

SPECIALIZATION ORDERS ON ATOM SPECTRA OF GROTHENDIECK CATEGORIES

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ABSTRACT. This is a lecture note of my talk in *Perspectives of Representation Theory of Algebras, Conference honoring Kunio Yamagata on the occasion of his 65th birthday (The 13th International Conference by Graduate School of Mathematics, Nagoya University)* on November 11–15, 2013 in Nagoya University. The talk was based on [Kan13].

1. INTRODUCTION

Throughout this talk, let \mathcal{A} be a Grothendieck category. It is known that the category $\text{Mod } \Lambda$ of right modules over a ring Λ and the category $\text{QCoh } X$ of quasi-coherent sheaves on a scheme X are Grothendieck categories (see [Con20, Lem 2.1.7]).

For a commutative ring R , we often consider the poset $(\text{Spec } R, \subset)$. Similarly, for a Grothendieck category \mathcal{A} , we construct a poset $(\text{ASpec } \mathcal{A}, \leq)$.

Theorem 1.1 (Hochster [Hoc69, Proposition 10] and Speed [Spe72, Corollary 1]). *Let P be a partially ordered set. Then P is isomorphic to the prime spectrum of some commutative ring with the inclusion relation if and only if P is an inverse limit of finite partially ordered sets in the category of partially ordered sets.*

Theorem 1.2 ([Kan13, Theorem 7.27]). *Any partially ordered set is isomorphic to the atom spectrum of some Grothendieck category as a partially ordered set.*

2. ATOM SPECTRUM

Definition 2.1. An object H in \mathcal{A} is called *monoform* if for any nonzero subobject L of H , there exists no common nonzero subobject of H and H/L , that is, there does not exist a nonzero subobject of H which is isomorphic to a subobject of H/L .

Definition 2.2. We say that monoform objects H_1 and H_2 in \mathcal{A} are *atom-equivalent* if there exists a common nonzero subobject of H_1 and H_2 .

Definition 2.3. Denote by $\text{ASpec } \mathcal{A}$ the quotient class of the class of monoform objects by the atom equivalence, and call it the *atom spectrum* of \mathcal{A} . We call an element of $\text{ASpec } \mathcal{A}$ an *atom* in \mathcal{A} . The equivalence class of a monoform object H in \mathcal{A} is denoted by \overline{H} .

Definition 2.4. Let M be an object in \mathcal{A} . Define a subset $\text{ASupp } M$ of $\text{ASpec } \mathcal{A}$ by

$$\text{ASupp } M = \{\alpha \in \text{ASpec } \mathcal{A} \mid \alpha = \overline{H} \text{ for a monoform subquotient } H \text{ of } M\},$$

and call it the *atom support* of M .

Proposition 2.5. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence in \mathcal{A} . Then we have $\text{ASupp } M = \text{ASupp } L \cup \text{ASupp } N$.*

Definition 2.6. Let $\alpha, \beta \in \text{ASpec } \mathcal{A}$. We denote by $\alpha \leq \beta$ that any object $M \in \mathcal{A}$ satisfying $\alpha \in \text{ASupp } M$ also satisfies $\beta \in \text{ASupp } M$.

The author is a Research Fellow of Japan Society for the Promotion of Science. This work is supported by Grant-in-Aid for JSPS Fellows 25·249.

Proposition 2.7. *Let R be a commutative ring. Then the map $\text{Spec } R \rightarrow \text{ASpec}(\text{Mod } R)$ defined by $\mathfrak{p} \mapsto \overline{R/\mathfrak{p}}$ gives a poset isomorphism between $(\text{Spec } R, \subset)$ and $(\text{ASpec}(\text{Mod } R), \leq)$.*

Proposition 2.8. *Let Λ be a right artinian ring. Then there exists a bijection between the set of isomorphism classes of simple Λ -modules and $\text{ASpec}(\text{Mod } \Lambda)$. The correspondence is given by $S \mapsto \overline{S}$.*

3. CONSTRUCTION OF GROTHENDIECK CATEGORIES

Theorem 1.2 is proven by the following steps.

- (1) From a poset P , construct a *colored quiver* Γ (technically).
- (2) From a colored quiver, associate a Grothendieck category \mathcal{A}_Γ .
- (3) Take a quotient category $\mathcal{A} = \mathcal{A}_\Gamma/\mathcal{X}$ by a localizing subcategory \mathcal{X} (if necessary).

In this talk, we mainly deal with the second step. We see key ideas in the first step by treating examples.

Definition 3.1. A sextuple $\Gamma = (Q_0, Q_1, C, s, t, u)$ is called a colored quiver if it satisfies the following conditions.

- (1) (Q_0, Q_1, s, t) is a quiver.
- (2) C is a set.
- (3) $u: Q_1 \rightarrow C$ is a map.
- (4) For any $v \in Q_0$ and $c \in C$, there are only finitely many $r \in Q_1$ satisfying $s(r) = v$ and $u(r) = c$.

Example 3.2. The diagram

$$v \xrightarrow{1} w \curvearrowright 2$$

represents a colored quiver.

We fix a field K .

Definition 3.3. Let $\Gamma = (Q_0, Q_1, C, s, t, u)$ be a colored quiver. Denote the free K -algebra on C by $F_C = K \langle f_c \mid c \in C \rangle$. Define a K -vector space M_Γ by $M_\Gamma = \bigoplus_{v \in Q_0} x_v K$, where $x_v K$ is a one-dimensional K -vector space generated by an element x_v . Regard M_Γ as a right F_C -module by defining the action of $f_c \in F_C$ as follows: for each vertex v in Q ,

$$x_v \cdot f_c = \sum_{\substack{r \in Q_1 \\ s(r)=v \\ u(r)=c}} x_{t(r)},$$

Denote by \mathcal{A}_Γ the smallest full subcategory of $\text{Mod } F_C$ closed under subobject, quotient object, and arbitrary direct sums, and containing M_Γ .

Lemma 3.4. *The poset $\text{ASpec } \mathcal{A}_\Gamma$ is isomorphic to the subset $\text{ASupp } M_\Gamma$ of $\text{ASpec}(\text{Mod } F_C)$.*

Example 3.5. Let Γ be the colored quiver

$$\begin{array}{c} v \quad . \\ \downarrow 1 \\ w \end{array}$$

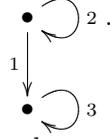
Then we have $F_C = K \langle f_1 \rangle = K[f_1]$ and $M_\Gamma = x_v K \oplus x_w K$, where $x_v f_1 = x_w$ and $x_w f_1 = 0$. M_Γ has a F_C -submodule S of the form $0 \oplus x_w K$. Then we have the exact sequence

$$0 \rightarrow S \rightarrow M_\Gamma \rightarrow S \rightarrow 0.$$

Hence we have

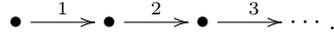
$$\text{ASpec } \mathcal{A}_\Gamma = \text{ASupp } M_\Gamma = \text{ASupp } S = \{\overline{S}\}.$$

Example 3.6. Let Γ be the colored quiver



Then $\text{ASpec } \mathcal{A}_\Gamma$ consists of two elements, and every element is maximal.

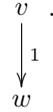
Example 3.7. Let Γ be the colored quiver



Then $\text{ASpec } \mathcal{A}_\Gamma = \{\overline{M}_\Gamma, \overline{K}\}$, and we have $\overline{M}_\Gamma < \overline{K}$.

We introduce a notation of *bold arrow*. The precise definition is given in [Kan13, Notation 7.20]. It is explained by using an example here.

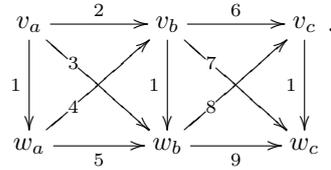
Example 3.8. Let Γ be the colored quiver



Then the diagram

$$\Gamma \Longrightarrow \Gamma \Longrightarrow \Gamma$$

represents the colored quiver



Lemma 3.9. Let Γ be a colored quiver. Let $\tilde{\Gamma}$ a colored quiver

$$\Gamma \Longrightarrow \Gamma \Longrightarrow \Gamma \Longrightarrow \cdots$$

Then $\text{ASpec } \mathcal{A}_{\tilde{\Gamma}} = \text{ASpec } \mathcal{A}_\Gamma \cup \{\overline{M}_\Gamma\}$, where \overline{M}_Γ is smaller than any element in $\text{ASpec } \mathcal{A}_\Gamma$.

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