Similarity between the Mandelbrot set and the Julia sets: a simplified proof *

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Abstract

This note gives a simplified proof of the similarity between the Mandelbrot set and the quadratic Julia sets at the Misiurewicz points following [Ka]. The result we concern is originally due to Tan Lei [TL]. We will just change the language to simplify the description of the proof.

The Julia sets and the Mandelbrot set. Let us consider the quadratic family

$$\{f_c(z) = z^2 + c \,:\, c \in \mathbb{C}\}.$$

The *Mandelbrot set* M is defined by

$$M \;:=\; \{c \in \mathbb{C} : |f_c^n(c)| \le 2 \;(\forall n \in \mathbb{N})\}.$$

For each $c \in \mathbb{C}$, the *filled Julia set* K_c is defined by

$$K_c \;:=\; \{z \in \mathbb{C} : |f_c^n(z)| \le \max\{2, |c|\} \;(\forall n \in \mathbb{N})\}.$$

The Julia set J_c is the boundary of K_c. It is known that M, K_c, and J_c are all non-empty and compact for any $c \in \mathbb{C}$.

Misiurewicz parameters. We say $c_0 \in \partial M$ is called a *Misiurewicz parameter* if the forward orbit of c_0 by f_{c_0} eventually lands on a repelling cycle. More precisely, there exist minimal $l, p \ge 1$ such that $a_0 := f_{c_0}^l(c_0)$ satisfies $a_0 = f_{c_0}^p(a_0)$ and $|(f_{c_0}^p)'(a_0)| > 1$.

Note that a_0 is *stable*: that is, there exists a neighborhood V of c_0 with the following property: there exists an conformal map $a : c \mapsto a(c)$ on V such that $a(c_0) = a_0$; $a(c) = f_c^p(a(c))$; and $|(f_c^p)'(a(c))| > 1$.

In the following we set $\lambda(c) := (f_c^p)'(a(c))$ and $\lambda_0 := \lambda(c_0)$.

A key lemma. This is our key lemma, which bridges dynamical and paramter plane:

Lemma 1 *Suppose $c_0 \in \partial M$ is a Misiurewicz parameter as above. For $k \in \mathbb{N}$, set $\rho_k := (f_{c_0}^{l+kp})'(c_0)$. Then we have:*

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(1) *The map* $\phi_k(w) = f_{c_0}^{l+kp}(c_0 + \rho_k w)$ *converges to a non-constant limit function* $\phi : \mathbb{C} \to \mathbb{C}$ *uniformly on any compact sets.*

(2) *There exists a constant* $Q \neq 0$ *such that the function*

$$\Phi_k(w) \;:=\; f_{c_0+Q\rho_k w}^{l+kp}(c_0 + Q\rho_k w)$$

converges to the same function $\phi(w)$ *uniformly on compact sets of* $\mathbb{C}$.

Proof. By a famous theorem by Kœnigs (see [Mi, Cor.8.12] and Appendix), the sequence of functions (indeed, polynomials)

$$w \;\longmapsto\; f_{c_0}^{kp}\left(a_0 + \frac{w}{\lambda_0^k}\right) \qquad (k \in \mathbb{N})$$

converges uniformly on compact sets of $\mathbb{C}$ to an entire function $\phi(w)$ with

$$\phi(\lambda_0 w) \;=\; f_{c_0}^{p} \circ \phi(w).$$

Note that this function satisfies $\phi(0) = a_0,\ \phi'(0) = 1.$ [1]

Now let us show (1): set $A_0 := (f_{c_0}^l)'(c_0)$, where $A_0 \neq 0$ since c_0 is not periodic. We also have $\rho_k = 1/(A_0\lambda_0^k)$. For sufficiently small Δz, we have the expansion

$$f_{c_0}^l(c_0 + \Delta z) \;=\; a_0 + A_0 \cdot \Delta z + o(\Delta z).$$

Fix an arbitrary large compact set $E \subset \mathbb{C}$ and take any $w \in E$. Then by setting $\Delta z = w/(A_0\lambda_0^k)$,

$$f_{c_0}^{kp}\left(a_0 + \frac{w}{\lambda_0^k}\right) \;\sim\; f_{c_0}^{l+kp}\left(c_0 + \frac{w}{A_0\lambda_0^k} + o(\lambda_0^{-k})\right) \;\sim\; f_{c_0}^{l+kp}(c_0 + \rho_k w) \;=\; \phi_k(w)$$

when $k \to \infty$. [2] Hence we have $\phi(w) \;=\; \lim_{k\to 0} \phi_k(w)$ on any compact sets.

Next we show (2): suppose that $Q \in \mathbb{C}^*$ is a constant and set $c := c_0 + Q\rho_k w$. We also set $\Phi_k(w) := f_c^{l+kp}(c)$ and $b(c) := f_c^l(c)$. Recall that $a(c)$ denotes a repelling periodic point (of f_c) of period p with $a(c_0) = a_0$, and $\lambda(c)$ denotes its multiplier.

Then the theorem by Kœnigs again implies the sequence of functions $\phi_k^c(w) := f_c^{kp}(a(c) + w/\lambda(c)^{kp})$ converges to an entire function $\phi^c(w)$ uniformly on compact sets. In particular, by the proof of the theorem, it is not difficult to check that the function $c \mapsto \phi^c(w)$ is holomorphic near $c = c_0$ when we fix a $w \in \mathbb{C}$.

Then we use a theorem by Douady and Hubbard [DH]: There exists a $B_0 \neq 0$ such that $b(c) - a(c) = B_0(c - c_0) + o(c - c_0)$. Hence for $c = c_0 + Q\rho_k w$ (taking w in a compact set), we have

$$b(c) \;=\; a(c) + B_0 Q\rho_k w + o(\rho_k) \;=\; a(c) + \frac{B_0 Q}{A_0} \cdot \frac{\lambda(c)^{kp}}{\lambda_0^{kp}} \cdot \frac{w}{\lambda(c)^{kp}} + o(\rho_k).$$

[1] Such a ϕ is called a *Poincaré function.*

[2] Here $A(w) \sim B(w)$ implies $A(w) - B(w) \to 0$ uniformly on E as $k \to \infty$.

Set $Q := A_0/B_0$. Since $\lambda(c)$ is a holomorphic function of c and thus $\lambda(c) = \lambda_0 + O(c - c_0)$, we have $|\lambda(c)/\lambda_0 - 1| = O(c - c_0)$. This implies that

$$\log \frac{\lambda(c)^{kp}}{\lambda(c_0)^{kp}} = k \cdot O(c - c_0) = O\left(\frac{k}{\lambda(c_0)^{kp}}\right) \to 0 \quad (k \to \infty)$$

for $c = c_0 + Q\rho_k w = c_0 + O(\lambda_0^{kp})$. Since $\Phi_k(w) = f_c^{kp}(b(c))$ and $\lim \phi^c(w) \to \phi(w)$ $(c \to c_0)$ uniformly on any compact sets of $\mathbb{C}$, we conclude that

$$\lim_{k\to\infty} \Phi_k(w) \;=\; \lim_{k\to\infty} f_c^{kp}\left(a(c) + \frac{w}{\lambda(c)^{kp}} + o(\rho_k)\right) \;=\; \phi(w),$$

where the convergence is uniform on any compact sets. ■

The Hausdorff topology. Let us briefly recall the *Hausdorff topology* of the set of non-empty compact sets $\mathrm{Comp}^*(\mathbb{C})$ of $\mathbb{C}$. For a sequence $\{K_k\}_{k\in\mathbb{N}} \subset \mathrm{Comp}^*(\mathbb{C})$, we say K_k converges to $K \in \mathrm{Comp}^*(\mathbb{C})$ as $k \to \infty$ if for any $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that if $k \geq k_0$, $K \subset \mathrm{N}_\epsilon(K_k)$ and $K_k \subset \mathrm{N}_\epsilon(K)$, where $\mathrm{N}_\epsilon(\cdot)$ is the open ϵ neighborhood in $\mathbb{C}$.

Set $\mathbb{D}(r) := \{|z| < r\}$. For a compact set K in $\mathbb{C}$, let $[K]_r$ denote the set $(K \cap \mathbb{D}(r)) \cup \partial\mathbb{D}(r)$. For $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$, let $a(K - b)$ denote the set of $a(z - b)$ with $z \in K$.

Similarity. Let c be a Misiurewicz parameter and $\phi : \mathbb{C} \to \mathbb{C}$ the meromorphic function given by Lemma 1. (Note that ϕ has no poles, since ϕ_k is all entire.) To simplify the notation we set $n_k := l + kp$, $f := f_{c_0}$, and $J := J_{c_0}$.

Now our main theorem is:

Theorem 2 (Similarity between M and J) *Set $\mathcal{J} := \phi^{-1}(J) \subset \mathbb{C}$ and let $Q \neq 0$ be the constant given in Lemma 1. Then for any large constant $r > 0$, we have*

(a) $\left[\rho_k^{-1}(J - c_0)\right]_r \to [\mathcal{J}]_r$

(b) $\left[Q^{-1}\rho_k^{-1}(M - c_0)\right]_r \to [\mathcal{J}]_r$

as $k \to \infty$ in the Hausdorff topology.

Proof of (a). Since $f^n(J) = J$, we have $\left[\rho_k^{-1}(J - c_0)\right]_r = \left[\phi_k^{-1}(J)\right]_r$. By $[\mathcal{J}]_r = [\phi^{-1}(J)]_r$ and uniform convergence of $\phi_k \to \phi$ on $\overline{\mathbb{D}(r)}$, the claim easily follows.

Proof of (b). Set $q := 1/Q$ and $\mathcal{M}_k := q\rho_k^{-1}(M - c_0)$. Fix any $\epsilon > 0$. Since the set $\overline{\mathbb{D}(r)} - \mathrm{N}_\epsilon(\mathcal{J})$ is compact, there exists an $N = N(\epsilon)$ such that $|f^N \circ \phi(w)| > 2$ for any $w \in \overline{\mathbb{D}(r)} - \mathrm{N}_\epsilon(\mathcal{J})$. By uniform convergence of $\Phi_k(w) \to \phi(w)$ on compact sets in $\mathbb{D}$ (Lemma 1), we have

$$|f^{N+n_k}_{c_0+Q\rho_k w}(c_0 + Q\rho_k w)| \;>\; 2$$

for all $k \gg 0$. This implies that $c_0 + Q\rho_k w \notin M$, equivalently, $w \notin \mathcal{M}_k$. Hence we have

$$[\mathcal{M}_k]_r \;\subset\; \mathrm{N}_\epsilon([\mathcal{J}]_r).$$

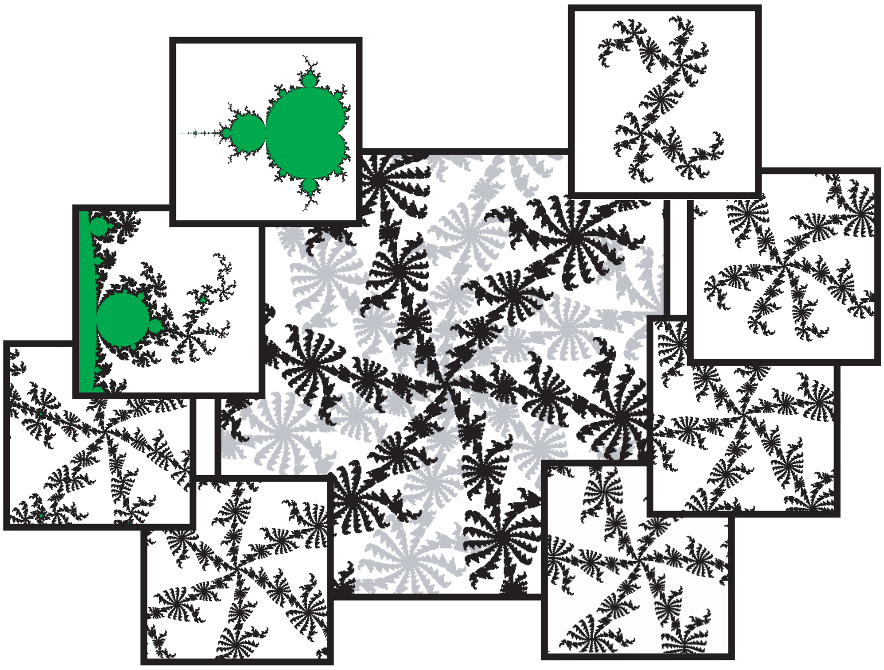

Figure 1: The central panel shows small peaces of M (in gray) and J_{c_0} (in black) centered at a Misiurewicz parameter c_0 in the same coordinate. By zooming out, we will see that they are actually different sets.

Next we show the opposite inclusions. Let us approximate $[\mathcal{J}]_r$ by a finite subset E of $[\mathcal{J}]_r$ such that the $\epsilon/2$ neighborhood of E covers $[\mathcal{J}]_r$. Now it is enough to prove that for any $w_0 \in E$, there exists a sequence $w_k \in [\mathcal{M}_k]_r$ such that $|w_0 - w_k| < \epsilon/2$ for $k \gg 0$.

Let Δ be a disk of radius $\epsilon/2$ centered at w_0. When $\Delta \cap \partial\mathbb{D}(r) \neq \emptyset$, we can take such w_k in $\partial\mathbb{D}(r)$. Hence we may assume that $\Delta \subset \mathbb{D}(r)$.

Since $\phi(w_0) \in J$ and repelling cycles are dense in J, we can choose a w_0' such that $\phi(w_0')$ is a repelling periodic point of some period m and $|w_0 - w_0'| < \epsilon/4$. This implies that the function $\chi : w \mapsto f^m(\phi(w)) - \phi(w)$ has a zero at $w = w_0'$.

Let us consider the function $\chi_k : w \mapsto f^m_{c_0+Q\rho_k w}(\Phi_k(w)) - \Phi_k(w)$, where $\Phi_k(w) = f^{l+kp}_{c_0+Q\rho_k w}(c_0 + Q\rho_k w)$ as in Lemma 1. By uniform convergence of Φ_k to ϕ on compact sets of $\mathbb{C}$, χ_k has a zero w_k in Δ and $|w_k - w_0'| < \epsilon/4$ for all $k \gg 0$. In particular, $c_k := c_0 + Q\rho_k w_k$ satisfies $f^{n_k+l}_{c_k}(c_k) = f^{n_k}_{c_k}(c_k)$ and thus $c_k \in M$. Hence we have a desired $w_k \in \mathcal{M}_k$ with $|w_k - w_0| < \epsilon/2$. ■

Note. The proof can be extended to semi-hyperblic parameters. See [Ka] for more details.

Appendix

Here we give a proof for the fact in the proof of Lemma 1 that $w \mapsto f^{kp}_{c_0}\left(a_0 + \frac{w}{\lambda_0^k}\right)$ converges as $k \to \infty$. Our proof is based on the normal family argument and the univalent function theory (see [Du] for example), which is different from the proof given in [Mi, Cor.8.12].

Proposition 3 *Let* $g : \mathbb{C} \to \mathbb{C}$ *be an entire function with* $g(0) = 0, g'(0) = \lambda$ *and* $|\lambda| > 1$*. Then the sequence* $\phi_n(w) = g^n(w/\lambda^n)$ *converges uniformly on compact sets in* $\mathbb{C}$*. Moreover, the limit function* $\phi : \mathbb{C} \to \mathbb{C}$ *satisfies* $g \circ \phi(w) = \phi(\lambda w)$ *and* $\phi'(0) = 1$*.*

Proof. Since $g(z) = \lambda z + O(z^2)$ near $z = 0$, there exists a disk $\Delta = \mathbb{D}(\delta) = \{|z| < \delta\}$ such that $g|\Delta$ is univalent and $\Delta \Subset g(\Delta)$. Hence we have a univalent branch g_0^{-1} of g that maps Δ into itself.

First we show that ϕ_n is univalent on $\mathbb{D}(\delta/4)$: Since the map $\phi_n^{-1} : w \mapsto \lambda^n g_0^{-n}(w)$ is well-defined on $\Delta = \mathbb{D}(\delta)$ and univalent, its image contains $\mathbb{D}(\delta/4)$ by the Koebe 1/4 theorem. Hence ϕ_n is univalent on $\mathbb{D}(\delta/4)$, and by the Koebe distortion theorem, the family $\{\phi_n\}_{n\geq 0}$ is locally uniformly bounded on $\mathbb{D}(\delta/4)$ and thus equicontinuous.

Next we show that ϕ_n has a limit on $\mathbb{D}(\delta/4)$: Fix an arbitrarily large $r > 0$ and an integer N such that $r < \delta|\lambda|^N/4$. By using the Koebe 1/4 theorem as above, the map $G_{N,k}(w) := \lambda^N g^k(w/\lambda^{N+k})$ ($k \in \mathbb{N}$) is univalent on the disk $\mathbb{D}(\delta|\lambda|^N/4)$. By the Koebe distortion theorem, there exists a constant $C > 0$ independent of N and k such that for any $w \in \mathbb{D}(r)$ and sufficiently large N we have $|G'_{N,k}(w) - 1| \leq C|w|/|\lambda|^N$. By integration we have $|G_{N,k}(w) - w| \leq Cr^2/(2|\lambda|^N)$ on $\mathbb{D}(r)$. In particular, $G_{N,k} \to \mathrm{id}$ uniformly on $\mathbb{D}(\delta/4)$ as $N \to \infty$. Since the family $\{\phi_n\}$ is equicontinuous on $\mathbb{D}(\delta/4)$, the relation $\phi_{N+k} = \phi_N \circ G_{N,k}$ implies that $\{\phi_n\}_{n\geq 0}$ is Cauchy and has a unique limit ϕ on any compact sets in $\mathbb{D}(\delta/4)$.

Let us check that the convergence extends to $\mathbb{C}$: (We will not use the functional equation $g^n \circ \phi(w) = \phi(\lambda^n w)$. Compare [Mi, Cor.8.12].) Since $|\phi_{N+k}(w) - \phi_N(w)| = |\phi_N(G_{N,k}(w)) - \phi_N(w)|$ and $|G_{N,k}(w) - w| = Cr^2/(2|\lambda|^N)$ on $\mathbb{D}(r)$, it follows that that the family $\{\phi_{N+k}\}_{k\geq 0}$ (with fixed N) is uniformly bounded on $\mathbb{D}(r)$. Hence $\{\phi_n\}_{n\geq 0}$ is normal on any compact set in $\mathbb{C}$ and any sequential limit coincides with the local limit ϕ on $\mathbb{D}(\delta/4)$.

The relations $g \circ \phi(w) = \phi(\lambda w)$ and $\phi'(0) = 1$ are immediate from $g \circ \phi_n(w) = \phi_{n+1}(\lambda w)$ and $\phi'_n(0) = 1$. ■

Remark. One can easily extend this proof to the case of meromorphic g by using the spherical metric.

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