Alternative Proof for the Convergence of Formal Solutions of Singular First Order Nonlinear Partial Differential Equations

Akira Shirai
(Sugiyama Jogakuen University, Nagoya, Japan)

The purpose of this talk is to give two proofs of Theorem which is written in the end of this abstract.

Let $\mathbb{C}$ be the set of complex numbers and $(t,x) = (t_1,\dots,t_d,x_1,\dots,x_n) \in \mathbb{C}^d_t \times \mathbb{C}^n_x$. We consider the following first order nonlinear PDE:

$$f(t,x,u,\partial_t u,\partial_x u) = 0 \qquad \text{with} \qquad u(0,x) \equiv 0, \tag{1}$$

where $u = u(t,x)$ is an unknown function, $\partial_t u = (\partial_{t_1} u,\dots,\partial_{t_d} u)$ $(\partial_{t_j} = \partial/\partial t_j)$ and $\partial_x u$ is similar to $\partial_t u$. We assume the following assumptions:

(A1) The function $f(t,x,u,\tau,\xi)$ $(\tau \in \mathbb{C}^d, \xi \in \mathbb{C}^n)$ is holomorphic in a neighborhood of the origin. Moreover, $f(t,x,u,\tau,\xi)$ is an entire function in τ variables for any fixed t,x,u and ξ.

(A2) The equation (1) is *singular* in t variables in the sense that

$$f(0,x,0,\tau,0) \equiv 0 \quad \text{and} \quad \frac{\partial f}{\partial \xi_k}(0,x,0,\tau,0) \equiv 0 \ (k=1,\dots,n). \tag{2}$$

(A3) The equation (1) has a formal solution of the form $u = \sum_{|\alpha|\geq 1} u_\alpha(x) t^\alpha$ with holomorphic coefficients, that is, $u_\alpha(x) \in \mathbb{C}\{x\}$ for all α.

Let $\varphi(x) = (\varphi_1(x),\dots,\varphi_d(x)) \in (\mathbb{C}\{x\})^d$ be the collection of coefficients of t_j of formal solution, and we set $\boldsymbol{a}(x) = (0,x,0,\varphi(x),0)$. Then the following theorem, which is the main theorem, holds under the assumptions (A1), (A2) and (A3).

Theorem (M. Miyake and A. Shirai). *Let $a_{ij}(x)$ $i,j = 1,\dots,d)$ be $a_{ij}(x) = f_{t_i\tau_j}(\boldsymbol{a}(x)) + f_{u\tau_j}(\boldsymbol{a}(x))\varphi_i(x)$ and $\{\lambda_1,\dots,\lambda_d\}$ be the eigenvalues of the matrix $(a_{ij}(0))_{i,j=1,\dots,d}$. If $\{\lambda_j\}$ satisfies the condition (3) and (4) below, then the fomal solution $u(t,x)$ of (1) is convergent in a neighborhood of the origin:*

$$\mathrm{Ch}(\lambda_1,\dots,\lambda_d) \not\ni 0 \qquad (\textit{Poincaré condition}) \tag{3}$$

where $\mathrm{Ch}(\lambda_1,\dots,\lambda_d)$ denotes the convex hull of $\{\lambda_1,\dots,\lambda_d\}$.

$$\sum_{j=1}^{d} \lambda_j \alpha_j + f_u(\boldsymbol{a}(0)) \neq 0 \qquad (\textit{Nonresonance condition}) \tag{4}$$

for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \geq 2$.