Asymptotic properties and inverse spectral problems for nonlinear Sturm-Liouville problems

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1 Introduction

We consider the following nonlinear Sturm-Liouville problem

$$-u''(t) + f(u(t)) = \lambda u(t), \quad t \in I, \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(0) = u(1) = 0, \tag{1.3}$$

where $I := (0, 1)$ and $\lambda > 0$ is a parameter. We emphasize that $f(u)$ is an **unknown** nonlinear term, which satisfies the following conditions (A.1)–(A.3):

(**A.1**) $f(u)$ is a function of C^1 for $u \geq 0$ satisfying $f(0) = f'(0) = 0$.

(**A.2**) $f(u)/u$ is strictly increasing for $u \geq 0$.

(**A.3**) $f(u)/u \to \infty$ as $u \to \infty$.

The typical examples of f which satisfy (A.1)–(A.3) are as follows.

$$\begin{aligned} f(u) &= u^p \qquad (p > 1), \\ f(u) &= u^p \log(u + 1) \qquad (p > 1), \\ f(u) &= u^p \cdot \left(1 - \frac{1}{1 + u^q}\right) \qquad (p > 1, q > 1), \\ f(u) &= u^p e^u \qquad (p > 1). \end{aligned}$$

(1) For any given $\alpha > 0$, there exists a unique solution pair of (1.1)–(1.3) $(\lambda, u) = (\lambda(\alpha), u_\alpha) \in \mathbf{R}_+ \times C^2(\bar{I})$ such that $\|u_\alpha\|_2 = \alpha$.

(2) The set $\{(\lambda(\alpha), u_\alpha) : \alpha > 0\}$ gives all solutions of (1.1)–(1.3), which is an unbounded C^1-bifurcation curve emanating from $(\pi^2, 0)$ in $\mathbf{R}_+ \times L^2(I)$ and $\lambda(\alpha)$ is C^1 and strictly increasing for $\alpha > 0$.

In this talk, we determine the unknown nonlinear term f from the L^2-bifurcation curve $\lambda(\alpha)$. We also mention some recent results for asymptotic formulas for $\lambda(\alpha)$ as $\alpha \to \infty$.

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