Convergence of formal solutions for singular first-order non-linear PDEs

—— Proof of Miyake-Shirai's theorem by the fixed point theorem ——

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Let $x = (x_1, \dots, x_d) \in \mathbb{C}^d$, $u \in \mathbb{C}$ and $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{C}^d$, and let $F(x, u, \xi)$ be a holomorphic function in a neighborhood of $(x, u, \xi) = (0, 0, 0) \in \mathbb{C}^{2d+1}$, which satisfies $F(0, 0, 0) = 0$. We study formal power series solutions for the following first-order non-linear partial differential equation:

$$\begin{cases} F(x, u(x), Du(x)) = 0, \\ u(0) = 0, \quad Du(0) = 0, \end{cases} \tag{1}$$

where $Du(x) = (D_{x_1}u(x), \dots, D_{x_d}u(x))$ $(D_{x_i}u(x) = (\partial/\partial x_i)u(x);\ i = 1,\ 2,\ \dots,\ d)$.

We consider the case where the equation is singular at the origin. Precisely, we deal with the case where $F(x, u, \xi)$ satisfies

$$D_{\xi_i}F(0, 0, 0) = 0 \quad (i = 1, 2, \dots, d). \tag{2}$$

In 2000, M. Miyake and A. Shirai studied the singular equation (1)-(2), and they gave conditions which ensure the existence, the uniqueness and the convergence of formal power series solutions $u(x) = \sum_{|\alpha|\geq 2} u_\alpha x^\alpha$ $(\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $\mathbb{N} = \{0, 1, 2, \dots\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $x^\alpha = {x_1}^{\alpha_1} \cdots {x_d}^{\alpha_d})$. Their theorem was proved by applying the majorant method developed by R. Gérard and H. Tahara. In this talk, we explain the proof of Miyake-Shirai's theorem by applying the fixed point theorem.