

Generalized multiple Dirichlet series and generalized multiple polylogarithms

by

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1 Introduction and statement of results

Let \mathbb{N} be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, \mathbb{Z} the ring of rational integers, \mathbb{Q} the field of rational numbers, \mathbb{R} the field of real numbers, and \mathbb{C} the field of complex numbers.

Let $s = \sigma + i\tau$ be a complex variable, and

$$(1.1) \quad \psi(s) = \sum_{n=0}^{\infty} \frac{a(n)}{(\beta + nw)^s}$$

be a function with complex coefficients $a(n)$, where $\beta, w \in \mathbb{R}$ with $0 < \beta \leq w$. We assume the following:

(Assumption I) *There exists a certain $q > 0$ such that $\psi(s)$ is convergent absolutely for $\sigma > q$.*

Throughout this paper we fix $\delta \in \mathbb{R}$ with $\delta > 0$ and let $u \in \mathbb{R}$ with $1 \leq u \leq 1 + \delta$. We let

$$(1.2) \quad \psi(s; u) = \sum_{n=0}^{\infty} \frac{a(n)u^{-n}}{(\beta + nw)^s}.$$

By Assumption I, we can check that if $1 < u \leq 1 + \delta$ then the right-hand side of (1.2) is convergent absolutely for any $s \in \mathbb{C}$, so $\psi(s; u)$ is holomorphic for all $s \in \mathbb{C}$. Corresponding to $\psi(s; u)$, let

$$(1.3) \quad G_1(t; \psi; u) = \sum_{n=0}^{\infty} a(n)u^{-n}e^{(\beta+nw)t},$$

where t is a complex variable. By Assumption I, we see that the series (1.3) is convergent when $\Re t < 0$. We further assume the following:

(Assumption II) *$\psi(s)$ can be continued analytically to the whole complex plane \mathbb{C} , and holomorphic for all $s \in \mathbb{C}$. In any fixed strip $\sigma_1 \leq \sigma \leq \sigma_2$, $\psi(s; u)$ is uniformly convergent to $\psi(s)$ as $u \rightarrow 1 + 0$. Furthermore there exists a certain $\theta_0 = \theta_0(\sigma_1, \sigma_2) \in \mathbb{R}$ with $0 \leq \theta_0 < \pi/2$ such that $\psi(s; u) = O(e^{\theta_0|\tau|})$ as $|\tau| \rightarrow \infty$.*

(Assumption III) *There exists a certain $\rho = \rho(\psi) > 0$ such that $G_1(t; \psi; u)$ can be continued holomorphically to*

$$(1.4) \quad \mathcal{D}(\rho) = \{t \in \mathbb{C} \mid |t| < \rho\}$$

for any $u \in [1, 1 + \delta]$.

We will give typical examples which satisfy Assumptions I-III in Section 2 (see Example 2.2).

In the present paper, we consider generalized multiple Dirichlet series as follows. Let $(\alpha_0, \alpha_1, \dots, \alpha_r) \in \mathbb{R}^{r+1}$, $(w_1, \dots, w_r) \in \mathbb{R}^r$ such that $\alpha_0 = 0$ and $0 < \alpha_k - \alpha_{k-1} \leq w_k$ ($1 \leq k \leq r$). Let $\mathcal{P}_r = \{\psi_1, \dots, \psi_r\}$, where

$$(1.5) \quad \psi_k(s) = \sum_{n=0}^{\infty} \frac{a_k(n)}{(\alpha_k - \alpha_{k-1} + nw_k)^s}.$$

We assume that $\psi_k(s)$ and the associated series $\psi_k(s; u)$, $G_1(t; \psi_k; u)$ (defined similarly to (1.2) and (1.3)) satisfy Assumptions I-III ($1 \leq k \leq r$). By Assumptions I and III, there exist $\{q_k = q(\psi_k) (> 0) \mid 1 \leq k \leq r\}$ and $\{\rho_k = \rho(\psi_k) (> 0) \mid 1 \leq k \leq r\}$. We let

$$(1.6) \quad \eta_r = \min_{1 \leq k \leq r} \left\{ \frac{\rho_k}{2^{r-1}} \right\}.$$

We define the generalized multiple Dirichlet series associated with \mathcal{P}_r by

$$(1.7) \quad \Psi_r(s_1, \dots, s_r; u) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{a_1(n_1) \cdots a_r(n_r) u^{-\sum_{\nu=1}^r n_\nu}}{\prod_{j=1}^r (\alpha_j + \sum_{\nu=1}^j n_\nu w_\nu)^{s_j}}$$

for $s_1, \dots, s_r \in \mathbb{C}$ and $u \in [1, 1 + \delta]$. The special case $u = 1$ and $a_j(n) = 1$ ($1 \leq j \leq r$) has been studied by the first author in [12, 13], which can be regarded as a generalization of both the Euler-Zagier multiple zeta function and the Barnes multiple zeta function. On the other hand, the special case $u = 1$, $\alpha_j = j$ and $w_j = 1$ ($1 \leq j \leq r$) has also been studied before; Arakawa-Kaneko [2] when a_j s are periodic functions on \mathbb{Z} , and Matsumoto-Tanigawa [14] for more general a_j s.

First we prove the following result by using the method introduced by Matsumoto-Tanigawa (see [14]; see also [11, 12, 13]). Indeed, this can be regarded as a generalization of Theorem 2 in [14].

Theorem 1.1. *For $s_1, \dots, s_r \in \mathbb{C}$ and $u \in [1, 1 + \delta]$, $\Psi_r(s_1, \dots, s_r; u)$ is convergent absolutely for $s_j = \sigma_j + i\tau_j \in \mathbb{C}$ ($1 \leq j \leq r$) with each $\sigma_j > q_j$. Furthermore $\Psi_r(s_1, \dots, s_r; u)$ can be continued analytically to the whole complex space \mathbb{C}^r and holomorphic on \mathbb{C}^r , and satisfies that*

$$(1.8) \quad \lim_{u \rightarrow 1+0} \Psi_r(s_1, \dots, s_r; u) = \Psi_r(s_1, \dots, s_r; 1)$$

for any $(s_1, \dots, s_r) \in \mathbb{C}^r$.

Remark 1.2. We can prove the meromorphic continuation of $\Psi_r(s_1, \dots, s_r; u)$ even if $\psi_k(s)$ has poles. When $u > 1$, the multiple series (1.7) is convergent absolutely, hence is holomorphic, for any $(s_1, \dots, s_r) \in \mathbb{C}^r$. When $u = 1$, if we assume that $\psi_k(s)$ has a pole of order at most one at $s = q_k$ and holomorphic elsewhere (and satisfies $\psi_k(s) = O(e^{\theta_0|\tau|})$) for $1 \leq k \leq r$, then we can show the following results, which generalize Theorem 1 in [14]:

The function $\Psi_r(s_1, \dots, s_r; 1)$ can be continued meromorphically to the whole space \mathbb{C}^r , and its possible singularities are located only on the subsets of \mathbb{C}^r each of which is defined by one of the following equations:

$$s_j + \dots + s_r = q_j + \delta_{j+1}q_{j+1} + \dots + \delta_r q_r - n$$

$$(1 \leq j \leq r, \delta_k = 0 \text{ or } 1 (2 \leq k \leq r), n \in \mathbb{N}_0).$$

Moreover, (i) if $j = r \geq 2$ and $q_r \in \mathbb{N}$, then $n \leq q_r - 1$, (ii) if $2 \leq j \leq r - 1$, $q_j \in \mathbb{N}$ and $\delta_{j+1} = \dots = \delta_r = 1$, then $n \leq q_r - 1$, (iii) if $j = r = 1$ or if $j = 1$ and $\delta_2 = \dots = \delta_r = 1$, then $n = 0$.

The proof of this fact can be given by the same method as in the proof of Theorem 1 in [14].

We further consider generalized multiple polylogarithms related to (1.5) as follows. Let $\mathbf{d}_r = (d_1, \dots, d_r) \in \mathbb{C}^r$ with $\Re d_j > q_j$ for each j . With the above notation, and for $u \in [1, 1 + \delta]$, let

$$(1.9) \quad F_r(t_1, \dots, t_r; \mathbf{d}_r; \mathcal{P}_r; u) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{a_1(n_1) \cdots a_r(n_r) u^{-\sum_{l=1}^r n_l} \prod_{j=1}^r e^{(\alpha_j + \sum_{\mu=1}^j n_\mu w_\mu) t_j}}{\prod_{j=1}^r (\alpha_j + \sum_{\mu=1}^j n_\mu w_\mu)^{d_j}}.$$

This multiple series is convergent when $\Re t_j \leq 0$ ($1 \leq j \leq r$). If we formally let $\psi_k(s) = \zeta(s)$ which is the Riemann zeta function and $d_k \in \mathbb{N}$ ($1 \leq k \leq r$) in (1.9), then $F_r(\log x_1, \dots, \log x_r; \mathbf{d}_r; \mathcal{P}_r; 1)$ is the multiple polylogarithm defined by Goncharov (see [6]; see also [4]). However $\zeta(s)$ does not satisfy Assumption II, so we will not consider the Goncharov multiple polylogarithms in this paper. Instead, we prove the following result.

Theorem 1.3. *For $\mathbf{d}_r \in \mathbb{C}^r$ with each $\Re d_j > q_j$ ($1 \leq j \leq r$) and $u \in [1, 1 + \delta]$, $F_r(t_1, \dots, t_r; \mathbf{d}_r; \mathcal{P}_r; u)$ is holomorphic for all $(t_1, \dots, t_r) \in \mathcal{D}(\eta_r)^r$, and satisfies that for $(t_1, \dots, t_r) \in \mathcal{D}(\eta_r)^r$,*

$$(1.10) \quad F_r(t_1, \dots, t_r; \mathbf{d}_r; \mathcal{P}_r; u) = \sum_{N_1, \dots, N_r=0}^{\infty} \Psi_r(d_1 - N_1, \dots, d_r - N_r; u) \frac{t_1^{N_1} \cdots t_r^{N_r}}{N_1! \cdots N_r!}.$$

Furthermore, for any $\xi \in \mathbb{R}$ with $0 < \xi < \eta_r$, (1.10) is uniformly convergent with respect to $(t_1, \dots, t_r, u) \in \overline{\mathcal{D}}(\xi)^r \times [1, 1 + \delta]$, where $\overline{\mathcal{D}}(\xi) = \{t \in \mathbb{C} \mid |t| \leq \xi\}$.

The special case $\psi_j(s) = \sum_{n \geq 1} (-1)^n n^{-s}$ ($1 \leq j \leq r$), $\mathbf{d}_r \in \mathbb{N}^r$ and $t_1 = \dots = t_{r-1} = 0$ has been studied by the second author. Indeed, $F_r(0, \dots, 0, t; \mathbf{d}_r; \mathcal{P}_r; u)$ played an important role in giving some evaluation formulas for Euler-Zagier sums (see [15]). In order to prove Theorem 1.3 and Proposition 2.1 (see below), we make use of the technique introduced in [15].

As applications, using Theorem 1.3, we prove certain estimation formulas for $\Psi_r(d_1 - N_1, \dots, d_r - N_r; 1)$ (see Proposition 5.1 and Example 5.2). We further give certain multiple analogues of both Berndt's and Katsurada's formulas for Dirichlet L -functions proved in [3, 9] (see Example 5.3).

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2 Generalized polylogarithms

First we consider the case of $r = 1$. Let $\psi(s)$ as defined by (1.1) and $F_1(t; d; \psi; u)$ as defined by (1.9). With the notation defined in Section 1, we can prove the following.

Proposition 2.1. *For $d \in \mathbb{C}$ with $\Re d > q$ and $u \in [1, 1 + \delta]$, $F_1(t; d; \psi; u)$ is holomorphic for all $t \in \mathcal{D}(\rho)$, and satisfies that for $t \in \mathcal{D}(\rho)$,*

$$(2.1) \quad F_1(t; d; \psi; u) = \sum_{N=0}^{\infty} \psi(d - N; u) \frac{t^N}{N!}.$$

Furthermore, for any $\xi \in \mathbb{R}$ with $0 < \xi < \rho$, (2.1) is uniformly convergent with respect to $(t, u) \in \overline{\mathcal{D}}(\xi) \times [1, 1 + \delta]$.

Proof. By Assumption III, we can let

$$(2.2) \quad G_1(t; \psi; u) = \sum_{n=0}^{\infty} \mathfrak{B}_n(\psi; u) \frac{t^n}{n!}$$

for $|t| < \rho$. We use the method of contour integrals (see, for example, [16] Proof of Theorem 4.2). We consider the path Υ which consists of the positive real axis $[\varepsilon, \infty]$ (top side), a circle C_ε around 0 of radius ε , and the positive real axis $[\varepsilon, \infty]$ (bottom side), where $0 < \varepsilon < \rho$. Note that we

interpret t^s to mean $\exp(s \log t)$, where the imaginary part of $\log t$ varies from 0 (on the top side of the real axis) to 2π (on the bottom side). Let

(2.3)

$$\begin{aligned} H_1(s; \psi; u) &= \int_{\Upsilon} G_1(-t; \psi; u) t^{s-1} dt \\ &= (e^{2\pi i s} - 1) \int_{\varepsilon}^{\infty} G_1(-t; \psi; u) t^{s-1} dt + \int_{C_{\varepsilon}} G_1(-t; \psi; u) t^{s-1} dt, \end{aligned}$$

which is, in view of (1.3), holomorphic for all $s \in \mathbb{C}$ if $0 < \varepsilon < \rho$. Putting $s = -n$ for $n \in \mathbb{N}_0$ and $\varepsilon = \xi$ with $0 < \xi < \rho$ in (2.3) and using (2.2), we have

$$H_1(-n; \psi; u) = \int_{C_{\xi}} G_1(-t; \psi; u) t^{-n-1} dt = \frac{(2\pi i) \mathfrak{B}_n(\psi; u) (-1)^n}{n!}.$$

From Assumption III, $G_1(t; \psi; u)$ is continuous for all $(t, u) \in \mathcal{D}(\rho) \times [1, 1+\delta]$. Hence there exists the value $\mathcal{M}_{\xi} = \max |G_1(-t; \psi; u)|$ on $\{t \in \mathbb{C} \mid |t| = \xi\} \times [1, 1+\delta]$. By the above equation, we have

$$(2.4) \quad \frac{|\mathfrak{B}_n(\psi; u)|}{n!} \leq \frac{1}{2\pi} \int_{C_{\xi}} |G_1(-t; \psi; u)| |t|^{-n-1} |dt| \leq \frac{\mathcal{M}_{\xi}}{\xi^n}$$

for any $n \in \mathbb{N}_0$ and $u \in [1, 1+\delta]$, where ξ is an arbitrary real number with $0 < \xi < \rho$.

On the other hand, let $s \in \mathbb{C}$ with $\Re s > \max(1, q)$. We see that the second term on the right-hand side of (2.3) tends to 0 as $\varepsilon \rightarrow 0$. Hence we have

$$\begin{aligned} (2.5) \quad H_1(s; \psi; u) &= (e^{2\pi i s} - 1) \int_0^{\infty} G_1(-t; \psi; u) t^{s-1} dt \\ &= (e^{2\pi i s} - 1) \sum_{n=0}^{\infty} a(n) u^{-n} \int_0^{\infty} t^{s-1} e^{-(\beta+nu)t} dt \\ &= (e^{2\pi i s} - 1) \Gamma(s) \psi(s; u), \end{aligned}$$

where the interchange of summation and integration is valid because $\Re s > q$. Hence

$$(2.6) \quad \psi(s; u) = \frac{1}{(e^{2\pi i s} - 1) \Gamma(s)} H_1(s; \psi; u) = \frac{\Gamma(1-s)}{2\pi i e^{\pi i s}} H_1(s; \psi; u),$$

because

$$\Gamma(s) \Gamma(1-s) = \frac{\pi}{\sin \pi s} = \frac{2\pi i}{e^{\pi i s} - e^{-\pi i s}}.$$

The relation (2.6) is valid for all $s \in \mathbb{C}$ by analytic continuation.

Next, for $d \in \mathbb{C}$ with $\Re d > q$ and $N \in \mathbb{N}_0$, we put $s = d - N$ in (2.3). Then we have

$$(2.7) \quad H_1(d - N; \psi; u) = (e^{2\pi id} - 1) \int_{\varepsilon}^{\infty} G_1(-t; \psi; u) t^{d-N-1} dt \\ + \int_{C_{\varepsilon}} G_1(-t; \psi; u) t^{d-N-1} dt.$$

For simplicity, we denote by I_1 and I_2 the first and second terms on the right-hand side of (2.7), respectively. Note that if $N \geq \Re d + 1$ then

$$(2.8) \quad \left| \int_{\varepsilon}^{\infty} e^{-(\beta+nw)t} t^{d-N-1} dt \right| \leq \frac{e^{-(\beta+nw)\varepsilon} \varepsilon^{\Re d - N - 1}}{\beta + nw}.$$

Hence we have

$$|I_1| \leq \varepsilon^{\Re d - N - 1} |e^{2\pi id} - 1| \sum_{n=0}^{\infty} \frac{|a(n)| e^{-(\beta+nw)\varepsilon}}{\beta + nw}.$$

On the other hand, by using the fact that

$$(2.9) \quad \int_{C_{\varepsilon}} t^p dt = \begin{cases} 2\pi i & (p = -1) \\ \varepsilon^{p+1} \left(\frac{e^{2\pi ip} - 1}{p+1} \right) & (p \neq -1) \end{cases}$$

for $p \in \mathbb{C}$ and by (2.2), we have

$$(2.10) \quad I_2 = \begin{cases} (2\pi i) \mathfrak{B}_{N-d}(\psi; u) \frac{(-1)^{N-d}}{(N-d)!} & (N-d \in \mathbb{N}_0), \\ \varepsilon^{d-N} (e^{2\pi id} - 1) \sum_{n=0}^{\infty} \frac{\mathfrak{B}_n(\psi; u) (-1)^n \varepsilon^n}{(n+d-N)n!} & (\text{otherwise}). \end{cases}$$

Note that the above infinite series in the second case is convergent because of the assumption $\varepsilon < \rho$ and (2.4). Hence we have

$$(2.11) \quad |I_2| \leq \begin{cases} 2\pi \frac{|\mathfrak{B}_{N-d}(\psi; u)|}{(N-d)!} & (N-d \in \mathbb{N}_0), \\ \varepsilon^{\Re d - N} |e^{2\pi id} - 1| \left| \sum_{n=0}^{\infty} \mathfrak{B}_n(\psi; u) \frac{(-1)^n \varepsilon^n}{(n+d-N)n!} \right| & (\text{otherwise}). \end{cases}$$

From (2.4) with $\xi = \varepsilon$, the first case of (2.11) yields that

$$|I_2| \leq (2\pi) \mathcal{M}_{\varepsilon} \varepsilon^{d-N}.$$

In the second case of (2.11), we let $\gamma_d = \min |d - m|$ for all $m \in \mathbb{Z}$. Using (2.4) with ξ such that $0 < \varepsilon < \xi < \rho$, we see that the second case of (2.11) yields that

$$|I_2| \leq \varepsilon^{\Re d - N} |e^{2\pi id} - 1| \frac{\mathcal{M}_{\xi}}{\gamma_d (1 - \varepsilon/\xi)}.$$

Hence it follows from (2.6)-(2.11) that there exists a constant $M > 0$ which depends on ε , d and ψ but is independent of N and u such that

$$(2.12) \quad \left| \frac{\psi(d-N; u)}{\Gamma(1+N-d)} \right| = \frac{1}{2\pi|e^{\pi id}|} \left| H_1(d-N; \psi; u) \right| \leq M\varepsilon^{-N}$$

for $N \in \mathbb{N}_0$ with $N \geq \Re d + 1$. Note that we can take an arbitrary ε such that $0 < \varepsilon < \rho$. Since $|s| \leq |\Re s| + |\Im s|$ for $s \in \mathbb{C}$, we have

$$\begin{aligned} |\Gamma(1+N-d)| &= |(N-d)(N-d-1)\cdots([\Re d]+1-d)\Gamma([\Re d]+1-d)| \\ &\leq (N-[\Re d]+[|\Im d|]+1)! |\Gamma([\Re d]+1-d)| \end{aligned}$$

for $N \in \mathbb{N}_0$ with $N \geq \Re d + 1$. Hence we have

$$(2.13) \quad \begin{aligned} \frac{|\psi(d-N; u)|}{N!} &\leq \frac{(N-[\Re d]+[|\Im d|]+1)! |\Gamma([\Re d]+1-d)|}{N!} \left| \frac{\psi(d-N; u)}{\Gamma(1+N-d)} \right| \\ &\leq \frac{(N-[\Re d]+[|\Im d|]+1)! |\Gamma([\Re d]+1-d)|}{N!} M\varepsilon^{-N}. \end{aligned}$$

Suppose $u \in (1, 1+\delta]$ and $t = i\theta$ with $\theta \in (-\rho, \rho) \subset \mathbb{R}$. Then there exists an $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < \rho$ and $|\theta| < \varepsilon$. From the definition (1.9) we have

$$(2.14) \quad \begin{aligned} F_1(i\theta; d; \psi; u) &= \sum_{n=0}^{\infty} \frac{a(n)u^{-n}}{(\beta+nw)^d} \sum_{N=0}^{\infty} \frac{(\beta+nw)^N (i\theta)^N}{N!} \\ &= \sum_{N=0}^{\infty} \psi(d-N; u) \frac{(i\theta)^N}{N!}. \end{aligned}$$

From (2.13) we can see that each side of (2.14) is uniformly convergent with respect to $u \in [1, 1+\delta]$ because $|\theta| < \varepsilon$. Hence we can let $u \rightarrow 1$ in each side of (2.14), namely (2.14) holds for $u = 1$ when $\theta \in (-\rho, \rho)$. We can define

$$F_1(t; d; \psi; u) = \sum_{N=0}^{\infty} \psi(d-N; u) \frac{t^N}{N!}$$

for any $u \in [1, 1+\delta]$ and $t \in \mathbb{C}$ with $|t| < \rho$. From (2.13), this is uniformly convergent with respect to $(t, u) \in \overline{\mathcal{D}}(\xi) \times [1, 1+\delta]$ when $0 < \xi < \rho$. Thus we have the assertion. \square

Example 2.2. Let $f : (\mathbb{Z}/m\mathbb{Z}) \rightarrow \mathbb{C}$ such that $\sum_{a=1}^m f(a) = 0$. It can be regarded as a periodic function defined on \mathbb{Z} . For example, any non-trivial primitive Dirichlet character and any non-trivial additive character defined mod m satisfy this condition. We define

$$(2.15) \quad L(s; f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

and

$$G_1(t; L; u) = \sum_{n=1}^{\infty} f(n)u^{-n}e^{nt} = \sum_{a=1}^m \frac{f(a)u^{-a}e^{at}}{1 - u^{-m}e^{mt}}$$

for $u \in [1, 1 + \delta]$. Then $L(s; f)$ and $G_1(t; L; u)$ satisfy Assumptions I-III. Note that $\rho = 2\pi/m$ and $q = 1$ in this case. For $d \in \mathbb{C}$ with $\Re d > 1$, let

$$F_1(t; d; L) = \sum_{n=1}^{\infty} \frac{f(n)e^{nt}}{n^d}.$$

It follows from Proposition 2.1 that $F_1(t; d; L)$ is holomorphic on $\mathcal{D}(2\pi/m)$ and satisfies that

$$(2.16) \quad F_1(t; d; L) = \sum_{N=0}^{\infty} L(d - N; f) \frac{t^N}{N!}.$$

In particular when f is a primitive Dirichlet character χ of conductor m , we know that $L(-2j-1, \chi) = 0$ if $\chi(-1) = -1$ and $L(-2j, \chi) = 0$ if $\chi(-1) = 1$ for $j \in \mathbb{N}_0$ (see, for example, [16] Chap. 4). Hence, applying (2.16) with $d = 2k$ and $d = 2k + 1$ for $k \in \mathbb{N}$ and using $\cos x = (e^{ix} + e^{-ix})/2$, we obtain,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\chi(n) \cos(n\theta)}{n^{2k}} &= \sum_{j=0}^{k-1} L(2k - 2j, \chi) \frac{(i\theta)^{2j}}{(2j)!} \quad (\chi(-1) = 1), \\ \sum_{n=1}^{\infty} \frac{\chi(n) \cos(n\theta)}{n^{2k+1}} &= \sum_{j=0}^k L(2k + 1 - 2j, \chi) \frac{(i\theta)^{2j}}{(2j)!} \quad (\chi(-1) = -1) \end{aligned}$$

for $\theta \in (-2\pi/m, 2\pi/m)$. These are typical examples of Berndt's result (see [3] Theorem 4.2; see also [5] (1.2.12)). Similarly, it follows from (2.16) that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\chi(n) \cos(n\theta)}{n^{2k+1}} &= \sum_{j=0}^{\infty} L(2k + 1 - 2j, \chi) \frac{(i\theta)^{2j}}{(2j)!} \quad (\chi(-1) = 1), \\ \sum_{n=1}^{\infty} \frac{\chi(n) \cos(n\theta)}{n^{2k}} &= \sum_{j=0}^{\infty} L(2k - 2j, \chi) \frac{(i\theta)^{2j}}{(2j)!} \quad (\chi(-1) = -1) \end{aligned}$$

for $k \in \mathbb{N}$ and $\theta \in (-2\pi/m, 2\pi/m)$. Using the functional equations for $L(s, \chi)$, we can confirm that these equations coincide with Katsurada's formulas for $L(s, \chi)$ (see [9] Theorem 3).

3 Proof of Theorem 1.1

Using the method introduced in [14] Section 2 (see also [11, 12, 13]), we give the proof of Theorem 1.1 by the induction on r . The case of $r = 1$

can be directly obtained from Assumptions I and II. Hence we assume that Theorem 1.1 holds for $r - 1$, and aim to prove the case of $r (\geq 2)$.

As mentioned in Section 1, let

$$\Psi_r(s_1, \dots, s_r; u) = \Psi_r(s_1, \dots, s_r; \psi_1, \dots, \psi_r; u)$$

be the function defined by (1.7). Since each $\psi_k(s)$ defined by (1.5) converges absolutely for $\Re s > q_k$ ($1 \leq k \leq r$), we can easily check that $\Psi_r(s_1, \dots, s_r; u)$ converges absolutely if $\sigma_k = \Re s_k > q_k$ ($1 \leq k \leq r$).

First we assume each $\sigma_k > q_k$ ($1 \leq k \leq r$). Recall the Mellin-Barnes formula

$$(3.1) \quad \Gamma(s)(1 + \lambda)^{-s} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s + z)\Gamma(-z)\lambda^z dz,$$

where $\Re s > 0$, $|\arg \lambda| < \pi$, $\lambda \neq 0$, $-\Re s < c < 0$, and the path of integration is the vertical line $\Re z = c$. By the above assumption, we may assume $-\sigma_r < c < -q_r$. Put $s = s_r$ and

$$\lambda = \frac{\alpha_r - \alpha_{r-1} + n_r w_r}{\alpha_{r-1} + n_1 w_1 + \dots + n_{r-1} w_{r-1}}$$

in (3.1). Then multiply the both sides by

$$\frac{a_1(n_1) \cdots a_r(n_r) u^{-\sum_{\nu=1}^r n_\nu}}{\prod_{j=1}^{r-2} (\alpha_j + \sum_{\nu=1}^j n_\nu w_\nu)^{s_j} (\alpha_{r-1} + \sum_{\nu=1}^{r-1} n_\nu w_\nu)^{s_{r-1} + s_r}}$$

and sum up with respect to n_1, \dots, n_r . Then we have

$$(3.2) \quad \Psi_r(s_1, \dots, s_r; u) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \\ \times \Psi_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + z; u) \psi_r(-z; u) dz.$$

Let $M \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$ a small positive number. We shall shift the path to $\Re z = M - \varepsilon$. We see that

$$\Psi_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + z; u) = O(1)$$

in the region $c \leq \Re z \leq M - \varepsilon$ because $\sigma_k > q_k$ ($1 \leq k \leq r - 2$), $-\sigma_r < c$ and

$$\sigma_{r-1} + \sigma_r + \Re z \geq \sigma_{r-1} + \sigma_r + c > \sigma_{r-1}.$$

From the well-known Stirling formula for $\Gamma(s)$, we have

$$(3.3) \quad |\Gamma(s)| = e^{-\frac{\pi}{2}|\tau|} (|\tau| + 1)^{\sigma - \frac{1}{2}} \left(1 + O\left(\frac{1}{|\tau| + 1}\right) \right)$$

as $|\tau| \rightarrow \infty$, where $s = \sigma + i\tau$. Hence, by Assumption II, we see that the integrand on the right-hand side of (3.2) tends to zero as $|\Im z| \rightarrow \infty$, namely this shifting is possible. By the assumption of induction, we see that Ψ_{r-1} is holomorphic on \mathbb{C}^{r-1} and ψ_r holomorphic on \mathbb{C} . Therefore we only have to count the residues of the poles of $\Gamma(-z)$ at $z = 0, 1, \dots, M-1$. Since the residue of the pole of $\Gamma(s_r + z)\Gamma(-z)/\Gamma(s_r)$ at $z = k$ equals to $-\binom{-s_r}{k}$, we obtain

$$\begin{aligned}
(3.4) \quad & \Psi_r(s_1, \dots, s_r; u) \\
&= \sum_{k=0}^{M-1} \binom{-s_r}{k} \Psi_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + k; u) \psi_r(-k; u) \\
&+ \frac{1}{2\pi i} \int_{(M-\varepsilon)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \\
&\quad \times \Psi_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + z; u) \psi_r(-z; u) dz.
\end{aligned}$$

The first term on the right-hand side is holomorphic on the whole \mathbb{C}^r space by the assumption of induction. On the other hand, $\Gamma(s_r + z)$ has no pole on the path $(M-\varepsilon)$, when $\Re(-s_r) = -\sigma_r < M-\varepsilon$, namely $\sigma_r > -M+\varepsilon$. Using (3.3) and Assumption II, we see that the second term on the right-hand side of (3.4) is convergent absolutely, so is holomorphic on the region

$$\left\{ (s_1, \dots, s_r) \in \mathbb{C}^r \mid \sigma_1 > q_1, \dots, \sigma_{r-1} > q_{r-1}, \sigma_r > -M + \varepsilon \right\},$$

where M is arbitrary.

Next we fix $s_r \in \mathbb{C}$ with $\sigma_r > -M + \varepsilon$, and consider the continuation with respect to s_k for $1 \leq k \leq r-1$. Since Ψ_{r-1} is holomorphic on \mathbb{C}^{r-1} , the integrand on the right-hand side of (3.4) is holomorphic for all $(s_1, \dots, s_{r-1}) \in \mathbb{C}^{r-1}$. So, if we prove that the second term on the right-hand side of (3.4) converges absolutely for any $(s_1, \dots, s_{r-1}) \in \mathbb{C}^{r-1}$ and $s_r \in \mathbb{C}$ with $\sigma_r > -M + \varepsilon$, then $\Psi_r(s_1, \dots, s_r; u)$ is holomorphic on the whole \mathbb{C}^r space because M is arbitrary. In order to prove this result, we need the following lemma.

Lemma 3.1. *For $r \in \mathbb{N}$ with $r \geq 2$, there exists a polynomial $P_r(X) \in \mathbb{R}[X]$ such that*

$$(3.5) \quad \Psi_r(s_1, \dots, s_r; u) = O\left(P_r(|\tau_r|)e^{\theta_0|\tau_r|}\right) \quad (|\tau_r| \rightarrow \infty)$$

for any $(s_1, \dots, s_{r-1}) \in \mathbb{C}^{r-1}$ and $u \in [1, 1+d]$, where the constant implied by the O -symbol depends on $\tau_1, \dots, \tau_{r-1}$.

Proof. We denote (3.5) by

$$\Psi_r(s_1, \dots, s_r; u) \ll P_r(|\tau_r|)e^{\theta_0|\tau_r|}.$$

We prove this lemma by the induction on $r(\geq 2)$. First we consider the case of $r = 2$. It follows from Assumption II and (3.4) that

$$\begin{aligned}
(3.6) \quad |\Psi_2(s_1, s_2; u)| &\leq \sum_{k=0}^{M-1} \left| \binom{-s_2}{k} \right| |\Psi_1(s_1 + s_2 + k; u) \psi_2(-k; u)| \\
&\quad + \frac{1}{2\pi} \left| \int_{(M-\varepsilon)} \frac{\Gamma(s_r + z) \Gamma(-z)}{\Gamma(s_r)} \Psi_1(s_1 + s_2 + z; u) \psi_2(-z; u) dz \right| \\
&\ll \sum_{k=0}^{M-1} \left| \binom{-s_2}{k} \right| e^{\theta_0 |\tau_2|} \\
&\quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\Gamma(s_r + z) \Gamma(-z)}{\Gamma(s_r)} \right| e^{\theta_0 |\tau_2 + y|} e^{\theta_0 |y|} dy,
\end{aligned}$$

where $z = x + iy$. For simplicity, we denote the last term on the right-hand side of (3.6) by I . Using (3.3), we have

$$\begin{aligned}
(3.7) \quad I &\ll e^{\frac{\pi}{2} |\tau_2|} (|\tau_2| + 1)^{-\sigma_2 + \frac{1}{2}} \\
&\quad \times \int_{-\infty}^{\infty} e^{(\theta_0 - \frac{\pi}{2}) |\tau_2 + y|} e^{(\theta_0 - \frac{\pi}{2}) |y|} (|\tau_2 + y| + 1)^{\sigma_2 + x - \frac{1}{2}} (|y| + 1)^{-x - \frac{1}{2}} dy.
\end{aligned}$$

Now we apply Lemma 4 in [12] with $A = B = \theta_0 - \frac{1}{2}\pi$, $p = \sigma_2 + x - \frac{1}{2}$ and $q = -x - \frac{1}{2}$. Then it follows from (3.7) that

$$\begin{aligned}
(3.8) \quad I &\ll e^{\frac{\pi}{2} |\tau_2|} (|\tau_2| + 1)^{-\sigma_2 + \frac{1}{2}} \\
&\quad \times \left[\left\{ 1 + (|\tau_2| + 1)^{\sigma_2 + x - \frac{1}{2}} \right\} (|\tau_2| + 1)^{-x + \frac{1}{2}} e^{(\theta_0 - \frac{\pi}{2}) |\tau_2|} \right. \\
&\quad \left. + \left\{ 1 + (|\tau_2| + 1)^{\sigma_2 + x - \frac{1}{2}} \right\} e^{(\theta_0 - \frac{\pi}{2}) |\tau_2|} \right].
\end{aligned}$$

Combining (3.6) and (3.8), we see that there exists $P_2(X) \in \mathbb{R}[X]$ such that

$$\Psi_2(s_1, s_2; u) \ll P_2(|\tau_2|) e^{\theta_0 |\tau_2|} \quad (|\tau_2| \rightarrow \infty).$$

Thus we have the assertion for $r = 2$.

Assume that the assertion for $r - 1$ holds. Substituting the asserted bounds into (3.4) and using Assumption II, we have

$$\begin{aligned}
\Psi_r(s_1, \dots, s_r; u) &\ll \sum_{k=0}^{M-1} \left| \binom{-\sigma_r + i\tau_r}{k} \right| P_{r-1}(|\tau_{r-1} + \tau_r|) e^{\theta_0 |\tau_{r-1} + \tau_r|} \\
&\quad + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left| \frac{\Gamma(s_r + z) \Gamma(-z)}{\Gamma(s_r)} \right| \\
&\quad \times P_{r-1}(|\tau_{r-1} + \tau_r + y|) e^{\theta_0 |\tau_{r-1} + \tau_r + y|} e^{\theta_0 |y|} dy.
\end{aligned}$$

By the same method as mentioned above, we can see that there exists $P_r(X) \in \mathbb{R}[X]$ such that

$$\Psi_r(s_1, \dots, s_r; u) \ll P_r(|\tau_r|)e^{\theta_0|\tau_r|}.$$

By induction, we obtain the proof of Lemma 3.1. \square

Now we can complete the proof of Theorem 1.1 as follows. If we fix any $(s_1, \dots, s_r) \in \mathbb{C}^r$, then it follows from Lemma 3.1 that

$$\Psi_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + z; u) \ll P_{r-1}(|\tau_{r-1} + \tau_r + y|)e^{\theta_0|\tau_{r-1} + \tau_r + y|}$$

as $|y| \rightarrow \infty$, where $z = x + iy$. Since s_{r-1} is fixed, this can be written as

$$(3.9) \quad \Psi_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + z; u) \ll \tilde{P}_{r-1}(|\tau_r + y|)e^{\theta_0|\tau_r + y|} \quad (|y| \rightarrow \infty),$$

where $\tilde{P}_{r-1}(X) \in \mathbb{R}[X]$. We denote the second term on the right-hand side of (3.4) by \tilde{I} . Then, by using (3.3), and by (3.8) and Assumption II, we have

$$\begin{aligned} \tilde{I} &\ll \int_{-\infty}^{\infty} \tilde{P}_{r-1}(y)e^{-\frac{\pi}{2}|y| - \frac{\pi}{2}|\tau_r + y|}e^{\theta_0|\tau_r + y|}e^{\theta_0|y|}dy \\ &= \int_{-\infty}^{\infty} \tilde{P}_{r-1}(y)e^{(\theta_0 - \frac{\pi}{2})(|\tau_r + y| + |y|)}dy \end{aligned}$$

for some $\tilde{P}_{r-1}(X) \in \mathbb{R}[X]$. Since $0 \leq \theta_0 < \pi/2$, \tilde{I} converges absolutely for any $(s_1, \dots, s_r) \in \mathbb{C}^r$. By (3.4), we see that $\Psi_r(s_1, \dots, s_r; u)$ is holomorphic on \mathbb{C}^r .

Lastly we prove (1.8). More strictly we prove that (1.8) is uniformly convergent with respect to s_j ($1 \leq j \leq r$) in any fixed strip $\sigma_{1j} \leq \Re s_j \leq \sigma_{2j}$ as $u \rightarrow 1 + 0$. The case of $r = 1$ follows from Assumption II. Hence we assume that the case of $r - 1$ holds and prove the case of r (≥ 2). Let $u \rightarrow 1 + 0$ in (3.4). From the assumption of induction, the integrand in the second term on the right-hand side of (3.4) is uniformly convergent with respect to z in any fixed strip $\sigma_1 \leq \Re z (= M - \varepsilon) \leq \sigma_2$ as $u \rightarrow 1 + 0$. Exchanging $\lim_{u \rightarrow 1 + 0}$ and the integral, and using the assumption of induction, we see that the right-hand side of (3.4) tends to

$$(3.10) \quad \begin{aligned} &\sum_{k=0}^{M-1} \binom{-s_r}{k} \Psi_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + k; 1) \psi_r(-k; 1) \\ &+ \frac{1}{2\pi i} \int_{(M-\varepsilon)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \\ &\quad \times \Psi_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + z; 1) \psi_r(-z; 1) dz \end{aligned}$$

as $u \rightarrow 1 + 0$. It is clear that this convergence is uniformly with respect to s_j in any fixed strip $\sigma_{1j} \leq \Re s_j \leq \sigma_{2j}$ ($1 \leq j \leq r$). From (3.4), we see that (3.10) coincides with $\Psi_r(s_1, \dots, s_r; 1)$. Hence the assertion in case of r holds. By induction, we have the assertion that (1.8) holds. Thus we obtain the proof of Theorem 1.1.

Remark 3.2. For any $N \in \mathbb{N}_0$, let $M = N + 1$ and $s_r \rightarrow -N$ in (3.4). Then the second term on the right-hand side of (3.4) tends to 0 because $\Gamma(s_r)$ has a pole at $s_r = -N$. Hence we obtain

$$(3.11) \quad \Psi_r(s_1, \dots, s_{r-1}, -N; u) \\ = \sum_{\nu=0}^N \binom{N}{\nu} \Psi_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + \nu - N; u) \psi_r(-\nu; u)$$

for $u \in [1, 1 + \delta]$ and $(s_1, \dots, s_{r-1}) \in \mathbb{C}^{r-1}$. In particular, let $\psi_j(s) = L(s; f_j)$ ($1 \leq j \leq r$) and $u = 1$, where each f_j is defined mod m_j and satisfies a certain condition (see Example 2.2). Then we can check that Assumptions I-III hold. In this case, $\Psi_r(s_1, \dots, s_r; 1)$ coincides with the multiple L -function

$$L_r(s_1, \dots, s_r; f_1, \dots, f_r) = \sum_{n_1, \dots, n_r=1}^{\infty} \frac{f_1(n_1) \cdots f_r(n_r)}{n_1^{s_1} (n_1 + n_2)^{s_2} \cdots (n_1 + \cdots + n_r)^{s_r}},$$

which has been studied in [2]. Hence (3.11) gives that

$$(3.12) \quad L_r(s_1, \dots, s_{r-1}, -N; f_1, \dots, f_r) \\ = \sum_{\nu=0}^N \binom{N}{\nu} L_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + \nu - N; f_1, \dots, f_{r-1}) L_1(-\nu; f_r)$$

for $(s_1, \dots, s_{r-1}) \in \mathbb{C}^{r-1}$. This result was proved by Kamano (see [8]) by using the method introduced in [1]. This case can be also derived directly from the relation (2.3) in [14].

4 Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3 by the induction on r .

The case of $r = 1$ is just what we argued in Proposition 2.1. Hence we assume that the assertion of Theorem 1.3 holds for $r - 1$ and prove the case of $r (\geq 2)$.

Let $\mathcal{P}_r = \{\psi_1, \dots, \psi_r\}$ which satisfy Assumptions I-III. Then we can take $\{q_k\}_{1 \leq k \leq r}$ and $\{\rho_k\}_{1 \leq k \leq r}$, and define η_{r-1} and η_r by (1.6). Let

$$(4.1) \quad \begin{aligned} G_r(t_1, \dots, t_r; \mathbf{d}_{r-1}; \mathcal{P}_r; u) &= F_{r-1}(t_1, \dots, t_{r-2}, t_{r-1} + t_r; \mathbf{d}_{r-1}; \mathcal{P}_{r-1}; u) G_1(t_r; \psi_r; u) \\ &= \sum_{n_1, \dots, n_r=0}^{\infty} \frac{a_1(n_1) \cdots a_r(n_r) u^{-\sum_{l=1}^r n_l} \prod_{j=1}^r e^{(\alpha_j + \sum_{\mu=1}^j n_\mu w_\mu) t_j}}{\prod_{j=1}^{r-1} (\alpha_j + \sum_{\mu=1}^j n_\mu w_\mu)^{d_j}}, \end{aligned}$$

which is convergent when $\Re t_j < 0$ ($1 \leq j \leq r$). By the assumption of induction, we see that $F_{r-1}(t_1, \dots, t_{r-1} + t_r; \mathbf{d}_{r-1}; \mathcal{P}_{r-1}; u)$ is holomorphic for $(t_1, \dots, t_r) \in \mathcal{D}(\eta_{r-1})^{r-2} \times \mathcal{D}(\eta_{r-1}/2)^2$, and $G_1(t_r; \psi_r; u)$ is holomorphic for $t_r \in \mathcal{D}(\rho_r)$. Since we have $\eta_r \leq \min(\eta_{r-1}/2, \rho_r)$, we can see that $G_r(t_1, \dots, t_r; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ is holomorphic for $(t_1, \dots, t_r) \in \mathcal{D}(\eta_r)^r$. Therefore, if we fix $t_r \in \mathcal{D}(\eta_r)$ then the function of real $(r-1)$ variables $G_r(i\theta_1, \dots, i\theta_{r-1}, t_r; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ is real-analytic for $(\theta_1, \dots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1} \subset \mathbb{R}^{r-1}$ (see, for example, [10] Corollary 2.3.7). Similarly, if we fix $(\theta_1, \dots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1}$, then we see that $G_r(\{i\theta_k\}, t_r; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ is holomorphic for $t_r \in \mathcal{D}(\eta_r)$. Hence we define $\{\mathfrak{B}_n(\{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u)\}_{n \geq 0}$ by

$$(4.2) \quad G_r(i\theta_1, \dots, i\theta_{r-1}, t_r; \mathbf{d}_{r-1}; \mathcal{P}_r; u) = \sum_{n=0}^{\infty} \mathfrak{B}_n(\{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) \frac{t_r^n}{n!}.$$

As well as in the proof of Proposition 2.1, we let

$$(4.3) \quad \begin{aligned} H_r(s; i\theta_1, \dots, i\theta_{r-1}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) &= \int_{\Upsilon} G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u) t^{s-1} dt \\ &= (e^{2\pi i s} - 1) \int_{\varepsilon}^{\infty} G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u) t^{s-1} dt \\ &\quad + \int_{C_{\varepsilon}} G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u) t^{s-1} dt, \end{aligned}$$

which is holomorphic for all $s \in \mathbb{C}$ if we fix $(\theta_1, \dots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1}$ and $0 < \varepsilon < \eta_r$.

Putting $s = -n$ for $n \in \mathbb{N}_0$ and $\varepsilon = \xi$ with $0 < \xi < \eta_r$ in (4.3), and using (4.2), we have

$$\begin{aligned} H_r(-n; i\theta_1, \dots, i\theta_{r-1}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) &= \int_{C_{\xi}} G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u) t^{-n-1} dt \\ &= \frac{(2\pi i) \mathfrak{B}_n(\{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) (-1)^n}{n!}. \end{aligned}$$

By the assumption of induction and (4.1), we see that the Taylor expansion series of $G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ around $t = 0$ is uniformly convergent

with respect to $(\theta_1, \dots, \theta_{r-1}, t, u) \in [-\xi, \xi]^{r-1} \times \overline{\mathcal{D}}(\xi) \times [1, 1 + \delta]$ when $\xi \in \mathbb{R}$ with $0 < \xi < \eta_r$. In particular, $G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ is continuous for $(\theta_1, \dots, \theta_{r-1}, t, u) \in [-\xi, \xi]^{r-1} \times \overline{\mathcal{D}}(\xi) \times [1, 1 + \delta]$. Hence there exists the value

$$\tilde{\mathcal{M}}_\xi = \max |G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u)| \quad \text{on } [-\xi, \xi]^{r-1} \times \{|t| = \xi\} \times [1, 1 + \delta]$$

when $\xi \in \mathbb{R}$ with $0 < \xi < \eta_r$. By the above equation, we have

$$(4.4) \quad \frac{|\mathfrak{B}_n(\{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u)|}{n!} \leq \frac{\tilde{\mathcal{M}}_\xi}{\xi^n}$$

for any $n \in \mathbb{N}_0$, $(\theta_1, \dots, \theta_{r-1}) \in [-\xi, \xi]^{r-1}$ and $u \in [1, 1 + \delta]$.

Define

$$(4.5) \quad \begin{aligned} \mathcal{Z}_r(\mathbf{d}_{r-1}, s; i\theta_1, \dots, i\theta_{r-1}; \mathcal{P}_r; u) &= F_r(i\theta_1, \dots, i\theta_{r-1}, 0; \mathbf{d}_{r-1}, s; \mathcal{P}_r; u) \\ &= \sum_{n_1, \dots, n_r=0}^{\infty} \frac{a_1(n_1) \cdots a_r(n_r) u^{-\sum_{\nu=1}^r n_\nu} \prod_{j=1}^{r-1} e^{(\alpha_j + \sum_{\mu=1}^j n_\mu w_\mu) i\theta_j}}{\prod_{j=1}^{r-1} (\alpha_j + \sum_{\nu=1}^j n_\nu w_\nu)^{d_j} (\alpha_r + \sum_{\nu=1}^r n_\nu w_\nu)^s} \end{aligned}$$

for $(\theta_1, \dots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1}$, $s \in \mathbb{C}$ with $\Re s > q_r$ and $u \in [1, 1 + \delta]$. Assuming $\Re s > \max(1, q_r)$ and using the same method as in the proof of Proposition 2.1, we have

$$(4.6) \quad \begin{aligned} \mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{i\theta_k\}; \mathcal{P}_r; u) &= \frac{1}{(e^{2\pi i s} - 1) \Gamma(s)} H_r(s; \{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) \\ &= \frac{\Gamma(1-s)}{2\pi i e^{\pi i s}} H_r(s; \{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u). \end{aligned}$$

Note that $H_r(s; \{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ is holomorphic for all $s \in \mathbb{C}$ if we fix $\{\theta_k\} \in (-\eta_r, \eta_r)^{r-1}$ (as mentioned above), and that poles of $\Gamma(1-s)$ coincide with $\mathbb{N} = \{1, 2, \dots\}$. Since $\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{i\theta_k\}; \mathcal{P}_r; u)$ is convergent absolutely for $s \in \mathbb{C}$ with $\Re s > q_r$, it follows from (4.6) that $\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{i\theta_k\}; \mathcal{P}_r; u)$ is defined and holomorphic for all $s \in \mathbb{C} \setminus \{1, 2, \dots, [q_r]\}$ if we fix $\{\theta_k\} \in (-\eta_r, \eta_r)^{r-1}$.

Furthermore, we can prove that $\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{i\theta_k\}; \mathcal{P}_r; u)$ has no pole as follows. We fix an arbitrary $s \in \mathbb{C}$. If $1 < u \leq 1 + \delta$ then from (1.7) and (4.5), and by substituting the Taylor expansion series for each $\exp((\alpha_j + \sum_{\nu=1}^j n_\nu w_\nu) i\theta_j)$ and changing the order of summations, we have

$$(4.7) \quad \begin{aligned} &\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{i\theta_k\}; \mathcal{P}_r; u) \\ &= \sum_{N_1, \dots, N_{r-1}=0}^{\infty} \Psi_r(d_1 - N_1, \dots, d_{r-1} - N_{r-1}, s; u) \frac{(i\theta_1)^{N_1} \cdots (i\theta_{r-1})^{N_{r-1}}}{N_1! \cdots N_{r-1}!}. \end{aligned}$$

We see that (4.3) is uniformly convergent with respect to $(\theta_1, \dots, \theta_{r-1}, u) \in [-\xi, \xi]^{r-1} \times [1, 1 + \delta]$, when we take any $\xi \in \mathbb{R}$ with $0 < \xi < \eta_r$. Hence, for $u \in [1, 1 + \delta]$, $H_r(s; \{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ is real-analytic for $(\theta_1, \dots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1}$. Put $\theta_1 = \dots = \theta_{r-1} = \theta$. Then for $u \in [1, 1 + \delta]$, $H_r(s; \{i\theta\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ is real-analytic for $\theta \in (-\eta_r, \eta_r)$, and its Taylor expansion series around $\theta = 0$ is uniformly convergent with respect to $(\theta, u) \in [-\xi, \xi] \times [1, 1 + \delta]$. It follows from (4.6) that $\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{i\theta\}; \mathcal{P}_r; u)$ also has these properties. Hence, for any $u \in [1, 1 + \delta]$, we define the one variable complex function $\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{t\}; \mathcal{P}_r; u)$ which is holomorphic for $t \in \mathcal{D}(\eta_r)$ and its Taylor expansion series around $t = 0$ is uniformly convergent with respect to $(t, u) \in \overline{\mathcal{D}}(\xi) \times [1, 1 + \delta]$. In particular, $\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{t\}; \mathcal{P}_r; u)$ is continuous for $(t, u) \in \overline{\mathcal{D}}(\xi) \times [1, 1 + \delta]$. Putting $\xi = \varepsilon$ with $0 < \varepsilon < \eta_r$, there exists the value

$$\mathcal{M}'_\varepsilon = \max |\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{t\}; \mathcal{P}_r; u)| \quad \text{on } \{t \in \mathbb{C} \mid |t| = \varepsilon\} \times [1, 1 + \delta].$$

Using the same method as in the proof of (4.4) and by (4.7) and the continuity of $\Psi_r(d_1 - N_1, \dots, d_{r-1} - N_{r-1}, s; u)$ on $u \in [1, 1 + \delta]$ (see Theorem 1.1), we see that

$$(4.8) \quad \left| \sum_{N_1 + \dots + N_{r-1} = n} \frac{\Psi_r(d_1 - N_1, \dots, d_{r-1} - N_{r-1}, s; u)}{N_1! \cdots N_{r-1}!} \right| \leq \frac{\mathcal{M}'_\varepsilon}{\varepsilon^n}$$

for $u \in [1, 1 + \delta]$ and $n \in \mathbb{N}_0$, where ε is an arbitrary real number with $0 < \varepsilon < \eta_r$. This means that the right-hand side of (4.7) is uniformly convergent with respect to $(\theta_1, \dots, \theta_{r-1}, u) \in [-\xi, \xi]^{r-1} \times [1, 1 + \delta]$ for any $\xi \in \mathbb{R}$ with $0 < \xi < \eta_r$. Hence we can let $u \rightarrow 1$ in (4.7), namely (4.7) holds for $u \in [1, 1 + \delta]$. Since s is an arbitrary complex number, $\mathcal{Z}_r(\mathbf{d}_{r-1}, s; \{i\theta_k\}; \mathcal{P}_r; u)$ has no pole, namely it is holomorphic for all $s \in \mathbb{C}$ when $u \in [1, 1 + \delta]$, and is real-analytic for $(\theta_1, \dots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1}$ when $s \in \mathbb{C}$ and $u \in [1, 1 + \delta]$.

For $d_r \in \mathbb{C}$ with $\Re d_r > q_r$ and $N \in \mathbb{N}_0$ with $N \geq \Re d_r + 1$, we put $s = d_r - N$ in (4.3). Then we have

$$(4.9) \quad \begin{aligned} & H_r(d_r - N; \{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) \\ &= (e^{2\pi i d_r} - 1) \int_\varepsilon^\infty G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u) t^{d_r - N - 1} dt \\ & \quad + \int_{\mathcal{C}_\varepsilon} G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u) t^{d_r - N - 1} dt. \end{aligned}$$

For simplicity, we denote by I_1 and I_2 the first and second terms of the right-hand side of (4.9), respectively. Since $N \geq \Re d_r + 1$, we have

$$\left| \int_\varepsilon^\infty e^{-(\alpha_r + \sum_{\mu=1}^r n_\mu w_\mu) t} t^{d_r - N - 1} dt \right| \leq \frac{e^{-(\alpha_r + \sum_{\mu=1}^r n_\mu w_\mu) \varepsilon} \varepsilon^{d_r - N - 1}}{\alpha_r + \sum_{\mu=1}^r n_\mu w_\mu}.$$

Hence we have

$$(4.10) \quad |I_1| \leq \varepsilon^{\Re d_r - N - 1} |e^{2\pi i d_r} - 1| \\ \times \sum_{n_1, \dots, n_r=0}^{\infty} \frac{|a_1(n_1) \cdots a_r(n_r)| e^{-(\alpha_r + \sum_{\mu=1}^r n_\mu w_\mu) \varepsilon}}{\prod_{j=1}^{r-1} (\alpha_j + \sum_{\mu=1}^j n_\mu w_\mu)^{\Re d_j} (\alpha_r + \sum_{\mu=1}^r n_\mu w_\mu)}.$$

On the other hand, by using (2.9), we have

$$I_2 = \begin{cases} (2\pi i) \mathfrak{B}_{N-d_r}(\{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) \frac{(-1)^{N-d_r}}{(N-d_r)!} & (N-d_r \in \mathbb{N}_0), \\ \varepsilon^{d_r - N} (e^{2\pi i d_r} - 1) \sum_{n=0}^{\infty} \frac{\mathfrak{B}_n(\{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) (-1)^n \varepsilon^n}{(n+d_r-N)n!} & (\text{otherwise}). \end{cases}$$

Note that the above infinite series in the second case is uniformly convergent with respect to $(\theta_1, \dots, \theta_{r-1}, u) \in [-\varepsilon, \varepsilon]^{r-1} \times [1, 1 + \delta]$ because of the assumption $\varepsilon < \eta_r$ and (4.4). Hence we have either

$$(4.11) \quad |I_2| \leq 2\pi \frac{|\mathfrak{B}_{N-d_r}(\{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u)|}{(N-d_r)!} \quad (N-d_r \in \mathbb{N}_0)$$

or

$$(4.12) \quad |I_2| \leq \varepsilon^{\Re d_r - N} |e^{2\pi i d_r} - 1| \left| \sum_{n=0}^{\infty} \mathfrak{B}_n(\{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) \frac{(-1)^n \varepsilon^n}{(n+d_r-N)n!} \right| \\ (\text{otherwise}).$$

As well as (2.12), it follows from (4.4), (4.6), (4.8)-(4.12) that there exists a constant $M > 0$ independent of N and $\{\theta_k\}$ such that

$$(4.13) \quad \left| \frac{\mathfrak{Z}_r(\mathbf{d}_{r-1}, d_r - N; \{i\theta_k\}; \mathcal{P}_r; u)}{\Gamma(1 + N - d_r)} \right| \\ \leq \frac{1}{2\pi |e^{\pi i d_r}|} \left| H_r(d_r - N; \{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) \right| \leq M \varepsilon^{-N}$$

for $N \in \mathbb{N}$ with $N \geq \Re d_r + 1$. Note that we can take an arbitrary ε such that $0 < \varepsilon < \eta_r$. As well as (2.13), we have

$$(4.14) \quad \frac{|\mathfrak{Z}_r(\mathbf{d}_{r-1}, d_r - N; \{i\theta_k\}; \mathcal{P}_r; u)|}{N!} \\ \leq \frac{(N - [\Re d_r] + [[\Im d_r]] + 1)! |\Gamma([\Re d_r] + 1 - d_r)|}{N!} M \varepsilon^{-N}$$

for $N \in \mathbb{N}$ with $N \geq \Re d_r + 1$ and $u \in [1, 1 + \delta]$.

Suppose $1 < u \leq 1 + \delta$ and $\theta_r \in (-\eta_r, \eta_r)$. Then by (1.9), and using the Taylor expansion series for $\exp((\alpha_r + \sum_{\nu=1}^r n_\nu w_\nu) i \theta_r)$, we have

$$(4.15) \quad \begin{aligned} F_r(i\theta_1, \dots, i\theta_{r-1}, i\theta_r; \mathbf{d}_r; \mathcal{P}_r; u) \\ = \sum_{N_r=0}^{\infty} \mathcal{Z}_r(\mathbf{d}_{r-1}, d_r - N_r; \{i\theta_k\}; \mathcal{P}_r; u) \frac{(i\theta_r)^{N_r}}{N_r!}. \end{aligned}$$

By (4.14), we see that the right-hand side of (4.15) is uniformly convergent with respect to $(\theta_r, u) \in [-\xi, \xi] \times [1, 1 + \delta]$ when $(\theta_1, \dots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1}$ and $0 < \xi < \eta_r$. Hence (4.15) holds for $u = 1$. As we mentioned above, (4.7) holds for any $s \in \mathbb{C}$, $(\theta_1, \dots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1}$, and $u \in [1, 1 + \delta]$. Hence we have

$$(4.16) \quad \begin{aligned} \mathcal{Z}_r(\mathbf{d}_{r-1}, d_r - N_r; \{i\theta_k\}; \mathcal{P}_r; u) \\ = \sum_{N_1, \dots, N_{r-1}=0}^{\infty} \Psi_r(d_1 - N_1, \dots, d_r - N_r; u) \frac{(i\theta_1)^{N_1} \dots (i\theta_{r-1})^{N_{r-1}}}{N_1! \dots N_{r-1}!} \end{aligned}$$

for $u \in [1, 1 + \delta]$. Hence (4.15) can also be written as

$$(4.17) \quad \begin{aligned} F_r(i\theta_1, \dots, i\theta_r; \mathbf{d}_r; \mathcal{P}_r; u) \\ = \sum_{N_1, \dots, N_r=0}^{\infty} \Psi_r(d_1 - N_1, \dots, d_r - N_r; u) \frac{(i\theta_1)^{N_1} \dots (i\theta_r)^{N_r}}{N_1! \dots N_r!} \end{aligned}$$

for $u \in [1, 1 + \delta]$, and (4.17) is uniformly convergent with respect to $(\theta_1, \dots, \theta_r, u) \in [-\xi, \xi]^r \times [1, 1 + \delta]$ for any $\xi \in \mathbb{R}$ with $0 < \xi < \eta_r$. Therefore, for $u \in [1, 1 + \delta]$, we can define

$$(4.18) \quad \begin{aligned} F_r(t_1, \dots, t_r; \mathbf{d}_r; \mathcal{P}_r; u) \\ = \sum_{N_1, \dots, N_r=0}^{\infty} \Psi_r(d_1 - N_1, \dots, d_r - N_r; u) \frac{t_1^{N_1} \dots t_r^{N_r}}{N_1! \dots N_r!} \end{aligned}$$

which is uniformly convergent with respect to $(t_1, \dots, t_r, u) \in \overline{\mathcal{D}}(\xi)^r \times [1, 1 + \delta]$ and is holomorphic for $(t_1, \dots, t_r) \in \mathcal{D}(\eta_r)^r$ (see, for example, [7] Section 2.2). Thus we obtain the case of r . By induction, we obtain the proof of Theorem 1.3.

5 Some applications

First we prove the following estimation formulas for $\Psi_r(d_1 - N_1, \dots, d_r - N_r; u)$ by using the same method as in the proof of Proposition 2.3.10 in [10].

Proposition 5.1. *With the same notation as in Theorem 1.3,*

$$(5.1) \quad \limsup_{N_1+\dots+N_r \rightarrow \infty} \left\{ \frac{|\Psi_r(d_1 - N_1, \dots, d_r - N_r; u)|}{N_1! \cdots N_r!} \right\}^{1/(N_1+\dots+N_r)} \leq \frac{1}{\eta_r}.$$

Proof. Assume that the left-hand side of (5.1) is greater than $1/\eta_r$. Then we take $\kappa \in \mathbb{R}$ with $\kappa > 1/\eta_r$ such that there exist infinitely many $(N_1, \dots, N_r) \in \mathbb{N}_0^r$ such that

$$\frac{|\Psi_r(d_1 - N_1, \dots, d_r - N_r; u)|}{N_1! \cdots N_r!} > \kappa^{N_1+\dots+N_r}.$$

This means that the right-hand side of (1.10) does not converge absolutely at $(1/\kappa, \dots, 1/\kappa) \in \mathcal{D}(\eta_r)^r$. This is contradiction. \square

Example 5.2. Let $\psi_j(s) = L(s; f_j)$ ($1 \leq j \leq r$) as considered in Remark 3.2. Then (5.1) gives

$$(5.2) \quad \limsup_{N_1+\dots+N_r \rightarrow \infty} \left\{ \frac{|L_r(d_1 - N_1, \dots, d_r - N_r; f_1, \dots, f_r)|}{N_1! \cdots N_r!} \right\}^{1/(N_1+\dots+N_r)} \leq \frac{1}{\eta_r},$$

where each $\Re d_j > 1$ ($1 \leq j \leq r$) and $\eta_r = \min_{1 \leq k \leq r} \{2\pi/2^{r-1}m_k\}$.

Secondly we give certain multiple analogues of both Berndt's and Katsurada's formulas considered in Example 2.2.

Example 5.3. As well as the above example, let $\psi_j(s) = L(s; f_j)$ ($1 \leq j \leq r$) and define a certain generalization of multiple polylogarithm by

$$(5.3) \quad \begin{aligned} & \mathcal{F}_r(t_1, \dots, t_r; \mathbf{d}_r; f_1, \dots, f_r) \\ &= \sum_{n_1, \dots, n_r=1}^{\infty} \frac{f_1(n_1) \cdots f_r(n_r) \prod_{j=1}^r e^{(\sum_{\mu=1}^j n_\mu)t_j}}{n_1^{d_1} (n_1 + n_2)^{d_2} \cdots (n_1 + \cdots + n_r)^{d_r}} \end{aligned}$$

for $d_1, \dots, d_r \in \mathbb{C}$ with $\Re d_j > 1$ ($1 \leq j \leq r$). Applying Theorem 1.3 with $\psi_j(s) = L(s; f_j)$ ($1 \leq j \leq r$) and $u = 1$, we see that $\mathcal{F}_r(t_1, \dots, t_r; \mathbf{d}_r; f_1, \dots, f_r)$ is defined and holomorphic for $(t_1, \dots, t_r) \in \mathcal{D}(\eta_r)^r$ such that

$$(5.4) \quad \begin{aligned} & \mathcal{F}_r(t_1, \dots, t_r; \mathbf{d}_r; f_1, \dots, f_r) \\ &= \sum_{N_1, \dots, N_r=0}^{\infty} L_r(d_1 - N_1, \dots, d_r - N_r; f_1, \dots, f_r) \frac{t_1^{N_1} \cdots t_r^{N_r}}{N_1! \cdots N_r!}, \end{aligned}$$

where $\eta_r = \min_{1 \leq k \leq r} \{2\pi/2^{r-1}m_k\}$. Putting $t_1 = \cdots = t_{r-1} = 0$ and $t_r = \pm i\theta$ for $\theta \in (-\eta_r, \eta_r)$ in (5.4), we have

$$(5.5) \quad \begin{aligned} & \sum_{n_1, \dots, n_r=1}^{\infty} \frac{f_1(n_1) \cdots f_r(n_r) \cos((n_1 + \cdots + n_r)\theta)}{n_1^{d_1} (n_1 + n_2)^{d_2} \cdots (n_1 + \cdots + n_r)^{d_r}} \\ &= \sum_{N=0}^{\infty} L_r(d_1, \dots, d_{r-1}, d_r - 2N; f_1, \dots, f_r) \frac{(i\theta)^{2N}}{(2N)!}. \end{aligned}$$

Remark 5.4. In the case $f_j(n) = (-1)^n$ ($1 \leq j \leq r$), $\mathcal{F}_r(i\theta_1, \dots, i\theta_r; f_1, \dots, f_r)$ has recently been used to prove what is called the parity result for Euler-Zagier sums (see [15]).

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