Finite cycles of indecomposable modules

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Preliminaries

- A – basic indecomposable artin algebra
  (over a fixed commutative artin ring K)
- mod A – category of finitely generated right A-modules
- ind A – full subcategory of mod A formed by all indecomposable modules
- rad_A – Jacobson radical of mod A
  (the ideal of mod A generated by all irreducible homomorphisms between modules in ind A)
- $\mathrm{rad}_A^\infty = \bigcap_{i\geq 1} \mathrm{rad}_A^i$ – infinite Jacobson radical of mod A

$\mathrm{rad}_A^\infty = 0 \overset{\text{Auslander}}{\Longleftrightarrow}$ A is of finite representation type

A is of infinite representation type $\xrightarrow{\text{Coelho-Marcos-Merklen-Skowroński}}$ $(\mathrm{rad}_A^\infty)^2 \neq 0$

Preliminaries

- A cycle in $\operatorname{ind} A$ is a sequence

$$(\star) \quad M_0 \xrightarrow{f_1} M_1 \to \cdots \to M_{r-1} \xrightarrow{f_r} M_r = M_0$$

  of nonzero nonisomorphisms in $\operatorname{ind} A$.
- A cycle $(\star)$ is said to be finite if the homomorphisms $f_1, \ldots, f_r$ do not belong to $\operatorname{rad}_A^{\infty}$.
- A module M in $\operatorname{ind} A$ is called directing if M does not lie on a cycle in $\operatorname{ind} A$.
- A module M in $\operatorname{ind} A$ is said to be cycle-finite if M is nondirecting and every cycle in $\operatorname{ind} A$ passing through M is finite.
- A is cycle-finite if all cycles in $\operatorname{ind} A$ are finite.

Note that every algebra of finite representation type is cycle-finite.

Support algebra of a module

For a module M in $\operatorname{ind} A$, consider:

- a decomposition $A = P_M \oplus Q_M$ of A in $\operatorname{mod} A$ such that the simple summands of the semisimple module $P_M / \operatorname{rad} P_M$ are exactly the simple composition factors of M
- the ideal $t_A(M)$ in A generated by the images of all homomorphisms from Q_M to A in $\operatorname{mod} A$

Then $\operatorname{Supp}(M) = A/t_A(M)$ is called the support algebra of M.

Theorem (Ringel)

If A is an algebra with all modules in $\operatorname{ind} A$ being directing, then A is of finite representation type.

Theorem (Ringel)

Let A be an algebra and M be a directing A-module. Then $\operatorname{Supp}(M)$ is a tilted algebra.

Support algebra of a module

Hence, if A is an algebra of infinite representation type, then ind A always contains a cycle.

Theorem (Peng–Xiao, Skowroński)

Let A be an algebra. Then Γ_A admits at most finitely many τ_A-orbits containing directing modules.

Γ_A – Auslander-Reiten quiver of A

Remark

In order to obtain information on the support algebras Supp(M) of nondirecting modules in ind A, it is natural to study properties of cycles in ind A containing M.

An object of study - the first approach

Problem: Let A be an algebra and M be a cycle-finite module in $\operatorname{ind} A$. Describe the support algebra $\operatorname{Supp}(M)$.

Remark (A – algebra, M – cycle-finite module in $\operatorname{ind} A$)

Every cycle in $\operatorname{ind} A$ passing through M has a refinement to a cycle of irreducible homomorphisms in $\operatorname{ind} A$ containing M and consequently M lies on an oriented cycle in Γ_A (we will consider a more general problem).

- ${}_c\Gamma_A$ – translation subquiver of Γ_A, called the cyclic quiver of A, obtained by removing from Γ_A all acyclic vertices and the arrows attached to them
- the connected components of ${}_c\Gamma_A$ are said to be cyclic components of Γ_A

Γ – cyclic component of Γ_A

$M, N \in \Gamma \overset{\text{M.–Skowroński}}{\Longleftrightarrow}$ M and N lie on a common oriented cycle in Γ_A

Support algebra of a component

For a cyclic component Γ of ${}_c\Gamma_A$, consider:

- a decomposition $A = P_\Gamma \oplus Q_\Gamma$ of A in $\operatorname{mod} A$ such that the simple summands of the semisimple module $P_\Gamma / \operatorname{rad} P_\Gamma$ are exactly the simple composition factors of indecomposable modules in Γ
- the ideal $t_A(\Gamma)$ in A generated by the images of all homomorphisms from Q_Γ to A in $\operatorname{mod} A$

Then $\operatorname{Supp}(\Gamma) = A/t_A(\Gamma)$ is called the support algebra of Γ.

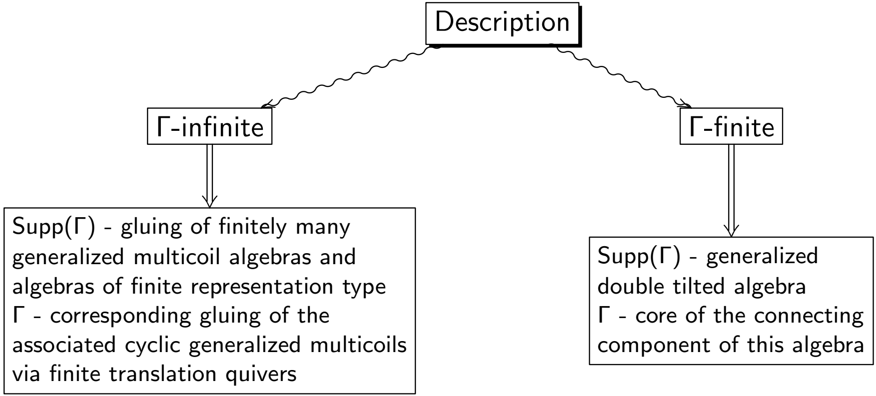
Remark

Observe that M belongs to a unique cyclic component $\Gamma(M)$ of Γ_A consisting entirely of cycle-finite indecomposable modules, and the support algebra $\operatorname{Supp}(M)$ of M is a quotient algebra of the support algebra $\operatorname{Supp}(\Gamma(M))$ of $\Gamma(M)$.

A cyclic component Γ of Γ_A containing a cycle-finite module is said to be a cycle-finite cyclic component of Γ_A.

An object of study

Problem: Let A be an algebra and Γ be a cycle-finite cyclic component of Γ_A. Describe the support algebra $\mathrm{Supp}(\Gamma)$.



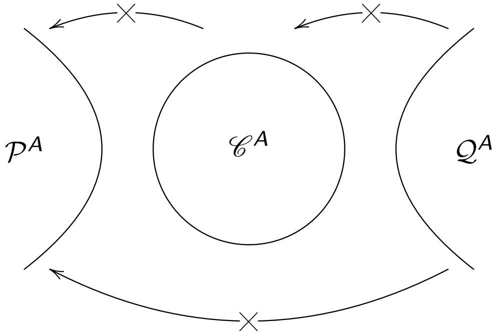
Separating family of components

- A an algebra
- $\mathscr{C} = (\mathscr{C}_i)_{i \in I}$ – family of connected components of Γ_A
- $\mathscr{C}$ is sincere if every simple module in mod A occurs as a composition factor of a module in $\mathscr{C}$
- $\mathscr{C}$ is generalized standard if $\mathrm{rad}_A^\infty(X, Y) = 0$ for all modules X and Y in $\mathscr{C}$
- $\mathscr{C} = (\mathscr{C}_i)_{i \in I}$ is said to be separating if the components in Γ_A split into three disjoint classes $\mathcal{P}^A$, $\mathscr{C}^A = \mathscr{C}$ and $\mathcal{Q}^A$ such that:
  1. $\mathscr{C}^A$ is sincere and generalized standard;
  2. $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\mathrm{Hom}_A(\mathcal{Q}^A, \mathscr{C}^A) = 0$, $\mathrm{Hom}_A(\mathscr{C}^A, \mathcal{P}^A) = 0$;
  3. any morphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ in mod A factors through $\mathrm{add}(\mathscr{C}^A)$.

Then we write: $\Gamma_A = \mathcal{P}^A \cup \mathscr{C}^A \cup \mathcal{Q}^A$ ($\mathscr{C}^A$ separates $\mathcal{P}^A$ from $\mathcal{Q}^A$).

$\mathcal{P}^A$ and $\mathcal{Q}^A$ are uniquely determined by $\mathscr{C}^A$

Separating family of components

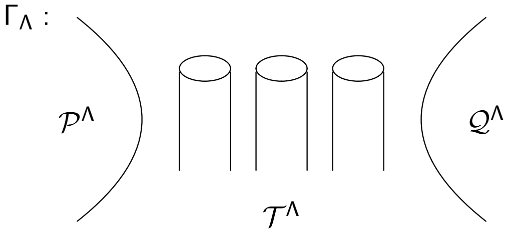


If $\mathscr{C}^A$ is generalized standard then components in $\mathscr{C}^A$ are pairwise orthogonal and almost periodic.

Concealed canonical algebras

Let Λ be a canonical algebra in the sense of Ringel. Then

- $\mathrm{gl.\,dim}\,\Lambda \leq 2$
- $\Gamma_\Lambda = \mathcal{P}^\Lambda \cup \mathcal{T}^\Lambda \cup \mathcal{Q}^\Lambda$

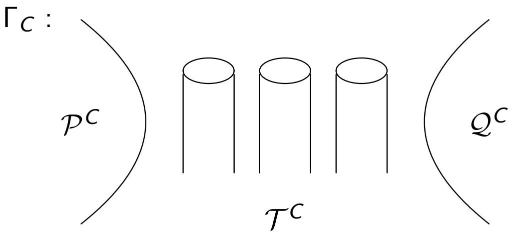


$\mathcal{T}^\Lambda$ – separating family of stable tubes

Let T – tilting Λ-module from the additive category $\mathrm{add}(\mathcal{P}^\Lambda)$ of $\mathcal{P}^\Lambda$

- C – concealed canonical algebra (of type Λ) : $C = \mathrm{End}_\Lambda(T)$
- $\Gamma_C = \mathcal{P}^C \cup \mathcal{T}^C \cup \mathcal{Q}^C$

Concealed canonical algebras



$\mathcal{T}^C = \mathrm{Hom}_\Lambda(T, \mathcal{T}^\Lambda)$ – separating family of stable tubes

Theorem (Lenzing–de la Peña)

Let A be an algebra. TFAE

1. *A is a concealed canonical algebra.*
2. *Γ_A admits a separating family of stable tubes.*

Quasitilted algebras

A – quasitilted: $\mathrm{gl.\,dim}\, A \leq 2$ and for any $X \in \operatorname{ind} A$ we have $\mathrm{pd}_A X \leq 1$ or $\mathrm{id}_A X \leq 1$

Theorem (Happel–Reiten)

Let A be a quasitilted algebra. Then A is either a tilted algebra or a quasitilted algebra of canonical type.

Theorem (Lenzing–Skowroński)

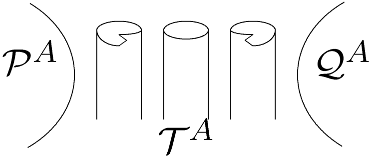
Let A be an algebra. TFAE

1. *A is a quasitilted algebra of canonical type.*
2. *A is a semiregular branch enlargement of a concealed canonical algebra C.*
3. *Γ_A admits a separating family of ray and coray tubes.*

Quasitilted algebras

A – quasitilted algebra of canonical type

Γ_A :



$\mathcal{T}^A$ – separating family of ray and coray tubes in mod A

Almost cyclic and coherent components

A – algebra, Γ – component of Γ_A

- Γ is almost cyclic if its cyclic part ${}_c\Gamma$ is a cofinite subquiver of Γ
- Γ is coherent if the following two conditions are satisfied:
  1. For each projective module P in Γ there is an infinite sectional path $P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to X_{i+2} \to \cdots$ in Γ ($X_i \neq \tau_A X_{i+2}$ for any $i \geq 1$)
  2. For each injective module I in Γ there is an infinite sectional path $\cdots \to Y_{j+2} \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I$ in Γ ($Y_{j+2} \neq \tau_A Y_j$ for any $j \geq 1$)

Note that the stable tubes, ray tubes and coray tubes of Γ_A are special types of coherent almost cyclic components.

Γ is almost cyclic and coherent $\overset{\text{M.–Skowroński}}{\Longleftrightarrow}$ Γ is a generalized multicoil (obtained from a finite family of stable tubes by a sequence of admissible operations (ad 1)-(ad 5) and their duals (ad 1^*)-(ad 5^*))

Generalized multicoil algebras

- $C_1, \ldots, C_m$ – concealed canonical algebras
- $\mathcal{T}^{C_1}, \ldots, \mathcal{T}^{C_m}$ – separating families of stable tubes of $\Gamma_{C_1}, \ldots, \Gamma_{C_m}$
- $C = C_1 \times \ldots \times C_m$
- A – generalized multicoil algebra if A is a generalized multicoil enlargement of a product C using modules from $\mathcal{T}^{C_1}, \ldots, \mathcal{T}^{C_m}$ and a sequence of admissible operations of types (ad 1)-(ad 5) and their duals (ad 1^*)-(ad 5^*)

Theorem (M.–Skowroński)

Let A be an algebra. TFAE

1. *A is a generalized multicoil algebra.*
2. *Γ_A admits a separating family of almost cyclic coherent components.*

Generalized multicoil algebras

Theorem (M.–Skowroński)

Let A be a generalized multicoil algebra. Then there are:

1. *unique quotient algebra $A^{(l)}$ of A which is a product of quasitilted algebras of canonical type having separating families of coray tubes and*
2. *unique quotient algebra $A^{(r)}$ of A which is a product of quasitilted algebras of canonical type having separating families of ray tubes*

s.t. Γ_A has a disjoint union decomposition $\Gamma_A = \mathcal{P}^A \cup \mathscr{C}^A \cup \mathcal{Q}^A$, where

- *$\mathcal{P}^A$ is the left part $\mathcal{P}^{A^{(l)}}$ in a decomposition $\Gamma_{A^{(l)}} = \mathcal{P}^{A^{(l)}} \cup \mathcal{T}^{A^{(l)}} \cup \mathcal{Q}^{A^{(l)}}$ of $\Gamma_{A^{(l)}}$ of $A^{(l)}$, with $\mathcal{T}^{A^{(l)}}$ a family of coray tubes separating $\mathcal{P}^{A^{(l)}}$ from $\mathcal{Q}^{A^{(l)}}$;*
- *$\mathcal{Q}^A$ is the right part $\mathcal{Q}^{A^{(r)}}$ in a decomposition $\Gamma_{A^{(r)}} = \mathcal{P}^{A^{(r)}} \cup \mathcal{T}^{A^{(r)}} \cup \mathcal{Q}^{A^{(r)}}$ of $\Gamma_{A^{(r)}}$ of $A^{(r)}$, with $\mathcal{T}^{A^{(r)}}$ a family of ray tubes separating $\mathcal{P}^{A^{(r)}}$ from $\mathcal{Q}^{A^{(r)}}$;*

Generalized multicoil algebras

Theorem (continuation)

- $\mathscr{C}^A$ *is a family of generalized multicoils separating* $\mathcal{P}^A$ *from* $\mathcal{Q}^A$*, obtained from stable tubes in the separating families* $\mathcal{T}^{C_1}, \dots, \mathcal{T}^{C_m}$ *of stable tubes of the Auslander-Reiten quivers* $\Gamma_{C_1}, \dots, \Gamma_{C_m}$ *of concealed canonical algebras* $C_1, \dots, C_m$ *by a sequence of admissible operations of types (ad 1)-(ad 5) and their duals (ad 1*)-(ad 5*), corresponding to the admissible operations leading from* $C = C_1 \times \dots \times C_m$ *to* A*;*
- $\mathscr{C}^A$ *consists of cycle-finite modules and contains all indecomposable modules of* $\mathcal{T}^{A^{(l)}}$ *and* $\mathcal{T}^{A^{(r)}}$*;*
- $\mathcal{P}^A$ *contains all indecomposable modules of* $\mathcal{P}^{A^{(r)}}$*;*
- $\mathcal{Q}^A$ *contains all indecomposable modules of* $\mathcal{Q}^{A^{(l)}}$*.*

$A^{(l)}$ – the left quasitilted algebra of A
$A^{(r)}$ – the right quasitilted algebra of A

Generalized multicoil algebras

Moreover, in the above notation, we have

- $\mathrm{gl.dim}\, A \leq 3$;
- $\mathrm{pd}_A X \leq 1$ for any indecomposable module X in $\mathcal{P}^A$;
- $\mathrm{id}_A Y \leq 1$ for any indecomposable module Y in $\mathcal{Q}^A$;
- $\mathrm{pd}_A M \leq 2$ and $\mathrm{id}_A M \leq 2$ for any indecomposable module M in $\mathscr{C}^A$.

A generalized multicoil algebra A is said to be tame if $A^{(l)}$ and $A^{(r)}$ are product of tilted algebras of Euclidean types or tubular algebras.

Note that every tame generalized multicoil algebra is a cycle-finite algebra.

For a subquiver Γ of Γ_A, we denote by $\mathrm{ann}_A(\Gamma)$ the intersection of the annihilators $\mathrm{ann}_A(X) = \{a \in A \mid Xa = 0\}$ of all indecomposable modules X in Γ, and call the quotient algebra $B(\Gamma) = A/\,\mathrm{ann}_A(\Gamma)$ the faithful algebra of Γ.

The first main result

Theorem

Let A be an algebra and Γ be a cycle-finite infinite component of ${}_c\Gamma_A$. Then there exist infinite full translation subquivers $\Gamma_1, \ldots, \Gamma_r$ of Γ such that the following statements hold.

1. *For each $i \in \{1, \ldots, r\}$, Γ_i is a cyclic coherent full translation subquiver of Γ_A.*
2. *For each $i \in \{1, \ldots, r\}$, $\mathrm{Supp}(\Gamma_i) = B(\Gamma_i)$ and is a generalized multicoil algebra.*
3. *$\Gamma_1, \ldots, \Gamma_r$ are pairwise disjoint full translation subquivers of Γ and $\Gamma^{cc} = \Gamma_1 \cup \ldots \cup \Gamma_r$ is a maximal cyclic coherent and cofinite full translation subquiver of Γ.*
4. *$B(\Gamma \setminus \Gamma^{cc})$ is of finite representation type.*
5. *$\mathrm{Supp}(\Gamma) = B(\Gamma)$.*

Generalized double tilted algebras

- Γ – component of Γ_A
- Γ – almost acyclic if all but finitely many modules of Γ are acyclic

Γ is an almost acyclic $\xleftrightarrow{\text{Reiten–Skowroński}}$ Γ admits a multisection

Note that for an almost acyclic component Γ of Γ_A, there exists a finite convex subquiver $c(\Gamma)$ of Γ (possibly empty), called the core of Γ, containing all modules lying on oriented cycles in Γ

Theorem (Reiten-Skowroński)

Let A be an algebra. TFAE

1. *Γ_A admits an almost acyclic separating component.*
2. *A is a generalized double tilted algebra.*

Generalized double tilted algebras

Theorem (Reiten-Skowroński)

Let B be a generalized double tilted algebra. Then Γ_B has a disjoint union decomposition $\Gamma_B = \mathcal{P}^B \cup \mathcal{C}^B \cup \mathcal{Q}^B$, where

- *$\mathcal{C}^B$ is an almost acyclic component separating $\mathcal{P}^B$ from $\mathcal{Q}^B$;*
- *There exist hereditary algebras $H_1^{(l)}, \dots, H_m^{(l)}$ and tilting modules $T_1^{(l)} \in \operatorname{mod} H_1^{(l)}, \dots, T_m^{(l)} \in \operatorname{mod} H_m^{(l)}$ such that the tilted algebras $B_1^{(l)} = \operatorname{End}_{H_1^{(l)}}(T_1^{(l)}), \dots, B_m^{(l)} = \operatorname{End}_{H_m^{(l)}}(T_m^{(l)})$ are quotient algebras of B and $\mathcal{P}^B$ is the disjoint union of all components of $\Gamma_{B_1^{(l)}}, \dots, \Gamma_{B_m^{(l)}}$ contained entirely in the torsion-free parts $\mathscr{Y}(T_1^{(l)}), \dots, \mathscr{Y}(T_m^{(l)})$ of $\operatorname{mod} B_1^{(l)}, \dots, \operatorname{mod} B_m^{(l)}$ determined by $T_1^{(l)}, \dots, T_m^{(l)}$;*

Generalized double tilted algebras

Theorem (continuation)

- *There exist hereditary algebras* $H_1^{(r)}, \dots, H_n^{(r)}$ *and tilting modules* $T_1^{(r)} \in \operatorname{mod} H_1^{(r)}, \dots, T_n^{(r)} \in \operatorname{mod} H_n^{(r)}$ *such that the tilted algebras* $B_1^{(r)} = \operatorname{End}_{H_1^{(r)}}(T_1^{(r)}), \dots, B_n^{(r)} = \operatorname{End}_{H_n^{(r)}}(T_n^{(r)})$ *are quotient algebras of* B *and* $\mathcal{Q}^B$ *is the disjoint union of all components of* $\Gamma_{B_1^{(r)}}, \dots, \Gamma_{B_n^{(r)}}$ *contained entirely in the torsion parts* $\mathscr{X}(T_1^{(r)}), \dots, \mathscr{X}(T_n^{(r)})$ *of* $\operatorname{mod} B_1^{(r)}, \dots, \operatorname{mod} B_n^{(r)}$ *determined by* $T_1^{(r)}, \dots, T_n^{(r)}$;
- *every indecomposable module in* $\mathcal{C}^B$ *not lying in the core* $c(\mathcal{C}^B)$ *of* $\mathcal{C}^B$ *is an indecomposable module over one of the tilted algebras* $B_1^{(l)}, \dots, B_m^{(l)}, B_1^{(r)}, \dots, B_n^{(r)}$;
- *every nondirecting indecomposable module in* $\mathcal{C}^B$ *is cycle-finite and lies in* $c(\mathcal{C}^B)$;

Generalized double tilted algebras

Theorem (continuation)

- $\mathrm{pd}_B X \leq 1$ *for all indecomposable modules* X *in* $\mathcal{P}^B$*;*
- $\mathrm{id}_B Y \leq 1$ *for all indecomposable modules* Y *in* $\mathcal{Q}^B$*;*
- *for all but finitely many indecomposable modules* M *in* $\mathcal{C}^B$*, we have* $\mathrm{pd}_B M \leq 1$ *or* $\mathrm{id}_B M \leq 1$.

$\mathcal{C}^B$ – connecting component of Γ_B
$B^{(l)} = B_1^{(l)} \times \ldots \times B_m^{(l)}$ – left tilted algebra of B
$B^{(r)} = B_1^{(r)} \times \ldots \times B_n^{(r)}$ – right tilted algebra of B
A generalized double tilted algebra B is said to be tame if the tilted algebras $B^{(l)}$ and $B^{(r)}$ are generically tame in the sense of Crawley-Boevey.

Note that every tame generalized double tilted algebra is a cycle-finite algebra.

The second main result

Theorem

Let A be an algebra and Γ be a cycle-finite finite component of ${}_c\Gamma_A$. Then the following statements hold.

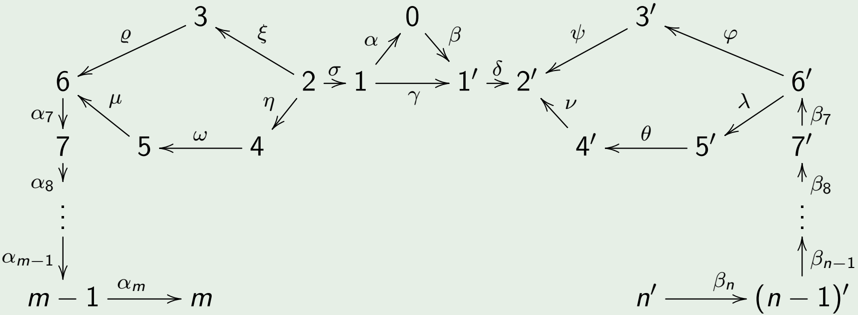
1. $\mathrm{Supp}(\Gamma)$ *is a generalized double tilted algebra.*
2. Γ *is the core* $c(\mathcal{C}^{B(\Gamma)})$ *of a unique almost acyclic connecting component* $\mathcal{C}^{B(\Gamma)}$ *of* $\Gamma_{B(\Gamma)}$.
3. $\mathrm{Supp}(\Gamma) = B(\Gamma)$.

Remark

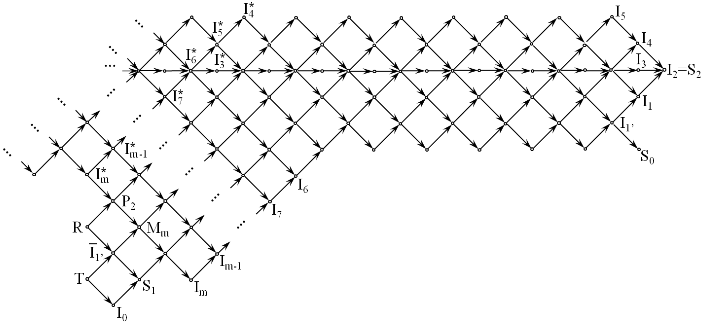
Every finite cyclic component Γ of an Auslander-Reiten quiver Γ_A contains both a projective module and an injective module, and hence Γ_A admits at most finitely many finite cyclic components.

Example

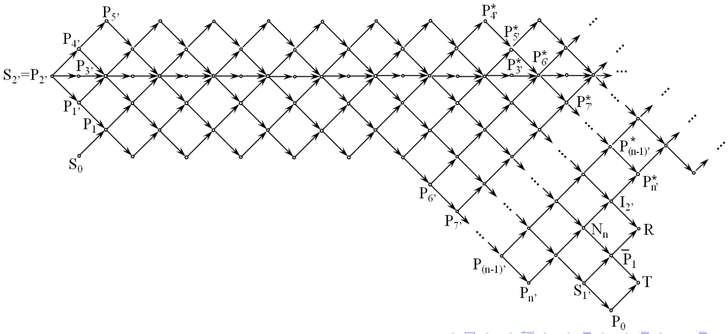
Let K be a field, $m, n \geq 8$ natural numbers, and $B_{m,n} = KQ_{m,n}/I_{m,n}$ the bound quiver algebra given by the quiver $Q_{m,n}$ of the form



and $I_{m,n}$ the ideal in the path algebra $KQ_{m,n}$ of $Q_{m,n}$ over K generated by the elements $\alpha\beta, \sigma\alpha, \beta\delta, \sigma\gamma\delta, \xi\varrho - \eta\omega\mu, \varphi\psi - \lambda\theta\nu$.



I_4^*
I_5^*
I_6^*
I_3^*
I_7^*
I_5
I_4
I_3
$I_2=S_2$
I_1
$I_{1'}$
S_0
I_{m-1}^*
I_m^*
P_2
R
M_m
$\bar{I}_{1'}$
T
S_1
I_0
I_m
I_{m-1}
I_6
I_7



$P_{5'}$
$P_{4'}$
$P_{3'}$
$S_{2'}=P_{2'}$
$P_{1'}$
P_1
S_0
$P_{4'}^*$
$P_{5'}^*$
$P_{3'}^*$
$P_{6'}^*$
$P_{7'}^*$
$P_{(n-1)'}^*$
$P_{n'}^*$
$I_{2'}$
R
N_n
$\bar{P}_1$
T
$S_{1'}$
P_0
$P_{6'}$
$P_{7'}$
$P_{(n-1)'}$
$P_{n'}$

Example

Denote by $\mathcal{C}_{m,n}$ the above component.

- $B_{m,n}$ is a generalized double tilted algebra
  - finite representation type $\iff m, n \in \{8, 9, 10\}$
  - tame $\iff m, n \in \{8, 9, 10, 11\}$
- $\Gamma_{B_{m,n}} = \mathcal{P}_{m,n} \cup \mathcal{C}_{m,n} \cup \mathcal{Q}_{m,n}$
- $\mathcal{C}_{m,n}$ is an almost acyclic component of $\Gamma_{B_{m,n}}$
- cyclic part $\Gamma_{m,n}$ of $\mathcal{C}_{m,n}$ is connected and consists of all indecomposable modules in $\mathcal{C}_{m,n}$ which lie on oriented cycles passing through the simple module S_0
- $\Gamma_{m,n}$ is a faithful cyclic component of $\Gamma_{B_{m,n}}$
- $\mathcal{C}_{m,n}$ is a faithful component of $\Gamma_{B_{m,n}}$
- $\mathrm{Supp}(\Gamma_{m,n}) = B_{m,n}$

The next results

An idempotent e of an algebra A is said to be convex provided e is a sum of pairwise orthogonal primitive idempotents of A corresponding to the vertices of a convex valued subquiver of the quiver Q_A of A.

Corollary

Let A be an algebra and Γ be a cycle-finite component of ${}_c\Gamma_A$. Then there exists a convex idempotent e_Γ of A such that $\mathrm{Supp}(\Gamma)$ is isomorphic to the algebra $e_\Gamma A e_\Gamma$.

Theorem

Let A be an algebra. Then, for all but finitely many isomorphism classes of cycle-finite modules M in ind A*, the following statements hold.*

1. $|\operatorname{Ext}_A^1(M,M)| \leq |\operatorname{End}_A(M)|$ *and* $\operatorname{Ext}_A^r(M,M) = 0$ *for* $r \geq 2$.
2. $|\operatorname{Ext}_A^1(M,M)| = |\operatorname{End}_A(M)|$ *if and only if there is a quotient concealed canonical algebra C of A and a stable tube $\mathcal{T}$ of Γ_C such that M is an indecomposable C-module in $\mathcal{T}$ of quasi-length divisible by the rank of $\mathcal{T}$.*

Hence, for all but finitely many isomorphism classes of cycle-finite modules M in a module category ind A, the Euler form

$$\chi_A(M) = \sum_{i=0}^{\infty} (-1)^i |\operatorname{Ext}_A^i(M,M)|$$

of M is well defined and nonnegative.

- A – algebra
- $K_0(A)$ – the Grothendieck group of A
- for a module M in $\operatorname{mod} A$, we denote by $[M]$ the image of M in $K_0(A)$

Theorem

Let A be an algebra. The following statements hold.

1. *There is a positive integer m such that, for any cycle-finite module M in $\operatorname{ind} A$ with $|\operatorname{End}_A(M)| \neq |\operatorname{Ext}_A^1(M, M)|$, the number of isomorphism classes of modules X in $\operatorname{ind} A$ with $[X] = [M]$ is bounded by m.*
2. *For all but finitely many isomorphism classes of cycle-finite modules M in $\operatorname{ind} A$ with $|\operatorname{End}_A(M)| = |\operatorname{Ext}_A^1(M, M)|$, there are infinitely many pairwise nonisomorphic modules X in $\operatorname{ind} A$ with $[X] = [M]$.*
3. *The number of isomorphism classes of cycle-finite modules M in $\operatorname{ind} A$ with $\operatorname{Ext}_A^1(M, M) = 0$ is finite.*

- X – nonprojective module in ind A
- $\alpha(X)$ – the number of indecomposable direct summands in the middle term

$$0 \to \tau_A X \to Y \to X \to 0$$

of the almost split sequence with the right term X

- A is an algebra of finite representation type and X a nonprojective module in ind A $\xrightarrow{\text{Bautista-Brenner}}$ $\alpha(X) \leq 4$
- $\alpha(X) = 4$ $\xrightarrow{\text{Bautista-Brenner}}$ Y admits a projective-injective indecomposable direct summand
- *Liu* – the same is true for any indecomposable nonprojective module X lying on an oriented cycle of Γ_A of any algebra A

Theorem

Let A be an algebra. Then, for all but finitely many isomorphism classes of nonprojective cycle-finite modules M in ind A, *we have $\alpha(M) \leq 2$.*

Theorem

Let A be a cycle-finite algebra. Then there exist tame generalized multicoil algebras $B_1, \dots, B_p$ and tame generalized double tilted algebras $B_{p+1}, \dots, B_q$ which are quotient algebras of A and the following statements hold.

1. $\operatorname{ind} A = \bigcup_{i=1}^{q} \operatorname{ind} B_i$.
2. *All but finitely many isomorphism classes of modules in* $\operatorname{ind} A$ *belong to* $\bigcup_{i=1}^{p} \operatorname{ind} B_i$.
3. *All but finitely many isomorphism classes of nondirecting modules in* $\operatorname{ind} A$ *belong to generalized multicoils of* $\Gamma_{B_1}, \dots, \Gamma_{B_p}$.

Theorem

Let A be a cycle-finite algebra. Then, for all but finitely many isomorphism classes of modules M in $\operatorname{ind} A$, *we have* $|\operatorname{Ext}_A^1(M, M)| \leq |\operatorname{End}_A(M)|$ *and* $\operatorname{Ext}_A^r(M, M) = 0$ *for $r \geq 2$.*

Problem

A is a tame algebra over an algebraically closed field, M is a directing module in ind $A \xRightarrow{\text{de la Peña}}$ Supp(M) is a tilted algebra being a gluing of at most two representation-infinite tilted algebras of Euclidean type

Open question: Let A be an algebra and Γ be a cycle-finite finite component in the cyclic quiver ${}_c\Gamma_A$. Is then Supp(Γ) gluing of at most two representation-infinite tilted algebras of Euclidean type?

- P. Malicki, *On the composition factors of indecomposable modules in almost cyclic coherent Auslander-Reiten components*, J. Pure Appl. Algebra **207** (2006), 469–490.
- P. Malicki, J. A. de la Peña and A. Skowroński, *Finite cycles of indecomposable modules*, arXiv:1306.0929v1 [math.RT].
- P. Malicki, J. A. de la Peña and A. Skowroński, *On the number of terms in the middle of almost split sequences over cycle-finite artin algebras*, Centr. Eur. J. Math., **12** (2014), 39–45.
- P. Malicki and A. Skowroński, *Almost cyclic coherent components of an Auslander-Reiten quiver*, J. Algebra **229** (2000), 695–749.
- P. Malicki and A. Skowroński, *On the indecomposable modules in almost cyclic coherent Auslander-Reiten components*, J. Math. Soc. Japan **63** (2011), 1121–1154.