Alternative Proof for the Convergence of Formal Solutions of
Singular First Order Nonlinear Partial Differential Equations

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The purpose of this talk is to give two proofs of Theorem which is written in
the end of this abstract.

Let $\mathbb{C}$ be the set of complex numbers and $(t, x) = (t_1, \ldots, t_d, x_1, \ldots, x_n) \in \mathbb{C}^d \times \mathbb{C}^n$. We consider the following first order nonlinear PDE:

$$f(t, x, u, \partial_t u, \partial_x u) = 0 \quad \text{with} \quad u(0, x) \equiv 0, \quad (1)$$

where $u = u(t, x)$ is an unknown function, $\partial_t u = (\partial_{t_1} u, \ldots, \partial_{t_d} u)$ $(\partial_{t_j} = \partial/\partial t_j)$ and $\partial_x u$ is similar to $\partial_t u$. We assume the following assumptions:

(A1) The function $f(t, x, u, \tau, \xi)$ $(\tau \in \mathbb{C}^d, \xi \in \mathbb{C}^n)$ is holomorphic in a neighborhood of the origin. Moreover, $f(t, x, u, \tau, \xi)$ is an entire function in $\tau$ variables for any fixed $t, x, u$ and $\xi$.

(A2) The equation (1) is singular in $t$ variables in the sense that

$$f(0, x, 0, \tau, 0) \equiv 0 \quad \text{and} \quad \frac{\partial f}{\partial \xi_k}(0, x, 0, \tau, 0) \equiv 0 \quad (k = 1, \ldots, n). \quad (2)$$

(A3) The equation (1) has a formal solution of the form $u = \sum_{|\alpha| \geq 1} u_\alpha(x) t^\alpha$ with holomorphic coefficients, that is, $u_\alpha(x) \in \mathbb{C}\{x\}$ for all $\alpha$.

Let $\varphi(x) = (\varphi_1(x), \ldots, \varphi_d(x)) \in (\mathbb{C}\{x\})^d$ be the collection of coefficients of $t_j$ of formal solution, and we set $a(x) = (0, x, 0, \varphi(x), 0)$. Then the following theorem, which is the main theorem, holds under the assumptions (A1), (A2) and (A3).

**Theorem** (M. Miyake and A. Shirai). Let $a_{ij}(x)$ $i, j = 1, \ldots, d$ be $a_{ij}(x) = f_{i \tau_j}(\mathbf{a}(x)) + f_{u \tau_j}(\mathbf{a}(x))\varphi_i(x)$ and $\{\lambda_1, \ldots, \lambda_d\}$ be the eigenvalues of the matrix $(a_{ij}(0))_{i,j=1,\ldots,d}$. If $\{\lambda_j\}$ satisfies the condition (3) and (4) below, then the formal solution $u(t, x)$ of (1) is convergent in a neighborhood of the origin:

$$\text{Ch}(\lambda_1, \ldots, \lambda_d) \neq 0 \quad \text{(Poincaré condition)} \quad (3)$$

where $\text{Ch}(\lambda_1, \ldots, \lambda_d)$ denotes the convex hull of $\{\lambda_1, \ldots, \lambda_d\}$.

$$\sum_{j=1}^{d} \lambda_j \alpha_j + f_u(\mathbf{a}(0)) \neq 0 \quad \text{(Nonresonance condition)} \quad (4)$$

for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \geq 2$. 