On the Cauchy problem for second-order hyperbolic operators with the coefficients of their principal parts depending only on the time variable

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1. Main results

Notations:
the time variable: $t \in \mathbb{R}$ dual $\tau \in \mathbb{R}$
the space variables: $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ dual $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$

Consider hyperbolic operators of second order whose symbols have the form

$$P(t, x, \tau, \xi) = \tau^2 - a(t, \xi) \tau + b(t, x, \xi) + c(t, x),$$

where

$$a(t, \xi) = \sum_{j,k} a_{j,k}(t) \xi_j \xi_k, \quad b(t, x, \xi) = \sum_{j=1}^n b_j(t) \xi_j, \quad a_{j,k}(t) \in C^\infty([0, \infty)), \quad b_j(t) \in C^\infty([0, \infty) \times \mathbb{R}^n),$$

and the Cauchy problem

$$\begin{aligned}
\left\{ 
& P(t, x, D_t, D_x) u(t, x) = f(t, x) \quad \text{in } [0, \infty) \times \mathbb{R}^n, \\
& D_j^l u(t, x)|_{t=0} = u_j(x) \quad \text{in } \mathbb{R}^n \quad (j = 0, 1)
\end{aligned}$$

(Def): We say that (CP) is $C^\infty$ well-posed if

$$\begin{aligned}
& \forall f \in C^\infty([0, \infty) \times \mathbb{R}^n), \quad \forall u_j \in C^\infty(\mathbb{R}^n) \quad (j = 0, 1), \quad \exists u \in C^\infty([0, \infty) \times \mathbb{R}^n) \text{ satisfying (CP)}. \quad \text{(Existence)}
\end{aligned}$$

(U) If $s > 0$, $u \in C^\infty([0, \infty) \times \mathbb{R}^n)$, $D_j^l u(t, x)|_{t=0} = 0$ in $\mathbb{R}^n$ $((j = 0, 1)$ &

$$\text{ supp } P(t, x, D_t, D_x) u \subset \{ t \geq s \}, \text{ then } \text{ supp } u \subset \{ t \geq s \}. \quad \text{(Uniqueness)}$$

Taking account of Lax-Mizohata theorem we assume that

$$(H) \quad a(t, \xi) \geq 0 \text{ for } (t, \xi) \in [0, \infty) \times \mathbb{R}^n$$

(see S. Mizohata, J. Math. Kyoto Univ. 1 (1961), 109–127). From Ivrii-Petkov’s result we can assume without loss of generality that

$$(F) \quad a(t, \xi) \neq 0 \text{ in } t \text{ for } \forall \xi \in \mathbb{R}^n \setminus \{0\}$$


Moreover, we assume that $a(t, \xi)$ satisfies the following condition (A):

$$(A) \quad \forall T > 0, \exists k_T \in \mathbb{Z}_+ (= \mathbb{N} \cup \{0\}) \text{ s.t.}$$

$$\sum_{k=0}^{k_T} |\partial^k_t a(t, \xi)| \neq 0 \text{ for } \forall (t, \xi) \in [0, T] \times S^{n-1}.$$
If the \( a_{j,k}(t) \) are real analytic on \([0, \infty)\), then the condition (A) is satisfied. For simplicity we assume that the \( a_{j,k}(t) \) are real analytic on \([0, \infty)\), in order to describe the condition (L) below in a simple form. Let \( \Omega \) be a neighborhood of \([0, \infty)\) in \( \mathbb{C} \) where the \( a_{j,k}(t) \) are analytic. Put

\[
\mathcal{R}(\xi) = \{(\Re \lambda)_+; \lambda \in \Omega \text{ and } a(\lambda, \xi) = 0\}
\]

for \( \xi \in \mathbb{R}^n \setminus \{0\} \), where \( a_+ = \max\{a,0\} \).

**Sufficiency:**

We assume in “Sufficiency” that

(A)' the \( a_{j,k}(t) \) are real analytic (for simplicity),

(B) \( \forall K \in \mathbb{R}^n, \exists \Omega_K: \) complex neighborhood of \([0, \infty)\) s.t. \( b_j(t, x) \) (1 \( \leq j \leq n \)) are analytic in \( \Omega_K \) for \( \forall x \in K \),

(L) \( \forall T > 0, \forall x \in \mathbb{R}^n, \exists C > 0 \) s.t.

\[
\min_{\tau \in \mathcal{R}(\xi)} |t - \tau| |b(t, x, \xi)| \leq C \sqrt{a(t, \xi)} \quad \text{for } \forall (t, \xi) \in [0, T] \times S^{n-1}.
\]

**Thm 1:** Under (B) and (L) (CP) is \( C^\infty \) well-posed.

**Remark:** \( \mathcal{R}(\xi) \) can be replaced in (L) by \( \mathcal{R}'(\xi) \) satisfying

\[
\sup_{\xi \in S^{n-1}} \#(\mathcal{R}'(\xi) \cap \{ t \leq T \}) < \infty \quad \text{for } \forall T > 0,
\]

where \# denotes the number of the elements of a set \( A \).

**Def:** (i) Let \( f \) be a function on \( \mathbb{R} \). We say that \( f(t) \) is a semi-algebraic function if the graph of \( f \) is a semi-algebraic set, i.e., the graph of \( f \) is a set defined by polynomial equations and inequalities. (ii) Let \( t_0 \in \mathbb{R}, U \) be a neighborhood of \( t_0 \) and \( f: U \to \mathbb{R} \). We say that \( f \) is semi-algebraic at \( t_0 \) if there is \( c > 0 \) such that \( \{(t, y) \in \mathbb{R}^2; y = f(t) \text{ and } |t - t_0| < c\} \) is a semi-algebraic set.

**Necessity:**

We assume in “Necessity” that (A)' and (B) are satisfied. Let \( t_0 \geq 0, x^0 \in \mathbb{R}^n \) and \( \xi^0 \in S^{n-1} \). If \( n \geq 3 \), we assume the following condition:

(A)'\(^n\)\(_{(t_0, x^0)}\) the \( a_{j,k}(t) \) and \( b_j(t, x^0) \) (1 \( \leq j \leq n \)) are semi-analytic at \( t_0 \).

The following condition is very similar to the condition (L):

(L)\(_{(t_0, x^0, \xi^0)}\) \( \exists U: \text{nbd of } t_0, \exists \Gamma: \text{conic nbd of } \xi^0, \exists C > 0 \) s.t.

\[
\min_{\tau \in \mathcal{R}(\xi)} |t - \tau| |b(t, x^0, \xi)| \leq C \sqrt{a(t, \xi)} \quad \text{for } \forall (t, \xi) \in U \times \Gamma.
\]

**Thm 2:** Assume that (A)' and (B) are satisfied. Moreover, we assume that (A)'\(^n\)\(_{(t_0, x^0)}\) is satisfied if \( n \geq 3 \). Then (L)\(_{(t_0, x^0, \xi^0)}\) is necessary for \( C^\infty \) well-posedness.

**Remark:** Assume that (A)' and (B) are satisfied, and that (A)'\(^n\)\(_{(t_0, x^0)}\) is valid for any \( t_0 \geq 0 \) and \( x^0 \in \mathbb{R}^n \) if \( n \geq 3 \). Then (CP) is \( C^\infty \) well-posed if and only if (L) is satisfied.
related results:

  
  \( CP \) is \( C^\infty \) well-posed if the coefficients do not depend on \( x \) and if (A) and the following condition are satisfied:
  
  \[
  |b(t, \xi)| \leq Ca(t, \xi)^{1/2-1/k}
  \]
  
  for \( (t, \xi) \in [0, \infty) \times S^{n-1} \).

  
  They tried to generalize C-I-O’s results to the case the lower order terms also depend on \( x \).

  
  The proof of Thm 2 is given in this paper.

2. Outline of Proof of Thm 1

We can assume without loss of generality that there is \( K \in \mathbb{R}^n \) such that \( \text{supp}_x b_j(t, x) \), \( \text{supp}_x c(t, x) \subset K \). Let \( t_0 \geq 0, O_{t_0} \) be the ring of power series centered at \( t_0 \) in one variable and

\[
M_{t_0} := \{ (\beta_1(t), \cdots, \beta_n(t)) \in O_{t_0}^n; \min_{\tau \in \mathbb{R}(\xi)} |t - \tau| \cdot |\sum_{j=1}^{n} \beta_j(t)\xi_j| \leq C \sqrt{a(t, \xi)} \}
\]

if \( t \) belongs to a neighborhood of \( t_0 \) in \( [0, \infty) \) and \( \xi \in S^{n-1} \).

Since \( O_{t_0} \)-submodule of \( O_{t_0}^n \) is finitely generated, there are \( \psi_j(t) = (\psi_{j,1}(t), \cdots, \psi_{j,n}(t)) \in M_{t_0} \) ( \( 1 \leq j \leq r_0 \)) such that

\[
M_{t_0} = \left\{ \sum_{j=1}^{r_0} c_j(t)\psi_j(t); c_j(t) \in O_{t_0} \ (1 \leq j \leq r_0) \right\}.
\]

The condition (L) implies that \( (b_1(t, x), \cdots, b_n(t, x)) \in M_{t_0} \) for each \( x \in \mathbb{R}^n \). So there are \( C^\infty \) functions \( c_j(t, x) \) of \( (t, x) \) such that \( b(t, x, \xi) = \sum_{j=1}^{r_0} c_j(t, x)\psi_j(t, \xi) \) in a neighborhood of \( t_0 \), where \( \psi_j(t, \xi) = \sum_{k=1}^{n} \psi_{j,k}(t)\xi_k \). Let \( T > 0 \). Then there are \( \varphi_j(t) = (\varphi_{j,1}(t), \cdots, \varphi_{j,n}(t)) \in (C^\infty(\mathbb{R}))^n \) and \( c_j(t, x) \in C^\infty([0, \infty) \times \mathbb{R}^n) \) ( \( 1 \leq j \leq r \)) such that

\[
\min_{\tau \in \mathbb{R}(\xi)} |t - \tau| \cdot |\varphi_j(t, \xi)| \leq C \sqrt{a(t, \xi)} \quad (t, \xi) \in [0, T] \times S^{n-1}, \ 1 \leq j \leq r,
\]

\[
b(t, x, \xi) = \sum_{j=1}^{r} c_j(t, x)\varphi_j(t, \xi) \quad (t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n),
\]
where $\varphi_j(t, \xi) = \sum_{k=1}^n \varphi_{j,k}(t)\xi_k$. Put

$$w_\rho(t, \xi) := a(t, \xi) + (\xi)^{2\rho}, \quad \rho := \frac{2}{k_0 + 2}.$$  

$$W_0(t, \xi) := \frac{(\xi)^{2\rho}}{\sqrt{w_\rho(t, \xi)}} + 1, \quad W_j(t, \xi) := \frac{\varphi_j(t, \xi)}{\sqrt{w_\rho(t, \xi)}} \quad (1 \leq j \leq r),$$

$$W(t, \xi) := \sum_{j=0}^r W_j(t, \xi), \quad \Phi(t, \xi) := \int_0^t \left( W(s, \xi) + \frac{|(\partial_t a)(s, \xi)|}{w_\rho(s, \xi)} \right) ds.$$  

Let us introduce the parameter $\varepsilon \in [0, 1]$ (to prove finite propagation property). Consider

$$\begin{cases}
(P(t, x, D_t, D_x) + \varepsilon \Delta_x)u_\varepsilon(t, x) = f(t, x) \quad (t \geq 0, \ x \in \mathbb{R}^n), \\
u_\varepsilon(0, x) = u_0(x), \quad (D_t u_\varepsilon)(0, x) = u_1(x) \quad (x \in \mathbb{R}^n).
\end{cases}$$

We use an Energy form

$$E_\varepsilon(t; \gamma, A, l) := \int_{\mathbb{R}^n} E_\varepsilon(t, \xi) K(t, \xi; \gamma, A, l) d\xi,$$

where $\gamma > 0$, $A > 0$, $l \in \mathbb{R}$, $t \in [0, T]$ and

$$E_\varepsilon(t, \xi) := |\partial_t v_\varepsilon(t, \xi)|^2 + (a(t, \xi) + \varepsilon|\xi|^2 + (\xi)^{2\rho})|v_\varepsilon(t, \xi)|^2,$$

$$K(t, \xi; \gamma, A, l) := \exp[-(\gamma t + A\Phi(t, \xi)) + l \log(\xi)],$$

$$v_\varepsilon(t, \xi) := F_x[u_\varepsilon(t, x)](\xi).$$

A simple calculation gives

$$\partial_t E_\varepsilon(t, \xi) - (\gamma + A\partial_t \Phi(t, \xi)) E_\varepsilon(t, \xi)$$

$$\leq |\tilde{f}(t, \xi)/W(t, \xi) - (\gamma + (A - 3)W(t, \xi) - 2)|\partial_t v_\varepsilon|^2$$

$$- \{(A - 1)(|\partial_t a(t, \xi)| + W(t, \xi)w_\rho(t, \xi)) + \gamma w_\rho(t, \xi)\}|v_\varepsilon|^2$$

$$+ |F_x[b(t, x)\partial_t u_\varepsilon(t, x)](\xi)|^2 + |F_x[b(t, x, D_x)u_\varepsilon](\xi)|^2/W(t, \xi) + |F_x[c(t, x)u_\varepsilon](\xi)|^2.$$  

**Lemma:** (i) $\Phi(T, \xi) \leq 3C_T(1 + \log(\xi)) \quad (\xi \in \mathbb{R}^n).$

(ii) $\forall \delta > 0, \exists c_\delta(T) > 0$ s.t.

$$(1 + \delta)^{-1}W(t, \xi) \leq W(t, \eta) \leq (1 + \delta)W(t, \xi) \quad \text{if} \ t \in [0, T], \ |\xi - \eta| \leq c_\delta(T)\langle \xi \rangle^\rho,$$

$$(1 + \delta)^{-1}\Phi(t, \xi) \leq \Phi(t, \eta) \leq (1 + \delta)\Phi(t, \xi)$$

$\text{if} \ t \in [0, T], \ |\xi - \eta| \leq c_\delta(T)\langle \xi \rangle^\rho/(1 + \log(\xi)).$

(iii) $\exists C_T > 0, \exists l_b > 0$ s.t. $\exp[\pm \Phi(t, \xi)] \leq C_T(\xi - \eta)^{l_b} \exp[\pm \Phi(t, \eta)].$

Using the above lemma, we have

$$\partial_t E_\varepsilon(t; \gamma, A, l) \leq \int |\tilde{f}(t, \xi)|^2 W(t, \xi)^{-1} K(t, \xi; \gamma, A, l) d\xi$$

if $A \geq 3A(b, T), \ \gamma \geq 3\gamma(b, b_0, c, T, A, l)$ and $t \in [0, T]$. Then a standard argument proves Thm 1.