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AN INTRODUCTION TO THE VALUE-DISTRIBUTION THEORY OF ZETA-FUNCTIONS

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Abstract. This is an expository survey on the value-distribution theory of zeta-functions, mainly of the Riemann zeta-function. We discuss denseness results, the universality, and limit theorems both in the complex plane and in function spaces.

Key words and phrases: density, Dirichlet series, limit theorem, Riemann zeta-function, universality.

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1. Basic properties of the Riemann zeta-function

The Riemann zeta-function is defined by the infinite series

$$\zeta(s)=\sum_{n=1}^{\infty} n^{-s}, \tag{1.1}$$

where $s=\sigma+it$ is a complex variable. This series is convergent absolutely in the half-plane $D_0=\{\sigma=\operatorname{Re} s>1\}$, uniformly in any compact subset of D_0, hence is holomorphic there. Moreover, as was discovered by L. Euler (1737), $\zeta(s)$ has the infinite product expression

$$\zeta(s)=\prod_{n=1}^{\infty}\left(1-p_n^{-s}\right)^{-1} \tag{1.2}$$

(the Euler product) in D_0, where p_n denotes the nth prime number. Consequently $\zeta(s)\neq 0$ in D_0.

The meromorphic continuation of $\zeta(s)$ was proved by B. Riemann (1859). He proved that $\zeta(s)$ is holomorphic in the whole complex plane $\mathbb{C}$ except for the only simple pole at $s=1$. Riemann also proved the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{1.3}$$

predicted by Euler, by which we can easily reduce the study of $\zeta(s)$ in the region $\sigma<0$ to that in D_0. In particular, we find that $\zeta(-2k)=0$ for any $k\in\mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers), and those are the only zeros in the region $\sigma<0$.

Riemann noticed that the behaviour of $\zeta(s)$ in the remaining region $0\leqslant\sigma\leqslant 1$ (the critical strip), especially the distribution of zeros of $\zeta(s)$ in this strip, is closely connected to the behaviour of prime numbers. Therefore the study of $\zeta(s)$ in this strip is very important in prime number theory. In particular, Riemann wrote that it is quite plausible that all the zeros in this strip are on the line $\sigma=1/2$ (the Riemann hypothesis). However, still now, no one knows whether the Riemann hypothesis is true or not. Generally speaking, the study of the behaviour of $\zeta(s)$ in this strip is extremely difficult. Many theorems on the zeros of $\zeta(s)$ in this strip are now known, but our knowledge is still far from the real understanding of $\zeta(s)$.

By function theory we can say that we can understand $\zeta(s)$ completely if we understand its zeros and poles completely. This implies that the properties of non-zero values of $\zeta(s)$ includes the information of zeros, that is, "what kind of distribution of zeros may give such properties of non-zero values?" This gives a motivation of the study of non-zero values of $\zeta(s)$. The purpose of the present article is to survey the theory of distribution of non-zero values of $\zeta(s)$ and more general zeta-functions.

We may expect that, by studying general distribution properties of $\zeta(s)$, we can arrive at the global understanding of $\zeta(s)$ (which may not be arrived at if we watch only the zeros). Moreover, the history of the distribution theory of non-zero values shows that the theory is connected with various other branches of mathematics, such as probability theory, ergodic theory, functional analysis, and the theory of almost periodic functions. Furthermore, several unexpected applications of theorems on the value-distribution theory have been discovered recently. Therefore we can say that the distribution theory of non-zero values is a rather rich and fruitful area.

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2. The denseness theorem of Bohr

The study of the distribution of non-zero values of $\zeta(s)$ was initiated by H. Bohr in 1910s. Because of the functional equation (1.3), it is not strange to restrict our consideration to the region $\sigma \geqslant 1/2$. If $\sigma > 1$, it is clear that $|\zeta(\sigma+it)| \leqslant \zeta(\sigma)$ for any $t \in \mathbb{R}$ (where $\mathbb{R}$ is the set of real numbers), that is, the orbit $\zeta(\sigma+it)$ is included in a compact subset of $\mathbb{C}$. However, the situation is different when $\sigma \leqslant 1$.

THEOREM 1 (Bohr-Courant, [8]). *For any σ satisfying $1/2 < \sigma \leqslant 1$, the set $\{\zeta(\sigma+it) \mid t \in \mathbb{R}\}$ is dense in $\mathbb{C}$.*

THEOREM 2 (Bohr, [7]). *For any σ satisfying $1/2 < \sigma \leqslant 1$, the set $\{\log\zeta(\sigma+it) \mid t \in \mathbb{R}\}$ is dense in $\mathbb{C}$.*

REMARK. Since we do not assume the Riemann hypothesis, we cannot exclude the possibility of existence of zeros of $\zeta(s)$ in the strip $1/2 < \sigma < 1$. Hence, in Theorem 2, we have to decide the branch of the logarithm. When $\sigma > 1$ we define

$$\log\zeta(s) = -\sum_{n=1}^{\infty} \mathrm{Log}(1 - p_n^{-s}),$$

where Log denotes the branch of the logarithm which is real on the positive real axis. Next, in the case $1/2 < \sigma \leqslant 1$, denote by $\mathcal{P}$ all (possible) poles and the zeros of $\zeta(s)$ in this strip, and exclude

$$\bigcup_{s=\sigma_0+it_0\in\mathcal{P}} \{\sigma + it_0 \mid 1/2 < \sigma \leqslant \sigma_0\}$$

from our consideration. When $\sigma + it$ is not in the above set, we define the value of $\log\zeta(\sigma+it)$ by the analytic continuation from the region $\sigma > 1$ along the horizontal line segment.

Actually Theorem 1 is a simple consequence of Theorem 2. In fact, let a be any non-zero complex number. For any $\varepsilon > 0$, Theorem 2 says that we can find a $t \in \mathbb{R}$ such that

$$|\log\zeta(\sigma+it)-\log a| < \frac{\varepsilon}{2|a|}.$$

Hence

$$\begin{aligned}|\zeta(\sigma+it)-a| &= |a|\cdot|\mathrm{e}^{\log\zeta(\sigma+it)-\log a}-1| \\ &\leqslant 2|a|\cdot|\log\zeta(\sigma+it)-\log a| < \varepsilon,\end{aligned}$$

which implies Theorem 1.

Now we briefly sketch how to prove Theorem 2. Let $N \in \mathbb{N}$, and consider the finite truncation

$$\zeta_N(s) = \prod_{n=1}^{N}\left(1-p_n^{-s}\right)^{-1} \tag{2.1}$$

of the Euler product (1.2). Then

$$\begin{aligned}\log\zeta_N(\sigma+it) &= -\sum_{n=1}^{N}\mathrm{Log}\left(1-p_n^{-\sigma-it}\right) \qquad (2.2)\\ &= -\sum_{n=1}^{N}\mathrm{Log}\left(1-p_n^{-\sigma}\mathrm{e}^{-it\log p_n}\right).\end{aligned}$$

Define the mapping $S_N : [0,1)^N \to \mathbb{C}$ by

$$S_N(\theta_1,\ldots,\theta_N) = -\sum_{n=1}^{N}\mathrm{Log}\left(1-p_n^{-\sigma}\mathrm{e}^{2\pi i\theta_n}\right). \tag{2.3}$$

Then from (2.2) and (2.3) we have

$$\log\zeta_N(\sigma+it) = S_N\left(\left\{-\frac{t}{2\pi}\log p_1\right\},\ldots,\left\{-\frac{t}{2\pi}\log p_N\right\}\right), \tag{2.4}$$

where $\{x\}$ denotes the fractional part of x.

When θ_n moves from 0 to 1, each term $\mathrm{Log}\left(1-p_n^{-\sigma}\mathrm{e}^{2\pi i\theta_n}\right)$ on the right-hand side of (2.3) describes a closed convex curve. Therefore

$$\mathcal{U}_N = \{S_N(\theta_1,\ldots,\theta_N) \ \ 0\leqslant\theta_n<1 \ \ (1\leqslant n\leqslant N)\}$$

is a "geometric sum" of convex curves. Since $1/2<\sigma\leqslant 1$, the series $\sum p_n^{-\sigma}$ is divergent. By using this fact, we can show that

$$\bigcup_N \mathcal{U}_N = \mathbb{C}.$$

Therefore, for any $a \in \mathbb{C}$, we can find a sufficiently large N and $\theta_n \in [0,1)$ $(1 \leqslant n \leqslant N)$ for which

$$S_N(\theta_1, \ldots, \theta_N) = a \tag{2.5}$$

holds.

An arithmetic key lemma in the proof of Theorem 2 is the following approximation lemma of L. Kronecker. Let $a_1, \ldots, a_N \in \mathbb{R}$, linearly independent over the rational number field $\mathbb{Q}$, and let $b_1, \ldots, b_N \in \mathbb{R}$. Then Kronecker's lemma asserts that, for any $\varepsilon > 0$, we can find a $t > 0$ and rational integers $m_1, \ldots, m_N$ for which

$$|ta_n - b_n - m_n| < \varepsilon \quad (1 \leqslant n \leqslant N) \tag{2.6}$$

holds. Since $\log p_1, \ldots, \log p_N$ are linearly independent over $\mathbb{Q}$ (which is an immediate consequence of uniqueness of the decomposition of integers into prime factors), we can apply Kronecker's lemma to $a_n = -\log p_n / 2\pi$ and $b_n = \theta_n$ to obtain that

$$\left| -\frac{t}{2\pi} \log p_n - \theta_n - m_n \right| < \varepsilon \quad (1 \leqslant n \leqslant N) \tag{2.7}$$

with some $t > 0$ and $m_1, \ldots, m_N$. Combining (2.4), (2.5) and (2.7) we find that, under a suitable choice of t, $|\log \zeta_N(\sigma + it) - a|$ can be arbitrarily small.

The only remaining task is to replace $\log \zeta_N(\sigma + it)$ by $\log \zeta(\sigma + it)$. This can be achieved by using the following mean value result. Let $\eta > 0$. Then, for any sufficiently large $N = N(\eta)$ and $T = T(N, \eta)$, one has

$$\int_1^T \left| \frac{\zeta(\sigma + it)}{\zeta_N(\sigma + it)} - 1 \right|^2 \mathrm{d}t \leqslant \eta T. \tag{2.8}$$

This can be shown by using a general mean value theorem of F. Carlson, and the (classically known) mean square result

$$\int_1^T |\zeta(\sigma + it)|^2 \mathrm{d}t = \mathrm{O}(T) \quad (\sigma > 1/2). \tag{2.9}$$

Inequality (2.8) implies that the value $\zeta_N(\sigma + it)$ is not far from $\zeta(\sigma + it)$ for almost all t. Therefore we obtain the assertion of Theorem 2.

It is to be noted that mean value estimates such as (2.9) are usually essentially used in the proof of value-distribution theorems.

3. Voronin's multidimensional denseness

The multidimensional generalization of Bohr's denseness theorem was proved by S. M. Voronin more than half-century later. Let

$$D_1 = \{s \in \mathbb{C} \mid 1/2 < \operatorname{Re} s \leqslant 1\}, \quad D = \{s \in \mathbb{C} \mid 1/2 < \operatorname{Re} s < 1\}.$$

His results can be stated as follows.
Theorem 3 (Voronin, [71]). *Let* $m \in \mathbb{N}$, $h > 0$.

(i) *For any* $s \in D_1$, *the set*

$$\{(\zeta(s+inh), \zeta'(s+inh), \ldots, \zeta^{(m-1)}(s+inh)) \mid n \in \mathbb{N}\}$$

is dense in $\mathbb{C}^m$.

(ii) *For any* $s_1, \ldots, s_m \in D_1$ (*distinct from each other*), *the set*

$$\{(\zeta(s_1+inh), \zeta(s_2+inh), \ldots, \zeta(s_m+inh)) \mid n \in \mathbb{N}\}$$

is dense in $\mathbb{C}^m$.

The following generalization of Theorem 1 is an immediate corollary of the assertion (i) of the above theorem.

Theorem 4. *For any* σ *satisfying* $1/2 < \sigma \leqslant 1$, *the set*

$$\{(\zeta(\sigma+it), \zeta'(\sigma+it), \ldots, \zeta^{(m-1)}(\sigma+it)) \mid t \in \mathbb{R}\} \tag{3.1}$$

dense in $\mathbb{C}^m$.

At first glance, this is just a simple generalization of Theorem 1. Actually, however, this result suggests a rather surprising fact that "any holomorphic function can be approximated by $\zeta(s)$".

Let $s_0 = \sigma_0 + it_0$ be any point in D. Let $U = \{s \in \mathbb{C} \mid |s - s_0| \leqslant a\}$ be a disc, $U \subset D$, and $f(s)$ be a function holomorphic on U. Then

$$f(s) = \sum_{k=0}^{\infty} \frac{f^{(k)}(s_0)}{k!}(s-s_0)^k \tag{3.2}$$

for $s \in U$. By Cauchy's estimate of the coefficients we have

$$\left| \frac{f^{(k)}(s_0)}{k!}(s-s_0)^k \right| \leqslant M a^{-k} |s-s_0|^k \tag{3.3}$$

for any k, where $M = \sup\{|f(s)| \mid s \in U\}$. Let

$$V_0 = \{s \in \mathbb{C} \mid |s - s_0| \leqslant \delta_0 a\} \subset U$$

where $0 < \delta_0 < 1$. Then the right-hand side of (3.3) is $\leqslant M\delta_0^k$ if $s \in V_0$. Therefore, for any $\varepsilon > 0$, we can find a sufficiently large $m = m(M, \varepsilon, \delta_0) \in \mathbb{N}$ for which

$$\left| f(s) - \sum_{k=0}^{m-1} \frac{f^{(k)}(s_0)}{k!}(s - s_0)^k \right| < \varepsilon \tag{3.4}$$

holds for $s \in V_0$.

On the other hand, Theorem 4 implies that we can find a $t_1 \in \mathbb{R}$ for which

$$|\zeta^{(k)}(\sigma_0 + it_1) - f^{(k)}(s_0)| < \varepsilon \mathrm{e}^{-\delta_0 a} \qquad (0 \leqslant k \leqslant m - 1) \tag{3.5}$$

holds. Hence

$$\begin{aligned} &\left| \sum_{k=0}^{m-1} \frac{f^{(k)}(s_0)}{k!}(s - s_0)^k - \sum_{k=0}^{m-1} \frac{\zeta^{(k)}(\sigma_0 + it_1)}{k!}(s - s_0)^k \right| \\ &\qquad \leqslant \frac{\varepsilon}{\mathrm{e}^{\delta_0 a}} \sum_{k=0}^{m-1} \frac{(\delta_0 a)^k}{k!} < \varepsilon \end{aligned} \tag{3.6}$$

for $s \in V_0$. Put $\tau = t_1 - t_0$. Then $\sigma_0 + it_1 = s_0 + i\tau$. The Taylor expansion of $\zeta(s + i\tau)$ as a function in s at $s = s_0$ is

$$\zeta(s + i\tau) = \sum_{k=0}^{\infty} \frac{\zeta^{(k)}(s_0 + i\tau)}{k!}(s - s_0)^k = \sum_{k=0}^{\infty} \frac{\zeta^{(k)}(\sigma_0 + it_1)}{k!}(s - s_0)^k. \tag{3.7}$$

Let $V_\delta = \{s \in \mathbb{C} \mid |s - s_0| \leqslant \delta a\}$, where $0 < \delta < \delta_0$. Again using Cauchy's estimate, we find that

$$\left| \frac{\zeta^{(k)}(\sigma_0 + it_1)}{k!}(s - s_0)^k \right| \leqslant A(\tau)\delta^k$$

for $s \in V_\delta$, where $A(\tau) = \sup\{|\zeta(s + i\tau)| \mid s \in U\}$. Hence, if we choose $\delta = \delta(m, \varepsilon, \tau)$ sufficiently small, we have

$$\left| \zeta(s + i\tau) - \sum_{k=0}^{m-1} \frac{\zeta^{(k)}(\sigma_0 + it_1)}{k!}(s - s_0)^k \right| < \varepsilon \tag{3.8}$$

for $s \in V_\delta$. From (3.4), (3.6) and (3.8) we obtain

$$\sup_{s \in V_\delta} |\zeta(s + i\tau) - f(s)| < 3\varepsilon. \tag{3.9}$$

This implies that $f(s)$ can be approximated by (a certain translation of) $\zeta(s)$ uniformly in the disc V_δ. This kind of property is called *universality* of $\zeta(s)$.

The above argument of deducing (3.9) from Theorem 4 is sketched in the author's textbook [56], written in Japanese. An unsatisfactory point of (3.9) is that it holds only in a small disc V_δ, and moreover, the radius δa of V_δ depends on τ. This point is overcome in the stronger version of the universality due to Voronin, which will be discussed in the next section.

4. The universality

Denote by $\mu_1\{A\}$ the 1-dimensional Lebesgue measure of the set A. We use the notation

$$\nu_T(\cdots) = \frac{1}{T}\mu_1\{\tau \in [0, T] \mid \cdots\}$$

for $T > 0$, where in place of dots we write a certain condition satisfied by τ. Let K be a compact subset of D, and $\mathcal{F}(K)$ be the family of functions which are non-vanishing, continuous on K and holomorphic in the interior of K. Then

Theorem 5 (The universality theorem of Voronin). *For any compact subset K of D with connected complement, we have*

$$\liminf_{T \to \infty} \nu_T \left\{ \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0 \tag{4.1}$$

for any $f \in \mathcal{F}(K)$ and any $\varepsilon > 0$.

This formulation of the universality theorem is due to Reich [65]. Voronin's original statement in [72] is restricted to the case when K is a disc in D, but the above form of the universality can be deduced by his argument.

It is a natural question how to generalize this remarkable result to more general zeta and L-functions. Let χ be a Dirichlet character mod q, and $L(s, \chi)$ the associated Dirichlet L-function. Voronin [72] already mentioned that his result can be generalized to $L(s, \chi)$. More strongly, the following "joint universality theorem" holds (see Voronin [73], Gonek [15], and Bagchi [2], [3]).

THEOREM 6. *Let $K_1, \dots, K_m$ be compact subsets of D with connected complements, and $f_j \in \mathcal{F}(K_j)$ $(1 \leqslant j \leqslant m)$. Let $\chi_1, \dots, \chi_m$ be pairwise non-equivalent Dirichlet characters. Then we have*

$$\liminf_{T\to\infty} \nu_T \left\{ \max_{1\leqslant j\leqslant m} \sup_{s\in K_j} |L(s+i\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0 \tag{4.2}$$

for any $\varepsilon > 0$.

In the proof of Theorem 2, we used the fact that $\log p_1, \dots, \log p_N$ are linearly independent over $\mathbb{Q}$. The same fact is also used in the proof of Theorem 5. To prove Theorem 6, the orthogonality property of Dirichlet characters is also applied.

Let $\zeta(s, \alpha)$ be the Hurwitz zeta-function with parameter α $(0 < \alpha \leqslant 1)$. If α is rational or transcendental, then $\zeta(s, \alpha)$ has the universality property, that is, the inequality of the form (4.1) with replacing $\zeta(s+i\tau)$ by $\zeta(s+i\tau, \alpha)$ holds (Bagchi [2], Gonek [15]). In this case the non-vanishing condition for $f(s)$ is not necessary. This is because we do not use the Euler product expansion in the proof; actually Hurwitz zeta-function in general has no Euler product. Instead, when α is rational, we write $\zeta(s, \alpha)$ as a linear combination of Dirichlet L-functions and apply Theorem 6. When α is transcendental, we use the fact that $\log(n + \alpha)$ $(n \in \mathbb{N})$ are linearly independent over $\mathbb{Q}$. When α is algebraic irrational, to prove the universality of $\zeta(s, \alpha)$ is an open problem.

In general, we call a Dirichlet series $Z(s)$ *strongly universal* if (4.1) with replacing $\zeta(s+i\tau)$ by $Z(s+i\tau)$ holds. Now it is known that various zeta and L-functions are strongly universal; Dedekind zeta-functions of algebraic number fields (Voronin [74], Gonek [15], Reich [66], [67]), Lerch zeta-functions (Laurinčikas [30], [31], see also Laurinčikas and Garunkštis [43]), Dirichlet series attached to finite Abelian groups (Laurinčikas [37], [39]), Rankin-Selberg L-functions (the author [54]), etc.

The universality in some general setting has also been studied. This direction was cultivated by Reich [65], and then, pursued further by Laurinčikas ([20], [21], [22], [23], [24], [32]). Joint universality theorems in general setting are discussed in Laurinčikas [25], [33]. See also the next section.

An important problem is to find some quantitative version of universality theorems. The first attempt to this direction is due to Good [14]. Recently, this problem has been studied by Laurinčikas [35], Steuding [69], and Garunkštis [11]. Steuding considered the upper density (defined by replacing lim inf by lim sup on the left-hand side of (4.1)) and obtained an upper bound of it, while Garunkštis, inspired by the work of Good [14], obtained a lower bound of the lower density when K is sufficiently small.

There are three textbooks ([19], [27], [43]) in which the universality is treated. Also there are several survey papers, such as Laurinčikas [36], [39], [41], the author [55]. Steuding's lecture note on this topic will appear soon.

5. The positive density method

There is a conjecture, due to Yu. V. Linnik and I. A. Ibragimov, which says that all functions defined by Dirichlet series, which can be continued analytically to the left of the abscissa of absolute convergence and satisfy some natural conditions there, would be strongly universal. In view of this conjecture, it is desirable to show universality property for more and more zeta and L-functions.

In this direction, one important turning point is the unconditional proof of the strong universality of automorphic L-functions attached to cusp forms for $SL(2,\mathbb{Z})$, due to Laurinčikas and the author [45]. Let $F(z)$ be a holomorphic normalized Hecke-eigen cusp form of weight κ for $SL(2,\mathbb{Z})$, and $c(n)$ the nth Fourier coefficient of $F(z)$. Then the associated Dirichlet series

$$L(s,F)=\sum_{n=1}^{\infty}c(n)n^{-s}$$

is convergent absolutely for $\sigma>(\kappa+1)/2$, and can be continued holomorphically to the whole plane. Let

$$D(\kappa)=\{s\in\mathbb{C}\mid \kappa/2<\sigma<(\kappa+1)/2\}.$$

Then

Theorem 7 ([45]). *For any compact subset K of $D(\kappa)$ with connected complement, we have*

$$\liminf_{T\to\infty}\nu_T\left\{\sup_{s\in K}|L(s+i\tau,F)-f(s)|<\varepsilon\right\}>0 \tag{5.1}$$

for any $f\in\mathcal{F}(K)$ and any $\varepsilon>0$.

The behaviour of $c(p)$, when p runs over the set of primes, is very complicated. This situation causes a trouble when one try to prove the above theorem analogously to Theorem 5. However, by a certain mean value result on $c(p)$ we can deduce that the set of primes, for which $|c(p)|$ is not so small, is of positive density. By using this fact, we can go through the obstacle to obtain Theorem 7.

This idea ("the positive density method") has then been repeatedly used in the study of universality. In fact, the method in [45] has been generalized to show the universality of L-functions of new forms in [48], and the joint universality of twisted automorphic L-functions in [46]. Laurinčikas and the author [44] used a subset of positive density, not in the set of primes, but in the set of positive integers, to prove a joint universality theorem for Lerch zeta-functions. (Note that the results in [44] are valid under the condition that $\alpha_1, \ldots, \alpha_n$ are algebraically independent over $\mathbb{Q}$, which is lacking in the statements. See [47].) Mishou [57], [58] used certain sets of primes of positive density to show that Hecke L-functions of algebraic number fields are strongly universal. The universality of Artin L-functions has been proved by Bauer [4].

The positive density method has been applied in a more general situation. Laurinčikas and Šleževičienė [50] applied this method to show a universality theorem for general zeta-functions with multiplicative coefficients. Šleževičienė [68] generalized this result to the joint case, which has been further applied to the proof of the universality of Estermann zeta-functions [13].

The universality of zeta-functions belonging to a certain subclass of Selberg class has been shown by Steuding [70]. Laurinčikas, Schwarz and Steuding [49] proved a universality theorem for a certain type of general Dirichlet series, and its joint version was obtained by Laurinčikas [40], [42].

For any holomorphic cusp form $F(z)$, the Ramanujan-Petersson conjecture $|c(p)| \leqslant 2p^{(\kappa-1)/2}$ was proved by P. Deligne, and this was used in [45]. However, the corresponding conjecture has not yet been proved for Maass forms. Nagoshi [63] tried to prove the universality for L-functions attached to Maass forms, and he proved it under a certain condition weaker than the Ramanujan conjecture. Then in [64], Nagoshi succeeded in proving the universality of Maass form L-functions unconditionally, by employing a certain fourth power mean estimate of the coefficients. The readers will notice that the total number of papers discussing universality of various zeta and L-functions is increasing very rapidly in recent years. It is impossible here to mention all relevant papers.

The universality property may be regarded as an ergodic property of zeta-functions in function spaces. Therefore it is natural to understand this property in a formulation of function spaces, and in fact, it is closely connected with functional limit theorems of zeta-functions. To explain this connection, however, it is better to begin with limit theorems on the complex plane, which goes back to Bohr's work.

6. Limit theorems on the complex plane

Let R be a given rectangle in the complex plane $\mathbb{C}$ with the edges parallel to the axes, and define

$$V(T;\sigma,R)=\mu_1\{t\in[0,T]\mid \log\zeta(\sigma+it)\in R\},$$

where $T>0$. As a refinement of his denseness theorems (Theorems 1 and 2), Bohr proved the following result in a collaborated work with Jessen:
Theorem 8 (Bohr-Jessen, [9]). *For any $\sigma>1/2$, there exists the limit*

$$W(\sigma,R)=\lim_{T\to\infty}\frac{1}{T}V(T;\sigma,R). \tag{6.1}$$

This limit $W(\sigma,R)$ may be regarded as the "probability" of how many values of $\log\zeta(s)$ on the line $\operatorname{Re}s=\sigma$ belong to R. The proof of this theorem given in [9] is based on the same idea as in the proof of Theorem 2 sketched in Section 2. In particular, the fact that each term on the right-hand side of (2.3) is convex is important in their proof. However, for more general zeta-functions, the corresponding term is not always convex. The author [51], [52], [53] presented alternative proofs which are free from convexity, hence succeeded in proving an analogous result for a certain general class of zeta-functions defined by Euler products.

It is more convenient to formulate the above type of limit theorems in the frame of modern probability theory.

We recall some basic notion of probability theory. The triple $(S,\mathcal{F},P)$, where S is an non-empty set, $\mathcal{F}$ is a σ-field consisting of some subsets of S, and P is a probability measure on S, is called a probability space. Hereafter we consider the case that S is a metric space and $\mathcal{F}=\mathcal{B}(S)$ is the family of all Borel subsets (that is the σ-field generated by all open subsets) of S. We call a set $A\in\mathcal{B}(S)$ a continuity set with respect to P if $P(\partial A)=0$, where ∂A is the boundary of A. Let P_n $(n\in\mathbb{N})$ and P be probability measures on $(S,\mathcal{B}(S))$. Then the following three assertions are equivalent:

(i) For any real bounded continuous function f on S,

$$\int_S f\mathrm{d}P_n\to\int_S f\mathrm{d}P\quad(n\to\infty), \tag{6.2}$$

(ii) For any continuity set A with respect to P,

$$\lim_{n\to\infty}P_n(A)=P(A), \tag{6.3}$$

(iii) For any open set G,

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G). \tag{6.4}$$

When these three assertions are valid, we say that P_n converges weakly to P, and we write $P_n \Rightarrow P$. This is the standard definition of weak convergence, but for our purpose, it is necessary to consider the family $\{P_T\}$ with a continuous parameter $T > 0$. When $T \to \infty$ continuously, the three statements analogous to the above are also equivalent, hence we can define the weak convergence $P_T \Rightarrow P$ (see the comment after Theorem 2.3 of [5]).

For any $\sigma > 1/2$ and $0 < \gamma < \delta$, let

$$W_{\gamma,\delta,\sigma}(A) = \frac{1}{\delta - \gamma} \cdot \mu_1\{\gamma < t < \delta \mid \log\zeta(\sigma + it) \in A\}$$

where $A \in \mathcal{B}(\mathbb{C})$. Then $W_{\gamma,\delta,\sigma}$ is a probability measure on $\mathbb{C}$, and we have

Theorem 9 (Borchsenius-Jessen, [10]). *For any $\sigma > 1/2$, there exists a probability measure $W_{\gamma,\sigma}$ on $\mathbb{C}$ for which $W_{\gamma,\delta,\sigma} \Rightarrow W_{\gamma,\sigma}$ holds as $\delta \to \infty$.*

When A is a rectangle R (with the edges parallel to the axes), we can show that R is a continuity set with respect to the limit measure (see, e. g., (4.1) of [52]). Hence from (6.3) and the above theorem we have

$$\lim_{T\to\infty} W_{1,T,\sigma}(R) = W_{1,\sigma}(R),$$

which implies Theorem 8.

The proof of Borchsenius and Jessen [10] is based on the theory of almost periodic functions. This theory was also created by Bohr in connection with the value-distribution theory of zeta-functions, though he did not use this theory in [9].

In Borchsenius and Jessen [10], an analogue of Theorem 9 for (not $\log\zeta(s)$ but) $\zeta(s)$ has been proved. From such an analogue, we can show the following corollary. Let $T > 0$, $\sigma > 1/2$, and define

$$P_{T,\sigma}(A) = \frac{1}{T}\mu_1\{t \in [0, T] \mid \zeta(\sigma + it) \in A\}$$

for any $A \in \mathcal{B}(\mathbb{C})$. Then

Theorem 10. *For any $\sigma > 1/2$, there exists a probability measure P_σ on $\mathbb{C}$ for which $P_{T,\sigma} \Rightarrow P_\sigma$ holds.*

This is the form stated as Theorem 4.1.1 of Laurinčikas [27]. This kind of limit theorems has now been shown in a very general situation; see Laurinčikas [38]. Discrete analogues of Theorem 10 for another general class of zeta-functions can be found in Kačinskaitė [17] and Kačinskaitė and Laurinčikas [18].

7. Limit theorems in function spaces

Let $H(D)$ be the set of all functions holomorphic on D. Introducing the topology of uniform convergence on compact subsets, we can regard $H(D)$ as a topological space. There exists a family of compact subsets K_n of D $(n \in \mathbb{N})$ such that

$$K_1 \subset K_2 \subset \cdots, \qquad D = \bigcup_{n=1}^{\infty} K_n,$$

and for any compact subset K of D, there exists an n for which $K \subset K_n$ holds. For any $f, g \in H(D)$, let

$$\rho_n(f,g) = \sup_{s \in K_n} |f(s) - g(s)|,$$

and define

$$\rho(f,g) = \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{\rho_n(f,g)}{1+\rho_n(f,g)}.$$

Then ρ is a metric on $H(D)$, which induces the above topology.

For any $A \in \mathcal{B}(H(D))$, define

$$P_T^*(A) = \frac{1}{T}\mu_1\{\tau \in [0,T] \mid \zeta(s+i\tau) \in A\}.$$

Then P_T^* is a probability measure on $H(D)$, and

Theorem 11 (Bagchi, [2]). *There exists a probability measure Q^* on $H(D)$ such that $P_T^* \Rightarrow Q^*$ as $T \to \infty$.*

This is clearly a “functional analogue” of Theorem 10. Moreover, Bagchi discovered an explicit form of the limit measure Q^*. Denote by γ the unit circle in the complex plane, and let

$$\Omega = \prod_p \gamma_p,$$

where p runs over all primes and $\gamma_p = \gamma$ for each p. This is naturally a compact Abelian group, hence there exists the unique Haar measure m_H on Ω with $m_H(\Omega) = 1$. Then $(\Omega, \mathcal{B}(\Omega), m_H)$ is a probability space. For $\omega \in \Omega$, denote by $\omega(p)$ the projection of ω to the coordinate space γ_p. Define

$$\zeta(s, \omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}. \tag{7.1}$$

Let $1/2 < \sigma_0 < 1$. Then $\zeta(\sigma_0, \omega)$ converges almost surely (with respect to the measure m_H), and equals to

$$\sum_{n=1}^{\infty} \omega(n) n^{-\sigma_0}, \tag{7.2}$$

where $\omega(n) = \omega(p_1)^{\alpha_1} \cdots \omega(p_r)^{\alpha_r}$ if $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Using the theory of Dirichlet series we find that the convergence of (7.2) implies the convergence of the series

$$\sum_{n=1}^{\infty} \omega(n) n^{-s}, \tag{7.3}$$

uniformly on any compact subset of the half-plane $\sigma > \sigma_0$. Since (7.3) equals to the right-hand side of (7.1), we can conclude that (7.1) converges almost surely on any compact subset of D, and hence an element of $H(D)$. Therefore $\omega \mapsto \zeta(s, \omega)$ is an $H(D)$-valued random element. Let Q_ζ be the distribution of this random element, that is

$$Q_\zeta(A) = m_H(\omega \in \Omega \mid \zeta(s, \omega) \in A) \quad (A \in \mathcal{B}(H(D))).$$

Then

Theorem 12 (Bagchi, [2]). *We have* $Q_\zeta = Q^*$.

From these theorems we can deduce an explicit form of P_σ in Theorem 10. In fact, let h be a mapping from $H(D)$ to $\mathbb{C}$ defined by $h(f) = f(\sigma)$ for any $f \in H(D)$. Then $P_{T,\sigma} = P_T^* \circ h^{-1}$. Since h is continuous, noting Theorem 1.1.16 of [27], from Theorems 11 and 12 we find that $P_\sigma = Q^* \circ h^{-1} = Q_\zeta \circ h^{-1}$, that is,

$$P_\sigma(A) = m_H(\omega \in \Omega \mid \zeta(\sigma, \omega) \in A)$$

for any $A \in \mathcal{B}(\mathbb{C})$. This argument (suggested by B. Grigelionis) is presented in Laurinčikas [26], and also in Notes of Chapter 5 of [27].

Bagchi used his Theorems 11 and 12 to give a new proof of Voronin's universality theorem. Here we sketch his argument.

The support $\mathrm{Supp}(P)$ of a probability measure P on a probability space S is defined by

$$\mathrm{Supp}(P)=\{x\in S\mid P(G)>0 \quad \text{for any neighbourhood } G \text{ of } x\}.$$

Another key of Bagchi's proof of the universality is the fact that

$$\mathrm{Supp}(Q_\zeta)=\{\varphi\in H(D)\mid \varphi(s)\neq 0 \text{ for any } s\in D, \text{ or } \varphi\equiv 0\}. \qquad (7.4)$$

This is a consequence of the following "denseness lemma" of Bagchi. For $s\in D$ and $a_p\in\gamma$, put $f_p(s)=-\log(1-a_p p^{-s})$.

Theorem 13 (Bagchi, [2]). *The set of all convergent series of the form* $\sum_p f_p(s)$ *is dense in* $H(D)$.

Now, let $f(s)\in\mathcal{F}(K)$, and assume that $f(s)$ can be continued to a holomorphic function on D, and $f(s)\neq 0$ for any $s\in D$. Since the set

$$G=\left\{g\in H(D)\;\middle|\;\sup_{s\in K}|g(s)-f(s)|<\varepsilon\right\}$$

is open, Theorems 11, 12 and (6.4) implies that

$$\liminf_{T\to\infty} P_T^*(G)\geqslant Q_\zeta(G). \qquad (7.5)$$

On the other hand, $f\in\mathrm{Supp}(Q_\zeta)$ by the assumption and (7.4). Since G is a neighbourhood of f, we have $Q_\zeta(G)>0$. From this and (7.5) we have

$$\liminf_{T\to\infty} P_T^*(G)>0,$$

which is exactly (4.1).

In general case, we use a theorem of Mergelyan, which asserts that any function $f\in\mathcal{F}(K)$ can be approximated by a certain polynomial uniformly on K. Then, applying the above argument to that polynomial, we can obtain (4.1).

The details of Bagchi's proof of (4.1) can be found in Chapter 6 of Laurinčikas [27].

In Sections 4 and 5, we mention many papers in which various generalizations of Voronin's universality theorem are treated. In most of those papers, the argument is based on Bagchi's idea. In order to generalize Bagchi's method, it is necessary to generalize Theorems 11, 12, and 13. Now, indeed, Theorems 11 and 12 have been generalized to a fairly general class of zeta-functions which have Euler products introduced by the author [52]; see Laurinčikas [28], [29].

Moreover, even in the case of Dirichlet series without Euler products, it is possible to prove an analogue of Theorems 11 and 12 under some suitable conditions. In this case, instead of the above Ω, the basic probability space is of the form

$$\widetilde{\Omega} = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$. The case of Lerch zeta-functions is written in [43]. For more general Dirichlet series, see Laurinčikas [34] (partly with Schwarz, Steuding and Genys). Various discrete versions and joint versions of those limit theorems have also been extensively studied.

On the other hand, in general, Theorem 13 cannot be easily generalized. This is because the proof of Theorem 13 is based on a deep arithmetical result, that is the asymptotic formula

$$\sum_{p \leqslant x} \frac{1}{p} = \log\log x + a_1 + \mathrm{O}(\exp(-a_2\sqrt{\log x}), \tag{7.6}$$

where the sum runs over all primes $\leqslant x$, and a_1, a_2 are constants with $a_2 > 0$. It is sometimes very difficult to obtain such an asymptotic formula in the case of more general zeta-functions. The positive density method explained in Section 5 is a technique which, in some sense, avoids the role of such an asymptotic formula in the proof of universality.

The original proof of the universality theorem due to Voronin himself is also still useful. For example, the argument in Mishou [57] and Nagoshi [63] are inspired by Voronin's method. Garunkštis [12] pointed out that it is possible to remove the rearrangement argument from Voronin's proof, which sheds some light on the effectivization problem. Voronin's idea is also important in Nagoshi's recent work on the joint universality of automorphic L-functions.

Finally, it is to be stressed that various interesting applications of universality theorems have already been discovered. Applications to the functional independence and the distribution of zeros of Dirichlet series were already studied by Voronin himself. Andersson [1] used the universality to disprove a conjecture of Ramachandra. The connections between universality and quantum physics were studied by [6], [16]. Recent series of joint papers of

Mishou and Nagoshi [60], [61], [62], and also Mishou [59], studied universality properties of L-functions of number fields, and deduced interesting arithmetical consequences, such as some distribution properties of class numbers of algebraic fields.

These applications suggest that the importance of limit theorems and universality theorems in the theory of zeta-functions will increase more and more in the near future.

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