde{a}_1)v$ . We choose  $b_1 \in \mathbb{Z}/8\mathbb{Z}$  such that  $2b_1 = \tilde{a}_1 - 1$  and define  $\tilde{v}_2$  to be the unique class such that

$$i(\tilde{v}_2) = v - \tilde{a}_1\xi_{3,1} + b_1V(v).$$

Then  $R(\tilde{v}_2) = \tilde{v}_1$  and  $F(\tilde{v}_2) = 0$  as desired.



We next define the class  $\tilde{v}_3$ . The image of  $\tilde{v}_2$  by the composition

$$\mathrm{TR}_3^2(\mathbb{S}, I; 2) \xrightarrow{i} \mathrm{TR}_3^2(\mathbb{S}; 2) \xrightarrow{S} \mathrm{TR}_3^3(\mathbb{S}; 2) \xrightarrow{\ell} \mathrm{TR}_3^3(\mathbb{Z}; 2)'$$

is equal to  $a_2\xi_{3,2}$ , for some  $a_2 \in \mathbb{Z}/8\mathbb{Z}$ . We claim that, in fact,  $a_2 \in 4\mathbb{Z}/8\mathbb{Z}$ . Indeed, since  $F(\tilde{v}_2) = 0$ , we have  $F(a_2\xi_{3,2}) = 0$ . But  $F(\xi_{3,2}) = \xi_{3,1}$  which shows that the modulo 4 reduction of  $a_2$  is zero as claimed. We let  $b_2 \in \mathbb{Z}/2\mathbb{Z}$  be the unique element such that  $4b_2 = a_2$  and define  $\tilde{v}_3$  to be the unique class such that

$$i(\tilde{v}_3) = \begin{cases} S(i(\tilde{v}_2)) + b_2(4\xi_{3,2} + dV^2(\eta^2) + 2V^2(v)) & \text{if } 4\xi_{3,1} = dV(\eta^2) \\ S(i(\tilde{v}_2)) + b_2(4\xi_{3,2} + dV^2(\eta^2)) & \text{if } 4\xi_{3,1} = \eta^2\xi_{1,1}. \end{cases}$$

The sum on the right-hand side is in the kernel of  $\ell$ , since both  $\ell(\eta^2) \in \mathrm{TR}_2^1(\mathbb{Z}; 2)$  and  $\ell(v) \in \mathrm{TR}_3^1(\mathbb{Z}; 2)$  are zero. We also have  $R(\tilde{v}_3) = \tilde{v}_2$  and  $F(\tilde{v}_3) = 0$  as desired. Indeed, if  $4\xi_{3,1} = dV(\eta^2)$ , then

$$\begin{aligned} i(F(\tilde{v}_3)) &= F(S(i(\tilde{v}_2)) + b_2(4\xi_{3,2} + dV^2(\eta^2) + 2V^2(v))) \\ &= S(i(F(\tilde{v}_2))) + b_2(4\xi_{3,1} + dV(\eta^2) + V(\eta^3) + 4V(v)), \end{aligned}$$

and if  $4\xi_{3,1} = \eta^2\xi_{3,1}$ , then

$$\begin{aligned} i(F(\tilde{v}_3)) &= F(S(i(\tilde{v}_2)) + b_2(4\xi_{3,2} + dV^2(\eta^2))) \\ &= S(i(F(\tilde{v}_2))) + b_2(4\xi_{3,1} + dV(\eta^2) + V(\eta^3)), \end{aligned}$$

and in either case, the sum is zero.

Finally, we let  $n \geq 4$  and assume that the class  $\tilde{v}_{n-1}$  has been defined. We find as before that the image of the class  $\tilde{v}_{n-1}$  by the composition

$$\mathrm{TR}_3^{n-1}(\mathbb{S}, I; 2) \xrightarrow{i} \mathrm{TR}_3^{n-1}(\mathbb{S}; 2) \xrightarrow{S} \mathrm{TR}_3^n(\mathbb{S}; 2) \xrightarrow{\ell} \mathrm{TR}_3^n(\mathbb{Z}; 2)'$$

is equal to  $a_{n-1}\xi_{3,n-1}$  with  $a_{n-1} \in 2^{n-1}\mathbb{Z}/2^n\mathbb{Z}$  and define  $\tilde{v}_n$  to be the unique class whose image by the map  $i$  is equal to

$$i(\tilde{v}_n) = S(i(\tilde{v}_{n-1})) - a_{n-1}\xi_{3,n-1}.$$

Then  $R(\tilde{v}_n) = \tilde{v}_{n-1}$  and  $F(\tilde{v}_n) = 0$ , since  $2^{n-1}\xi_{3,n-2} = 0$ , for  $n \geq 4$ .

It remains to prove that the group  $\mathrm{TR}_4^n(\mathbb{S}, I; 2)$  is generated by the homotopy classes  $dV^s(\tilde{v})$  with  $0 \leq s < n$ . The sequence

$$\mathbb{H}_4(C_{2^{n-1}}, T(\mathbb{S}, I)) \rightarrow \mathrm{TR}_4^n(\mathbb{S}, I; 2) \rightarrow \mathrm{TR}_4^{n-1}(\mathbb{S}, I; 2) \rightarrow 0,$$

which is exact by Corollary 21, together with Proposition 22 show that  $\mathrm{TR}_4^n(\mathbb{S}, I; 2)$  is generated by the classes  $dV^s(\tilde{v})$ ,  $1 \leq s < n$ , and  $\bar{\kappa}$ . Indeed, since the boundary map

$$\partial: \mathrm{TR}_5^n(\mathbb{Z}; 2) \rightarrow \mathrm{TR}_4^n(\mathbb{S}, I; 2)$$

commutes with the Verschiebung, it follows that  $V^{n-1}(\bar{\kappa}) = c\bar{\kappa}$ , for some  $c \in 2^{n-1}\mathbb{Z}$ . Hence, it suffices to show that there exists a class  $x_n \in \mathrm{TR}_3^n(\mathbb{S}, I; 2)$  with  $dx_n = \bar{\kappa}$ . The statement for  $n = 1$  is trivial, since the group  $\mathrm{TR}_4^1(\mathbb{S}, I; 2)$  is zero. We postpone the proof of the statement for  $n = 2$  to Lemma 26 and here prove the induction step. So we let  $n \geq 3$  and assume that there exists a class  $x_{n-1} \in \mathrm{TR}_3^{n-1}(\mathbb{S}, I; 2)$  with  $dx_{n-1} = \bar{\kappa}$ . We use Corollary 21 to choose a class  $x'_n \in \mathrm{TR}_3^n(\mathbb{S}, I; 2)$  with  $R(x'_n) = x_{n-1}$ . Then the exact sequence above and Proposition 22 show that

$$dx'_n = \bar{\kappa} + adV^{n-1}(\tilde{v}) + bV^{n-1}(\kappa) = adV^{n-1}(\tilde{v}) + (1 + bc)\bar{\kappa},$$

for some integers  $a$  and  $b$ . Since  $1 + bc$  is a 2-adic unit, the class

$$x_n = (1 + bc)^{-1}(x'_n - aV^{n-1}(\tilde{v}))$$

is well-defined and satisfies  $dx_n = \bar{\kappa}$  as desired. □

One wonders whether the class  $\tilde{v}$ , which was defined in the proof, satisfies that  $d\tilde{v} = \bar{\kappa}$ . This would imply that  $Fd\tilde{v} = d\tilde{v}$ , since  $\kappa$  is in the image of the cyclotomic trace map.

The following result was used in the proof of Theorem 25.

**Lemma 26.** *Connes' operator*

$$d: \mathrm{TR}_3^2(\mathbb{S}, I; 2) \rightarrow \mathrm{TR}_4^2(\mathbb{S}, I; 2)$$

is surjective.

*Proof.* The groups  $\mathrm{TR}_q^2(\mathbb{S}, I; 2)$  for  $q \leq 5$  are given by

$$\begin{aligned} \mathrm{TR}_0^2(\mathbb{S}, I; 2) &= 0, \\ \mathrm{TR}_1^2(\mathbb{S}, I; 2) &= \mathbb{Z}/2\mathbb{Z} \cdot \tilde{\eta} \oplus \mathbb{Z}/2\mathbb{Z} \cdot V(\tilde{\eta}), \\ \mathrm{TR}_2^2(\mathbb{S}, I; 2) &= \mathbb{Z}/2\mathbb{Z} \cdot d\tilde{\eta} \oplus \mathbb{Z}/2\mathbb{Z} \cdot dV(\tilde{\eta}) \oplus \mathbb{Z}/2\mathbb{Z} \cdot \eta\tilde{\eta} \oplus \mathbb{Z}/2\mathbb{Z} \cdot V(\eta\tilde{\eta}), \\ \mathrm{TR}_3^2(\mathbb{S}, I; 2) &= \mathbb{Z}/2\mathbb{Z} \cdot dV(\eta\tilde{\eta}) \oplus \mathbb{Z}/8\mathbb{Z} \cdot \tilde{v} \oplus \mathbb{Z}/8\mathbb{Z} \cdot V(\tilde{v}), \\ \mathrm{TR}_4^2(\mathbb{S}, I; 2) &= \mathbb{Z}/2\mathbb{Z} \cdot \bar{\kappa} \oplus \mathbb{Z}/2\mathbb{Z} \cdot dV(\tilde{v}), \\ \mathrm{TR}_5^2(\mathbb{S}, I; 2) &= 0. \end{aligned}$$

Hence, the lemma is equivalent to the statement that in the spectral sequence

$$E_{s,t}^2 = H_s(C_2, \mathrm{TR}_t^2(\mathbb{S}, I; 2)) \Rightarrow \mathbb{H}_{s+t}(C_2, TR^2(\mathbb{S}, I)),$$

the  $d^2$ -differential  $d^2: E_{3,3}^2 \rightarrow E_{1,4}^2$  is surjective. We first argue that this is equivalent to the statement that  $\mathbb{H}_5(C_2, TR^2(\mathbb{S}, I))$  has order 4. The elements  $\bar{\kappa}z_0$  and  $dV(\tilde{v})z_0$

in  $E_{0,4}^2$  are infinite cycles and represent the homotopy classes  $V^2(\bar{\kappa})$  and  $2dV^2(\tilde{\nu})$  of  $\mathbb{H}_4(C_2, TR^2(\mathbb{S}, I))$ . We claim that these classes are non-zero and generate a subgroup of order 4. To see this, we consider the norm maps from Proposition 4,

$$\mathbb{H}_4(C_2, TR^2(\mathbb{S}, I)) \xrightarrow{N_2} TR_4^3(\mathbb{S}, I; 2) \xleftarrow{N_1} \mathbb{H}_4(C_4, T(\mathbb{S}, I)).$$

It will suffice to show that the subgroup of the middle group generated by the images of the classes  $V^2(\bar{\kappa})$  and  $dV^2(\tilde{\nu})$  has order 4. This subgroup is equal to the subgroup generated by the images of the classes  $V^2(\bar{\kappa})$  and  $dV^2(\tilde{\nu})$  of the right-hand group. The right-hand map is injective, since  $TR_5^2(\mathbb{S}, I; 2)$  is zero. (The left-hand map is also injective, since  $TR_5^1(\mathbb{S}, I; 2)$  is zero.) Hence, it suffices to show that the subgroup of the right-hand group generated by the classes  $V^2(\bar{\kappa})$  and  $dV^2(\tilde{\nu})$  has order 4. But this is proved in Proposition 22. The claim follows. We conclude that in the spectral sequence under consideration, the differentials

$$d^r: E_{r,5-r}^r \rightarrow E_{0,4}^r$$

are zero, for all  $r \geq 2$ . It follows that the groups  $E_{0,5}^\infty, E_{2,3}^\infty, E_{3,2}^\infty, E_{4,1}^\infty$ , and  $E_{5,0}^\infty$  have orders 0, 2, 2, 0, and 0, respectively, and that for all  $r \geq 3$ , the differentials

$$d^r: E_{r+1,4-r}^r \rightarrow E_{1,4}^r$$

are zero. We conclude that the differential  $d^2: E_{3,3}^2 \rightarrow E_{1,4}^2$  is surjective if and only if the group  $\mathbb{H}_5(C_2, TR^2(\mathbb{S}, I))$  has order 4.

The order of the group  $\mathbb{H}_5(C_2, TR^2(\mathbb{S}, I))$  is divisible by 4 and to show that it is equal to 4, we consider the following diagram with exact rows and columns:

$$\begin{array}{ccccc} TR_7^1(\mathbb{S}; 2) & \xrightarrow{0} & \mathbb{H}_6(C_2, TR^2(\mathbb{S}; 2)) & \longrightarrow & TR_6^3(\mathbb{S}; 2) \\ \downarrow & & \downarrow \ell & & \downarrow \\ TR_7^1(\mathbb{Z}; 2) & \xrightarrow{\delta''} & \mathbb{H}_6(C_2, TR^2(\mathbb{Z}; 2)) & \longrightarrow & TR_6^3(\mathbb{Z}; 2) \\ \downarrow & & \downarrow \partial & & \downarrow \\ TR_6^1(\mathbb{S}, I; 2) & \xrightarrow{\delta'} & \mathbb{H}_5(C_2, TR^2(\mathbb{S}, I)) & \longrightarrow & TR_5^3(\mathbb{S}, I; 2) \\ \downarrow & & \downarrow & & \downarrow \\ TR_6^1(\mathbb{S}; 2) & \xrightarrow{0} & \mathbb{H}_5(C_2, TR^2(\mathbb{S}; 2)) & \longrightarrow & TR_5^3(\mathbb{S}; 2). \end{array}$$

It follows from Theorems 11 and 18 that the group  $TR_5^3(\mathbb{S}, I; 2)$  is equal to  $\mathbb{Z}/2\mathbb{Z} \cdot 2\xi_{5,2}$ . Hence, it will suffice to show that the image of the map  $\delta'$  has order at most 2. Since the lower left-hand horizontal map in the diagram is zero, we conclude that

the image of the map  $\delta'$  is contained in the image of the map  $\partial$ . Therefore, it suffices to show that the image of the map  $\partial$  has order at most 2.

The group  $\text{TR}_6^3(\mathbb{Z}; 2)$  is zero by Proposition 13 and the group  $\text{TR}_7^1(\mathbb{Z}; 2)$  is cyclic of order 4. It follows that the group  $\mathbb{H}_6(C_2, \text{TR}^2(\mathbb{Z}; 2))$  is cyclic and has order either 0, 2, or 4. If the order is either 0 or 2, we are done, so assume that the order is 4. We must show that 2 times a generator is contained in the image of the map  $\ell$  in the diagram. To this end, we consider the diagram

$$\begin{array}{ccc} \mathbb{H}_6(C_4, \text{TR}^2(\mathbb{S}; 2)) & \xrightarrow{F} & \mathbb{H}_6(C_2, \text{TR}^2(\mathbb{S}; 2)) \\ \downarrow \ell & & \downarrow \ell \\ \mathbb{H}_6(C_4, \text{TR}^2(\mathbb{Z}; 2)) & \xrightarrow{F} & \mathbb{H}_6(C_2, \text{TR}^2(\mathbb{Z}; 2)). \end{array}$$

We first show that the lower horizontal map  $F$  is surjective. The assumption that the lower right-hand group has order 4 implies that a generator of this group is represented in the spectral sequence

$$E_{s,t}^2 = H_s(C_2, \text{TR}_t^2(\mathbb{Z}; 2)) \Rightarrow \mathbb{H}_{s+t}(C_2, \text{TR}^2(\mathbb{Z}; 2))$$

by the element  $\lambda z_3 \in E_{3,3}^2$ . This element is the image by the map of spectral sequences induced by the map  $F$  of the element  $\lambda z_3 \in E_{3,3}^2$  in the spectral sequence

$$E_{s,t}^2 = H_s(C_4, \text{TR}_t^2(\mathbb{Z}; 2)) \Rightarrow \mathbb{H}_{s+t}(C_4, \text{TR}^2(\mathbb{Z}; 2)).$$

We must show that the latter element  $\lambda z_3$  is an infinite cycle. For degree reasons, the only possible non-zero differential is  $d^3: E_{3,3}^3 \rightarrow E_{0,5}^3$ . The target group is equal to  $\mathbb{Z}/2\mathbb{Z} \cdot \kappa z_0$ , and the generator  $\kappa z_0$  represents the homotopy class  $V^2(\kappa)$  in  $\mathbb{H}_5(C_4, \text{TR}^2(\mathbb{Z}; 2))$ . To see that this class is non-zero, we consider the norm maps

$$\mathbb{H}_4(C_4, \text{TR}^2(\mathbb{Z}; 2)) \xrightarrow{N_2} \text{TR}_5^4(\mathbb{Z}; 2) \xleftarrow{N_1} \mathbb{H}_4(C_8, T(\mathbb{Z})).$$

We may instead prove that the image of the class  $V^2(\kappa)$  by the left-hand map is non-zero. This image class, in turn, is equal to the image of the class  $V^2(\kappa)$  by the right-hand map which is injective since  $\text{TR}_6^3(\mathbb{Z}; 2)$  is zero. Now, Proposition 15 shows that the class  $V^2(\kappa)$  in the right-hand group is non-zero. We conclude that the lower horizontal map  $F$  in the square diagram is surjective as stated.

Finally, we show that the image of the left-hand vertical map  $\ell$  in the square diagram contains two times the homotopy class represented by the element  $\lambda z_3$ . In fact, the image of the composition

$$\mathbb{H}_5(C_4, T(\mathbb{S})) \xrightarrow{S} \mathbb{H}_5(C_4, \text{TR}^2(\mathbb{S}; 2)) \xrightarrow{\ell} \mathbb{H}_5(C_4, \text{TR}^2(\mathbb{Z}; 2))$$

of the Segal-tom Dieck splitting and the map  $\ell$  contains 2 times the class represented by  $\lambda_{z_3}$ . Indeed, by Proposition 10, the element  $\nu_{z_3} \in E_{3,3}^2$  of the spectral sequence

$$E_{s,t}^2 = H_s(C_4, \text{TR}_t^1(\mathbb{S}; 2)) \Rightarrow \mathbb{H}_{s+t}(C_4, T(\mathbb{S}))$$

is an infinite cycle whose image by the map of spectral sequence induced by the composition of the maps  $S$  and  $\ell$  is equal  $2\lambda_{z_3} \in E_{3,3}^2 = \mathbb{Z}/4\mathbb{Z} \cdot \lambda_{z_3}$ . This completes the proof. □

### 8 The Groups $\text{Wh}_q^{\text{Top}}(S^1)$ for $q \leq 3$

In this section, we complete the proof of Theorem 1 of the Introduction. It follows from [15, Theorem 1.2] that the odd-primary torsion subgroup of  $\text{Wh}_q^{\text{Top}}(S^1)$  is zero, for  $q \leq 3$ . Hence, it suffices to consider the homotopy groups with  $\mathbb{Z}_2$ -coefficients. We implicitly consider homotopy groups with  $\mathbb{Z}_2$ -coefficients.

As we explained in the introduction, there is a long-exact sequence

$$\dots \rightarrow \text{Wh}_q^{\text{Top}}(S^1) \rightarrow \widetilde{\text{TR}}_q(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2) \xrightarrow{1-F} \widetilde{\text{TR}}_q(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2) \rightarrow \dots$$

where the middle and on the right-hand terms are the cokernel of the assembly map

$$\alpha: \text{TR}_q(\mathbb{S}; I; 2) \oplus \text{TR}_{q-1}(\mathbb{S}; I; 2) \rightarrow \text{TR}_q(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2).$$

Moreover, since the groups  $\text{TR}_q^n(\mathbb{S}, I; 2)$  are finite, for all integers  $q$  and  $n \geq 1$ , the limit system  $\{\text{TR}_q^n(\mathbb{S}, I; 2)\}$  satisfies the Mittag-Leffler condition, and Corollary 3 then shows that the same holds for the limit system  $\{\widetilde{\text{TR}}_q^n(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2)\}$ . It follows that, for all integers  $q$ , the canonical map

$$\widetilde{\text{TR}}_q(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2) \rightarrow \lim_n \widetilde{\text{TR}}_q^n(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2)$$

is an isomorphism. Finally, Theorem 2 expresses the right-hand side in terms of the groups  $\text{TR}_q^m(\mathbb{S}, I; 2)$  which we evaluated in Theorem 25, for  $q \leq 3$ .

**Theorem 27.** *The groups  $\text{Wh}_0^{\text{Top}}(S^1)$  and  $\text{Wh}_1^{\text{Top}}(S^1)$  are zero.*

*Proof.* We first note that, as an immediate consequence of Theorems 2 and 25, the group  $\widetilde{\text{TR}}_0(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2)$  is zero. Moreover, we showed in Theorem 25 that the Frobenius map  $F: \text{TR}_1^m(\mathbb{S}, I; 2) \rightarrow \text{TR}_1^{m-1}(\mathbb{S}, I; 2)$  is zero, and hence,

$$1 - F: \widetilde{\text{TR}}_1(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2) \rightarrow \widetilde{\text{TR}}_1(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2),$$

is the identity map. This shows that the group  $\text{Wh}_0^{\text{Top}}(S^1)$  is zero as stated. To prove that  $\text{Wh}_1^{\text{Top}}(S^1)$  is zero, it remains to prove that the map

$$1 - F: \widetilde{\text{TR}}_2(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2) \rightarrow \widetilde{\text{TR}}_2(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2),$$

is surjective. So let  $\omega = (\omega^{(n)})$  be an element on the right-hand side. We find an element  $\omega' = (\omega'^{(n)})$  such that  $(R - F)(\omega') = \omega$ . By Theorem 2, we can write  $\omega^{(n)}$  uniquely as a sum

$$\sum_{j \in \mathbb{Z} \setminus \{0\}} (a_{0,j}^{(n)} [x]_n^j + b_{0,j}^{(n)} [x]_n^j d \log[x]_n) + \sum_{\substack{1 \leq s < n \\ j \in \mathbb{Z} \setminus 2\mathbb{Z}}} (V^s(a_{s,j}^{(n)} [x]_{n-s}^j) + dV^s(b_{s,j}^{(n)} [x]_{n-s}^j))$$

with  $a_{s,j}^{(n)} \in \text{TR}_2^{n-s}(\mathbb{S}, I; 2)$  and  $b_{s,j}^{(n)} \in \text{TR}_1^{n-s}(\mathbb{S}, I; 2)$ . We first consider the four types of summands separately.

First, if  $\omega^{(n)} = V^s(a^{(n)}[x]^j)$  with  $s \geq 1$ , we let  $\omega' = \omega$ . Then

$$(R - F)(\omega'^{(n+1)}) = (R - F)(V^s(a^{(n+1)}[x]^j)) = V^s(a^{(n)}[x]^j),$$

since  $FV = 2$  and  $2a^{(n)} = 0$ . We note that here  $j$  may be any integer.

Second, if  $\omega^{(n)} = dV^s(b^{(n)}[x]^j)$ , where  $j$  and  $s \geq 1$  are integers, we define

$$\omega'^{(n)} = - \sum_{s \leq r < n-1} dV^{r+1}(b^{(n-1-r+s)}[x]^j) - \sum_{s \leq r < n} V^r(\eta b^{(n-r+s)}[x]^j).$$

Then we have  $R(\omega'^{(n+1)}) = \omega'^{(n)}$  and

$$\begin{aligned} (R - F)(\omega'^{(n+1)}) &= - \sum_{s \leq r < n-1} dV^{r+1}(b^{(n-1-r+s)}[x]^j) - \sum_{s \leq r < n} V^r(\eta b^{(n-r+s)}[x]^j) \\ &\quad + \sum_{s \leq r < n} dV^r(b^{(n-r+s)}[x]^j) + \sum_{s \leq r < n} V^r(\eta b^{(n-r+s)}[x]^j) \\ &= dV^s(b^{(n)}[x]^j) \end{aligned}$$

as desired.

Third, if  $\omega^{(n)} = b^{(n)}[x]^j d \log[x]$ , we let  $\omega' = \omega$ . Then  $(R - F)(\omega'^{(n+1)}) = \omega^{(n)}$ , since  $F(b^{(n)}) = 0$ .

Fourth, we consider the case  $\omega^{(n)} = a^{(n)}[x]^j$ . Then  $a^{(n)} \in \text{TR}_2^n(\mathbb{S}, I; 2)$  and we showed in Theorem 25 that this group is an  $\mathbb{F}_2$ -vector space with a basis given by the classes  $V^s(\eta\tilde{\eta})$  and  $dV^s(\eta\tilde{\eta})$ , where  $0 \leq s < n$ . If  $a^{(n)} = V^s(\eta\tilde{\eta})$  with  $0 \leq s < n$ , then we let  $\omega' = \omega$ . Then  $(R - F)(\omega'^{(n+1)}) = \omega^{(n)}$ , since  $F(\tilde{\eta}) = 0$ . Next, suppose that  $a^{(n)} = dV^s(\tilde{\eta})$  with  $1 \leq s < n$ . Then

$$dV^s(\tilde{\eta})[x]^j = dV^s(\tilde{\eta}[x]^{2^s j}) - jV^s(\tilde{\eta})[x]^j d \log[x].$$

and we have already considered the two terms on the right-hand side. Hence, also in this case, there exists  $\omega'$  such that  $(R - F)(\omega'^{(n+1)}) = \omega^{(n)}$ . Similarly, in the remaining case  $\omega^{(n)} = (d\tilde{\eta})[x]^j$ , the calculation

$$\begin{aligned} (R - F)(dV(\tilde{\eta}[x]^j)) &= dV(\tilde{\eta}[x]^j) - d(\tilde{\eta}[x]^j) - \eta\tilde{\eta}[x]^j \\ &= dV(\tilde{\eta}[x]^j) - (d\tilde{\eta})[x]^j + j\tilde{\eta}[x]^j d \log[x] - \eta\tilde{\eta}[x]^j \end{aligned}$$

shows that there exists  $\omega'$  such that  $(R - F)(\omega'^{(n+1)}) = \omega^{(n)}$ . Indeed, we have already considered  $dV(\tilde{\eta}[x]^j)$ ,  $\tilde{\eta}[x]^j d \log[x]$ , and  $\eta\tilde{\eta}[x]^j$ .

Finally, we can write every element  $\omega = (\omega^{(n)})$  of  $\widetilde{\text{TR}}_2(\mathbb{S}[x]^{\pm 1}, I[x^{\pm 1}]; 2)$  as a series  $\omega = \sum_{i \in I} \omega_i$ , where each  $\omega_i$  is an element of the one of the four types considered above, and where, for every  $n \geq 1$ , all but finitely many of the  $\omega_i^{(n)}$  are zero. Now, for every  $i \in I$ , we have constructed an element  $\omega'_i = (\omega'_i)^{(n)}$  such that  $(R - F)(\omega'_i) = \omega_i$ . Moreover, the element  $\omega'_i$  has the property that, if  $\omega_i^{(n)} = 0$ , then also  $\omega'_i{}^{(n)} = 0$ . It follows that, for all  $n \geq 1$ , all but finitely many of the  $\omega'_i{}^{(n)}$ . Hence, the series  $\omega' = \sum_{i \in I} \omega'_i$  defines an element with  $(R - F)(\omega') = \omega$  as desired.  $\square$

**Theorem 28.** *There is a canonical isomorphism*

$$\text{Wh}_2^{\text{Top}}(S^1) \xrightarrow{\sim} \bigoplus_{r \geq 1} \bigoplus_{j \in \mathbb{Z} \setminus 2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

*Proof.* We first evaluate the kernel of the map  $1 - F$  in the long-exact sequence at the beginning of the section. Let  $\omega = (\omega^{(n)})$  be an element of  $\widetilde{\text{TR}}_2(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2)$ . Then  $\omega$  lies in the kernel of  $1 - F$  if and only if the coefficients

$$\begin{aligned} a_{s,j}^{(n)} &= a_{s,j}(\omega^{(n)}) \in \text{TR}_2^{n-s}(\mathbb{S}, I; 2) \\ b_{s,j}^{(n)} &= b_{s,j}(\omega^{(n)}) \in \text{TR}_1^{n-s}(\mathbb{S}, I; 2) \end{aligned}$$

satisfy the equations of Corollary 3. In the case at hand, the equations imply that the coefficients above are determined by the coefficients  $b_{1,j}^{(n)}$ . Indeed, if we write  $j$  as  $2^u j'$  with  $j'$  odd, then we have

$$a_{s,j}^{(n)} = \begin{cases} F^u(db_{1,j'}^{(n+1+u)} + \eta b_{1,j'}^{(n+1+u)}) & (s = 0) \\ \eta b_{1,j}^{(n+1-s)} & (1 \leq s < n) \end{cases}$$

$$b_{s,j}^{(n)} = \begin{cases} 0 & (s = 0 \text{ and } j \text{ even}) \\ -j b_{1,j}^{(n+1)} & (s = 0 \text{ and } j \text{ odd}) \\ b_{1,j}^{(n+1-s)} & (1 \leq s < n). \end{cases}$$

The coefficients  $b_{1,j}^{(n)}$ , however, are not unrestricted, since for every  $n \geq 1$ , all but finitely many of the coefficients  $a_{s,j}^{(n)}$  and  $b_{s,j}^{(n)}$  are zero. We write

$$b_{1,j}^{(n)} = \sum_{0 \leq r < n-1} c_{r,j} V^r(\tilde{\eta})$$

and consider the coefficients

$$c_{r,j} = c_{r,j}(\omega) \in \mathbb{Z}/2\mathbb{Z}.$$

Since  $R(b_{1,j}^{(n+1)}) = b_{1,j}^{(n)}$  and  $R(\tilde{\eta}) = \tilde{\eta}$ , the coefficients  $c_{r,j}$  depend only on the integers  $r \geq 0$  and  $j \in \mathbb{Z} \setminus \mathbb{Z}$  and not on  $n$ . They determine and are determined by the coefficients  $a_{s,j}^{(n)}$  and  $b_{s,j}^{(n)}$ .

The requirement that for all  $n \geq 1$ , all but finitely many of the  $b_{s,j}^{(n)}$  be zero implies that there exists a finite subset  $I = I(\omega) \subset \mathbb{Z}/2\mathbb{Z}$  such that  $c_{r,j}$  is zero, unless  $j \in I$ . We fix  $j \in I$  and consider  $a_{0,2^u j}^{(n)}$ , with  $u \geq 0$ . We calculate

$$\begin{aligned} a_{0,2^u j}^{(n)} &= F^u(d b_j^{(n+1+u)} + \eta b_j^{(n+1+u)}) \\ &= \sum_{0 \leq r < n+u} c_{r,j} F^u(d V^r(\tilde{\eta}) + V^r(\eta \tilde{\eta})) \\ &= \sum_{0 \leq r < u} c_{r,j} F^{u-r}(d \tilde{\eta} + \eta \tilde{\eta}) + \sum_{u \leq r < u+n} c_{r,j} (d V^{r-u}(\tilde{\eta}) + V^{r-u}(\eta \tilde{\eta})) \\ &= \sum_{0 \leq r < u} c_{r,j} d \tilde{\eta} + \sum_{u \leq r < u+n} c_{r,j} (d V^{r-u}(\tilde{\eta}) + V^{r-u}(\eta \tilde{\eta})). \end{aligned}$$

Now, for all  $n \geq 1$ , there exists  $N^{(n)} = N^{(n)}(\omega)$  such that for all  $j \in I$  and all  $u \geq N^{(n)}$ , the coefficient  $a_{0,2^u j}^{(n)}$  is zero. We assume that  $N^{(n)}$  is chosen minimal. Since

$$R: \text{TR}_2^n(\mathbb{S}, I; 2) \rightarrow \text{TR}_2^{n-1}(\mathbb{S}, I; 2)$$

is surjective and takes  $a_{0,2^u j}^{(n)}$  to  $a_{0,2^u j}^{(n-1)}$ , we have  $N^{(n)} \geq N^{(n-1)}$ . Considering the coefficients of  $d \tilde{\eta}$  and  $\eta \tilde{\eta}$  in the sum above, we find that for all  $u \geq N^{(n)}$ ,

$$\begin{aligned} \sum_{0 \leq r < u+1} c_{r,j} &= 0 && \text{(coefficient of } d \tilde{\eta}) \\ c_{u,j} &= 0 && \text{(coefficients of } \eta \tilde{\eta}). \end{aligned}$$

But these equations are satisfied also for  $u \geq N^{(n-1)}$  which implies that we also have  $N^{(n)} \leq N^{(n-1)}$ . We conclude that there exists an integer  $N = N(\omega) \geq 0$  independent of  $n$  such that  $c_{u,j} = 0$ , for  $u \geq N$ , and that the coefficient  $c_{0,j}$  is equal to the sum of the coefficients  $c_{r,j}$  with  $r \geq 1$ . Conversely, suppose we are given coefficients  $c_{r,j}$  all but finitely many of which are zero. Then, for every  $n \geq 1$ , all but finitely many of the corresponding coefficients  $a_{s,j}^{(n)}$  and  $b_{s,j}^{(n)}$  are zero. This shows that the map



$$\ker(1 - F: \widetilde{\text{TR}}_2(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2) \rightarrow \widetilde{\text{TR}}_2(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2)) \rightarrow \bigoplus_{r \geq 1} \bigoplus_{j \in \mathbb{Z} \setminus \mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

that to  $\omega$  assigns  $(c_{r,j}(\omega))$  is an isomorphism.

It remains to show that the map

$$1 - F: \widetilde{\text{TR}}_3(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2) \rightarrow \widetilde{\text{TR}}_3(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2)$$

is surjective. Given the element  $\omega = (\omega^{(n)})$  on the right-hand side, we find an element  $\omega' = (\omega'^{(n)})$  on the left-hand side such that  $(R - F)(\omega') = \omega$ . As in the proof of Theorem 27, we first consider several cases separately.

First, if  $\omega^{(n)} = dV^s(b^{(n)}[x]^j)$ , where  $j$  and  $s \geq 1$  are integers, we define

$$\omega'^{(n)} = - \sum_{s \leq r < n-1} dV^{r+1}(b^{(n-1-r+s)}[x]^j) - \sum_{s \leq r < n} V^r(\eta b^{(n-r+s)}[x]^j).$$

Then we find that  $R(\omega'^{(n+1)}) = \omega'^{(n)}$  and  $(R - F)(\omega'^{(n+1)}) = \omega^{(n)}$  by calculations entirely similar to the ones in the proof of Theorem 27.

Second, if  $\omega^{(n)} = b^{(n)}[x]^j d \log[x]$ , we consider three cases separately. In the case  $\omega^{(n)} = V^s(\eta\tilde{\eta})[x]^j d \log[x]$  with  $0 \leq s < n$ , we let  $\omega' = \omega$ . Then  $(R - F)(\omega') = \omega$  since  $F(\tilde{\eta}) = 0$ . In the case  $\omega^{(n)} = dV^s(\tilde{\eta})[x]^j d \log[x]$ , where  $1 \leq s < n$ , we note that  $\omega^{(n)} = dV^s(\tilde{\eta}[x]^{2^s j} d \log[x])$  and define

$$\omega'^{(n)} = - \sum_{s \leq r < n-1} dV^{r+1}(\tilde{\eta}[x]^{2^s j} d \log[x]) - \sum_{s \leq r < n} V^r(\eta\tilde{\eta}[x]^{2^s j} d \log[x]).$$

Then  $R(\omega'^{(n+1)}) = \omega'^{(n)}$  and  $(R - F)(\omega'^{(n+1)}) = \omega^{(n)}$  as before. In the remaining case  $\omega^{(n)} = (d\tilde{\eta})[x]^j d \log[x]$ , the calculation

$$\begin{aligned} & (R - F)(dV(\tilde{\eta}[x]^j d \log[x])) \\ &= dV(\tilde{\eta}[x]^j d \log[x]) - (d\tilde{\eta})[x]^j d \log[x] - \eta\tilde{\eta}[x]^j d \log[x] \end{aligned}$$

shows that there exists  $\omega'$  with  $(R - F)(\omega') = \omega$ . Indeed, we have already considered  $dV(\tilde{\eta}[x]^j d \log[x])$  and  $\eta\tilde{\eta}[x]^j d \log[x]$ .

Third, if  $\omega^{(n)} = a^{(n)}[x]^j$ , we consider two cases separately. In the first case, we have  $\omega^{(n)} = V^s(\tilde{v})[x]^j$  with  $0 \leq s < n$  and define

$$\omega'^{(n)} = V^s(\tilde{v})[x]^j + F(V^s(\tilde{v})[x]^j) + F^2(V^s(\tilde{v})[x]^j).$$

Then  $R(\omega'^{(n+1)}) = \omega'^{(n)}$  and  $(R - F)(\omega'^{(n+1)}) = \omega^{(n)}$  because  $F^3 V^s(\tilde{v}) = 0$ . In the second case,  $\omega^{(n)} = dV^s(\eta\tilde{\eta})[x]^j$ , we calculate

$$dV^s(\eta\tilde{\eta})[x]^j = dV^s(\eta\tilde{\eta}[x]^{2^s j}) - jV^s(\eta\tilde{\eta})[x]^j d \log[x].$$

Since we have already considered the two terms on the right-hand side, it follows that there exists  $\omega'$  with  $(R - F)(\omega') = \omega$ .

Finally, we consider  $\omega^{(n)} = V^s(a^{(n)}[x]^j)$  with  $1 \leq s < n$ . For  $s \geq 3$ , we define

$$\omega'^{(n)} = V^s(a^{(n)}[x]^j) + FV^s(a^{(n+1)}[x]^j) + F^2V^s(a^{(n+2)}[x]^j).$$

Then  $R(\omega'^{(n+1)}) = \omega'^{(n)}$  and  $(R - F)(\omega'^{(n+1)}) = \omega^{(n)}$  since  $8a^{(n+3)} = 0$ . For  $s = 0$  and  $s = 1$ , the calculation

$$\begin{aligned} (R - F)(V(a^{(n+1)}[x]^j)) &= V(a^{(n)}[x]^j) - 2a^{(n+1)}[x]^j \\ (R - F)(V^2(a^{(n+1)}[x]^j) + FV^2(a^{(n+2)}[x]^j)) &= V^2(a^{(n)}[x]^j) - 4a^{(n+2)}[x]^j, \end{aligned}$$

shows that there exists  $\omega'$  with  $(R - F)(\omega') = \omega$ . Indeed, we have already considered  $2a^{(n+1)}[x]^j$  and  $4a^{(n+2)}[x]^j$  above.

The elements  $\omega'$  with  $(R - F)(\omega') = \omega$  which we constructed above have the property that, if  $\omega^{(n)}$  is zero, then  $\omega'^{(n)}$  is zero. It follows as in the proof of Theorem 27 that the map  $1 - F$  in question is surjective.  $\square$

**Theorem 29.** *There is a canonical isomorphism*

$$\text{Wh}_3^{\text{Top}}(\mathcal{S}^1) \xrightarrow{\sim} \bigoplus_{r \geq 0} \bigoplus_{j \in \mathbb{Z} \sim 2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{r \geq 1} \bigoplus_{j \in \mathbb{Z} \sim 2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

*Proof.* We first show that the kernel of the map  $1 - F$  in the long-exact sequence at the beginning of the section is canonically isomorphic to the group that appears on the right-hand side in the statement. So we let  $\omega = (\omega^{(n)})$  be an element of  $\widetilde{\text{TR}}_3(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2)$  that lies in the kernel of  $1 - F$ . The equations of Corollary 3 again show that the coefficients

$$\begin{aligned} a_{s,j}^{(n)} &= a_{s,j}(\omega^{(n)}) \in \text{TR}_3^{n-s}(\mathbb{S}, I; p) \\ b_{s,j}^{(n)} &= b_{s,j}(\omega^{(n)}) \in \text{TR}_2^{n-s}(\mathbb{S}, I; p) \end{aligned}$$

are completely determined by the coefficients  $b_{1,j}^{(n)}$ . Indeed, we find

$$\begin{aligned} a_{s,j}^{(n)} &= \begin{cases} F^u(db_{1,j'}^{(n+1+u)} + \eta b_{1,j'}^{(n+1+u)}) & (s = 0) \\ \eta b_{1,j}^{(n+1-s)} & (1 \leq s < n) \end{cases} \\ b_{s,j}^{(n)} &= \begin{cases} jF^u(b_{1,j'}^{(n+1+u)}) & (s = 0) \\ b_{1,j}^{(n+1-s)} & (1 \leq s < n), \end{cases} \end{aligned}$$

where  $j = 2^u j'$  with  $j'$  odd. For example, if  $1 \leq s < n$ , then

$$\begin{aligned}
 a_{s,j}^{(n)} &= 2a_{s+1,j}^{(n+1)} + \eta b_{s+1,j}^{(n+1)} = 2(2a_{s+2,j}^{(n+2)} + \eta b_{s+2,j}^{(n+2)}) + \eta b_{s+1,j}^{(n+1)} \\
 &= 2(2(2a_{s+3,j}^{(n+3)} + \eta b_{s+3,j}^{(n+3)}) + \eta b_{s+2,j}^{(n+2)}) + \eta b_{s+1,j}^{(n+1)} \\
 &= \eta b_{s+1,j}^{(n+1)} = \eta b_{1,j}^{(n+1-s)}
 \end{aligned}$$

since  $\text{TR}_3^{n-s}(\mathbb{S}, I; 2)$  is annihilated by 8. We now write

$$b_{1,j}^{(n)} = \sum_{0 \leq r < n-1} c_{r,j} V^r(\eta \tilde{\eta}) + \sum_{0 \leq r < n-1} c'_{r,j} dV^r(\tilde{\eta}),$$

where the coefficients  $c_{r,j} = c_{r,j}(\omega)$  and  $c'_{r,j} = c'_{r,j}(\omega)$  are independent on  $n$ . It is clear that the  $c_{r,j}$  and  $c'_{r,j}$  are non-zero for only finitely many values of the odd integer  $j$ . We fix such a  $j$  and evaluate the coefficients  $a_{0,2^u j}^{(n)}$  and  $b_{0,2^u j}^{(n)}$  for  $u \geq 1$  as functions of the coefficients  $c_{r,j}$  and  $c'_{r,j}$ .

$$\begin{aligned}
 a_{0,2^u j}^{(n)} &= F^u(db_j^{(n+1+u)} + \eta b_j^{(n+1+u)}) \\
 &= \sum_{0 \leq r < n+u} c_{r,j} (F^u dV^r(\eta \tilde{\eta}) + \eta F^u V^r(\eta \tilde{\eta})) \\
 &\quad + \sum_{0 \leq r < n+u} c'_{r,j} (F^u dV^r(\tilde{\eta}) + \eta F^u V^r(\tilde{\eta})) \\
 &= \sum_{u \leq r < n+u} c_{r,j} (dV^{r-u}(\eta \tilde{\eta}) + V^{r-u}(\eta^2 \tilde{\eta})) \\
 b_{0,2^u j}^{(n)} &= jF^u(b_j^{(n+1+u)}) \\
 &= \sum_{0 \leq r < n+u} jc_{r,j} F^u V^r(\eta \tilde{\eta}) + \sum_{0 \leq r < n+u} jc'_{r,j} F^u dV^r(\tilde{\eta}) \\
 &= \sum_{0 \leq r < u} jc'_{r,j} d\tilde{\eta} + \sum_{u \leq r < n+u} jc'_{r,j} (dV^{r-u}(\tilde{\eta}) + V^{r-u}(\eta \tilde{\eta})).
 \end{aligned}$$

We claim that the elements  $dV^{r-u}(\eta \tilde{\eta})$  and  $V^{r-u}(\eta^2 \tilde{\eta})$  with  $u < r < n + u$  form a linearly independent set. Indeed, the map

$$i_*: \text{TR}_3^{n+u}(\mathbb{S}, I; 2) \rightarrow \text{TR}_3^{n+u}(\mathbb{S}; 2)$$

is injective by Proposition 13, and Lemma 23 shows that

$$\begin{aligned}
 i_*(dV^{r-u}(\eta \tilde{\eta})) &= dV^{r-u}(\eta^2) + dV^{r-u+1}(\eta^2) \\
 i_*(V^{r-u}(\eta^2 \tilde{\eta})) &= V^{r-u}(\eta^3) + V^{r-u+1}(\eta^3) = 4V^{r-u}(v) + 4V^{r-u+1}(v).
 \end{aligned}$$

The claim then follows from Theorem 11. We now conclude as in the proof of Theorem 28 that the map that to  $\omega$  assigns  $((c_{r,j}(\omega)), (c'_{r,j}(\omega)))$  defines an isomorphism

$$\ker(1 - F: \widetilde{\mathrm{TR}}_3(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2) \rightarrow \widetilde{\mathrm{TR}}_3(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2)) \\ \xrightarrow{\sim} \bigoplus_{r \geq 0} \bigoplus_{j \in \mathbb{Z} \setminus \mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{r \geq 1} \bigoplus_{j \in \mathbb{Z} \setminus \mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

Finally, we argue as in the proof of Theorem 27 that the map

$$1 - F: \widetilde{\mathrm{TR}}_4(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2) \rightarrow \widetilde{\mathrm{TR}}_4(\mathbb{S}[x^{\pm 1}], I[x^{\pm 1}]; 2)$$

is surjective. Given  $\omega = (\omega^{(n)})$  on the right-hand side, we find  $\omega' = (\omega'^{(n)})$  on the left-hand side with  $(R - F)(\omega') = \omega$ .

First, if  $\omega^{(n)} = dV^s(b^{(n)}[x]^j)$ , where  $1 \leq s < n$  and  $j$  are integers, we define

$$\omega'^{(n)} = - \sum_{s \leq r < n-1} dV^{r+1}(b^{(n-1-r+s)}[x]^j) - \sum_{s \leq r < n} V^r(\eta b^{(n-r+s)}[x]^j).$$

Then we have  $R(\omega'^{(n+1)}) = \omega'^{(n)}$  and  $(R - F)(\omega'^{(n+1)}) = \omega^{(n)}$  as desired.

Second, if  $\omega^{(n)} = b^{(n)}[x]^j d \log[x]$ , we consider two cases separately. In the case  $\omega^{(n)} = dV^s(\eta\tilde{\eta})[x]^j d \log[x]$  with  $1 \leq s < n$ , we write  $\omega^{(n)} = dV^s(\eta\tilde{\eta}[x]^{2^s j}) d \log[x]$  and define

$$\omega'^{(n)} = - \sum_{s \leq r < n-1} dV^{r+1}(\eta\tilde{\eta}[x]^j d \log[x]) - \sum_{s \leq r < n} V^r(\eta^2 \tilde{\eta}[x]^j d \log[x]).$$

Then  $R(\omega'^{(n+1)}) = \omega'^{(n)}$  and  $(R - F)(\omega'^{(n+1)}) = \omega^{(n)}$  as before. In the case where  $\omega^{(n)} = V^s(\tilde{\nu})[x]^j d \log[x]$  with  $0 \leq s < n$ , we define

$$\omega'^{(n)} = V^s(\tilde{\nu})[x]^j d \log[x] + F(V^s(\tilde{\nu})[x]^j d \log[x]) + F^2(V^s(\tilde{\nu})[x]^j d \log[x]).$$

Then  $R(\omega'^{(n+1)}) = \omega'^{(n)}$  and  $(R - F)(\omega'^{(n+1)}) = \omega^{(n)}$  since  $8\tilde{\nu}$  and  $F\tilde{\nu}$  are zero.

Finally, we consider  $\omega^{(n)} = dV^s(\tilde{\nu})[x]^j$  with  $0 \leq s < n$ . For  $s \geq 1$ ,

$$dV^s(\tilde{\nu})[x]^j = dV^s(\tilde{\nu}[x]^{2^s j}) - jV^s(\tilde{\nu})[x]^j d \log[x]$$

and the two terms on the right-hand side were considered above. It follows that there exists  $\omega'$  with  $(R - F)(\omega') = \omega$ . For  $s = 0$ , we calculate

$$(R - F)(dV(\tilde{\nu}[x]^j)) = dV(\tilde{\nu}[x]^j) - (d\tilde{\nu})[x]^j + j\tilde{\nu}[x]^j d \log[x] - \eta\tilde{\nu}[x]^j \\ (R - F)(\eta\tilde{\nu}[x]^j) = \eta\tilde{\nu}[x]^j.$$

This shows that also for  $\omega^{(n)} = (d\tilde{\nu})[x]^j$ , there exists  $\omega'$  such that  $(R - F)(\omega') = \omega$ . Indeed, we have already considered the remaining classes on the right-hand side.

The elements  $\omega'$  with  $(R - F)(\omega') = \omega$  which we constructed above have the property that, if  $\omega^{(n)}$  is zero, then  $\omega'^{(n)}$  is zero. It follows as in the proof

of Theorem 27 that the map  $1 - F$  in question is surjective. This completes the proof.  $\square$

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