Dimensions of triangulated categories with respect to subcategories

This talk is based on joint work with

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1 Introduction

- 2003, A. Bondal and M. van den Bergh,
  Generators and representability of functors in commutative and noncommutative geometry.
- 2008, R. Rouquier, *Dimensions of triangulated categories*.

Problem 1. *Give a precise value of the dimension for a given triangulated category.*

Aim

- Introduce a concept of dimension of a triangulated category with respect to a fixed full subcategory.
- Give upper bounds of our relative dimensions of derived categories in terms of global dimensions.

All subcategories are full and closed under isomorphisms.

2 Relative dimension

- $\mathcal{T}$: a triangulated category with a shift functor $[1]$.
- $\mathcal{X}$, $\mathcal{Y}$: subcategories of $\mathcal{T}$.
- $\mathcal{X} * \mathcal{Y} := \{M \in \mathcal{T} \mid X \to M \to Y \to X[1]$with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}\}$.
  By octahedral axiom, $(\mathcal{X} * \mathcal{Y}) * \mathcal{Z} = \mathcal{X} * (\mathcal{Y} * \mathcal{Z})$ holds (see [BBD]).
- $\langle \mathcal{X} \rangle := \mathrm{add}\{X[i] \mid X \in \mathcal{X}, i \in \mathbb{Z}\}$.
- For a positive integer n,

$$\langle \mathcal{X} \rangle_n := \mathrm{add}(\underbrace{\langle \mathcal{X} \rangle * \langle \mathcal{X} \rangle * \cdots * \langle \mathcal{X} \rangle}_{n \text{ copies}}).$$

Definition 2 (AAITY). The *(triangle) dimension of $\mathcal{T}$ w.r.t. $\mathcal{X}$*.

$$\mathcal{X}\text{-}\,\mathrm{tri.dim}\,\mathcal{T} := \inf\{n \geq 0 \mid \mathcal{T} = \langle \mathcal{X} \rangle_{n+1}\}.$$

Remark 3. • (Avramov-Buchweitz-Iyengar-Miller)

$$\mathcal{X}\text{-}\operatorname{tri.dim}\mathcal{T} = \sup_{M\in\mathcal{T}}\left\{\mathsf{level}_{\mathcal{T}}^{\mathcal{X}}(M)\right\} - 1,$$

where $\mathsf{level}_{\mathcal{T}}^{\mathcal{X}}(M) := \inf\{n \geq 1 \mid M \in \langle\mathcal{X}\rangle_n\}$ is the $\mathcal{X}$-*level of* M.

• (Rouquier) The *(triangle) dimension of* $\mathcal{T}$.

$$\operatorname{tri.dim}\mathcal{T} = \inf_{M\in\mathcal{T}}\{(\operatorname{add} M)\text{-}\operatorname{tri.dim}\mathcal{T}\}.$$

Proposition 4 (Rouquier, Krause-Kussin)**.**

Let $\mathcal{A}$ be an abelian category with enough projective objects. Then

$$(\operatorname{proj}\mathcal{A})\text{-}\operatorname{tri.dim}\mathsf{D}^{\mathrm{b}}(\mathcal{A}) = \operatorname{gl.dim}\mathcal{A}.$$

Proposition 5 (Rouquier, Krause-Kussin).

Let Λ be a noetherian algebra and M a generator of Λ.

$$\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \leq \operatorname{gl.dim} \operatorname{End}_{\Lambda}(M).$$

Lemma 6. *Let $F : \mathcal{T} \to \mathcal{T}'$ be a dense triangle functor.*

$$\mathcal{T} = \langle \mathcal{X} \rangle_n \implies \mathcal{T}' = \langle F\mathcal{X} \rangle_n.$$

In particular, $F\mathcal{X}\text{-}\operatorname{tri.dim} \mathcal{T}' \leq \mathcal{X}\text{-}\operatorname{tri.dim} \mathcal{T}$.

Proposition 7 (AAITY, cf. Han).

Let $\mathcal{A}$ be an abelian category with enough projective objects. Then

$$\mathsf{D}^{\mathrm{b}}(\mathcal{A}) = \langle \Omega\mathcal{A} \rangle_2.$$

In particular, $(\Omega\mathcal{A})\text{-}\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\mathcal{A}) \leq 1$.

3 Main theorem

- $\mathcal{A}$: an abelian category.
- $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$: the bounded derived category of $\mathcal{A}$.

  When $\mathcal{A}$ has enough projective objects,
- $\mathrm{proj}\,\mathcal{A}$: the subcategory of $\mathcal{A}$ consisting of projective objects of $\mathcal{A}$.
- $\mathrm{gl.dim}\,\mathcal{A}$: the global dimension of $\mathcal{A}$.
- $\mathcal{X}$: an additive subcategory of $\mathcal{A}$.
- $\mathrm{mod}\,\mathcal{X}$: the category of finitely presented $\mathcal{X}$-modules.
  - An $\mathcal{X}$-module $:=$ an additive contravariant functor $\mathcal{X} \to \mathrm{Ab}$.
  - An $\mathcal{X}$-module F is *finitely presented* if

    $\exists$ an exact sequence

$$\mathrm{Hom}_{\mathcal{X}}(-, X_1) \to \mathrm{Hom}_{\mathcal{X}}(-, X_0) \to F \to 0$$

    with $X_0, X_1 \in \mathcal{X}$.

Definition 8 (Auslander-Smalø). $\mathcal{X}$ is *contravariantly finite* if for $\forall\ M \in \mathcal{A}$, $\exists\ X \in \mathcal{X}$ and $\exists\ f : X \to M$ s.t.

$$\mathrm{Hom}_{\mathcal{A}}(-, X) \stackrel{f\bullet}{\to} \mathrm{Hom}_{\mathcal{A}}(-, M) \to 0$$

is exact on $\mathcal{X}$.

Remark 9. (Auslander) $\mathcal{X}$: contravariantly finite $\Longrightarrow \mathrm{mod}\,\mathcal{X}$: abelian.

Definition 10. $\mathcal{X}$ *generates* $\mathcal{A}$ if for $\forall\ M \in \mathcal{A}$, $\exists\ X \twoheadrightarrow M$ with $X \in \mathcal{X}$.

Theorem 11 (AAITY). *Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ a contravariantly finite subcategory that generates $\mathcal{A}$.*

$$\mathcal{X}\text{-}\,\mathrm{tri.dim}\,\mathsf{D}^{\mathrm{b}}(\mathcal{A}) \leq \mathrm{gl.dim}(\mathrm{mod}\,\mathcal{X}).$$

Sketch. $\mathcal{X}\text{-}\,\mathrm{tri.dim}\,\mathsf{D}^{\mathrm{b}}(\mathcal{A}) \leq (\mathrm{proj}(\mathrm{mod}\,\mathcal{X}))\text{-}\,\mathrm{tri.dim}\,\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,\mathcal{X}).$ □

4 Application

- Λ : a noetherian ring.
- $\operatorname{mod}\Lambda$: the abelian category of finitely generated right Λ-modules.
- For $T \in \operatorname{mod}\Lambda$, $\mathcal{X}_T := \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X,T) = 0$ for all $i > 0\}$.

Definition 12. T is *cotilting* if

(1) $\operatorname{inj.dim} T < \infty$.

(2) $\operatorname{Ext}^i_\Lambda(T,T) = 0$ for all $i > 0$ (i.e., $T \in \mathcal{X}_T$).

(3) For $\forall\ X \in \mathcal{X}_T$, $\exists$ an exact sequence

$$0 \to X \to T' \to X' \to 0$$

with $T' \in \operatorname{add} T$ and $X' \in \mathcal{X}_T$.

Example 13. Let Λ be an Iwanaga-Gorenstein ring.

$\Longrightarrow \Lambda_\Lambda$ is cotilting and $\mathcal{X}_\Lambda = \mathrm{CM}(\Lambda)$.

<u>**Fact**</u> (Auslander-Buchweitz approximation theory)

- $\mathcal{X}_T$ is a contravariantly finite subcategory of $\operatorname{mod}\Lambda$.
- $\mathcal{X}_T$ generates $\operatorname{mod}\Lambda$.

Theorem 14 (AAITY). *Let T be a cotilting Λ-module.*

$$\mathcal{X}_T\text{-}\operatorname{tri.dim}\mathsf{D}^{\mathrm{b}}(\operatorname{mod}\Lambda) \leq \max\{1, \operatorname{inj.dim} T\}.$$

Sketch. • $\operatorname{gl.dim}(\operatorname{mod}\mathcal{X}_T) \leq \max\{2, \operatorname{inj.dim} T\}$ (Iyama).

• $\operatorname{inj.dim} T \leq 1 \Longrightarrow \Omega(\operatorname{mod}\Lambda) \subset \mathcal{X}_T$ and $\mathsf{D}^{\mathrm{b}}(\operatorname{mod}\Lambda) = \langle \mathcal{X}_T \rangle_2$. □

Remark 15. If $\operatorname{inj.dim} T = 0$, then $\mathcal{X}_T\text{-}\operatorname{tri.dim}\mathsf{D}^{\mathrm{b}}(\operatorname{mod}\Lambda) \not\leq \operatorname{inj.dim} T$.

<u>Counter example</u> :

Let Λ be a f.d. non-semisimple self-injective algebra.

$\Longrightarrow \Lambda_\Lambda$ is cotilting with $\operatorname{inj.dim}\Lambda = 0$ and $\mathcal{X}_\Lambda = \operatorname{mod}\Lambda$.

However, $\langle \operatorname{mod}\Lambda \rangle$ is different from $\mathsf{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$.