SPECIALIZATION ORDERS ON ATOM SPECTRA OF GROTHENDIECK CATEGORIES

RYO KANDA

ABSTRACT. This is a lecture note of my talk in *Perspectives of Representation Theory of Algebras, Conference honoring Kunio Yamagata on the occasion of his 65th birthday (The 13th International Conference by Graduate School of Mathematics, Nagoya University)* on November 11–15, 2013 in Nagoya University. The talk was based on [Kan13].

1. Introduction

Throughout this talk, let $\mathcal{A}$ be a Grothendieck category. It is known that the category $\operatorname{Mod}\Lambda$ of right modules over a ring Λ and the category $\operatorname{QCoh} X$ of quasi-coherent sheaves on a scheme X are Grothendieck categories (see [Con20, Lem 2.1.7]).

For a commutative ring R, we often consider the poset $(\operatorname{Spec} R, \subset)$. Similarly, for a Grothendieck category $\mathcal{A}$, we construct a poset $(\operatorname{ASpec}\mathcal{A}, \leq)$.

Theorem 1.1 (Hochster [Hoc69, Proposition 10] and Speed [Spe72, Corollary 1])**.** *Let P be a partially ordered set. Then P is isomorphic to the prime spectrum of some commutative ring with the inclusion relation if and only if P is an inverse limit of finite partially ordered sets in the category of partially ordered sets.*

Theorem 1.2 ([Kan13, Theorem 7.27])**.** *Any partially ordered set is isomorphic to the atom spectrum of some Grothendieck category as a partially ordered set.*

2. Atom spectrum

Definition 2.1. An object H in $\mathcal{A}$ is called *monoform* if for any nonzero subobject L of H, there exists no common nonzero subobject of H and H/L, that is, there does not exist a nonzero subobject of H which is isomorphic to a subobject of H/L.

Definition 2.2. We say that monoform objects H_1 and H_2 in $\mathcal{A}$ are *atom-equivalent* if there exists a common nonzero subobject of H_1 and H_2.

Definition 2.3. Denote by $\operatorname{ASpec}\mathcal{A}$ the quotient class of the class of monoform objects by the atom equivalence, and call it the *atom spectrum* of $\mathcal{A}$. We call an element of $\operatorname{ASpec}\mathcal{A}$ an *atom* in $\mathcal{A}$. The equivalence class of a monoform object H in $\mathcal{A}$ is denoted by $\overline{H}$.

Definition 2.4. Let M be an object in $\mathcal{A}$. Define a subset $\operatorname{ASupp} M$ of $\operatorname{ASpec}\mathcal{A}$ by

$$\operatorname{ASupp} M = \{\alpha \in \operatorname{ASpec}\mathcal{A} \mid \alpha = \overline{H} \text{ for a monoform subquotient } H \text{ of } M\},$$

and call it the *atom support* of M.

Proposition 2.5. *Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\mathcal{A}$. Then we have* $\operatorname{ASupp} M = \operatorname{ASupp} L \cup \operatorname{ASupp} N$.

Definition 2.6. Let $\alpha, \beta \in \operatorname{ASpec}\mathcal{A}$. We denote by $\alpha \leq \beta$ that any object $M \in \mathcal{A}$ satisfying $\alpha \in \operatorname{ASupp} M$ also satisfies $\beta \in \operatorname{ASupp} M$.

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Proposition 2.7. *Let R be a commutative ring. Then the map* $\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$ *defined by* $\mathfrak{p} \mapsto \overline{R/\mathfrak{p}}$ *gives a poset isomorphism between* $(\operatorname{Spec} R, \subset)$ *and* $(\operatorname{ASpec}(\operatorname{Mod} R), \leq)$.

Proposition 2.8. *Let Λ be a right artinian ring. Then there exists a bijection between the set of isomorphism classes of simple Λ-modules and* $\operatorname{ASpec}(\operatorname{Mod} \Lambda)$. *The correspondence is given by* $S \mapsto \overline{S}$.

3. Construction of Grothendieck categories

Theorem 1.2 is proven by the following steps.

(1) From a poset P, construct a *colored quiver* Γ (technically).
(2) From a colored quiver, associate a Grothendieck category $\mathcal{A}_\Gamma$.
(3) Take a quotient category $\mathcal{A} = \mathcal{A}_\Gamma / \mathcal{X}$ by a localizing subcategory $\mathcal{X}$ (if necessary).

In this talk, we mainly deal with the second step. We see key ideas in the first step by treating examples.

Definition 3.1. A sextuple $\Gamma = (Q_0, Q_1, C, s, t, u)$ is called a colored quiver if it satisfies the following conditions.

(1) (Q_0, Q_1, s, t) is a quiver.
(2) C is a set.
(3) $u\colon Q_1 \to C$ is a map.
(4) For any $v \in Q_0$ and $c \in C$, there are only finitely many $r \in Q_1$ satisfying $s(r) = v$ and $u(r) = c$.

Example 3.2. The diagram

$$\begin{tikzcd} v \arrow[r, "1"] & w \arrow[loop right, "2"] \end{tikzcd}$$

represents a colored quiver.

We fix a field K.

Definition 3.3. Let $\Gamma = (Q_0, Q_1, C, s, t, u)$ be a colored quiver. Denote the free K-algebra on C by $F_C = K \langle f_c \mid c \in C \rangle$. Define a K-vector space M_Γ by $M_\Gamma = \bigoplus_{v \in Q_0} x_v K$, where $x_v K$ is a one-dimensional K-vector space generated by an element x_v. Regard M_Γ as a right F_C-module by defining the action of $f_c \in F_C$ as follows: for each vertex v in Q,

$$x_v \cdot f_c = \sum_{\substack{r \in Q_1 \\ s(r)=v \\ u(r)=c}} x_{t(r)},$$

Denote by $\mathcal{A}_\Gamma$ the smallest full subcategory of $\operatorname{Mod} F_C$ closed under subobject, quotient object, and arbitrary direct sums, and containing M_Γ.

Lemma 3.4. *The poset* $\operatorname{ASpec} \mathcal{A}_\Gamma$ *is isomorphic to the subset* $\operatorname{ASupp} M_\Gamma$ *of* $\operatorname{ASpec}(\operatorname{Mod} F_C)$.

Example 3.5. Let Γ be the colored quiver

$$\begin{tikzcd} v \arrow[d, "1"] \\ w \end{tikzcd} .$$

Then we have $F_C = K \langle f_1 \rangle = K[f_1]$ and $M_\Gamma = x_v K \oplus x_w K$, where $x_v f_1 = x_w$ and $x_w f_1 = 0$. M_Γ has a F_C-submodule S of the form $0 \oplus x_w K$. Then we have the exact sequence

$$0 \to S \to M_\Gamma \to S \to 0.$$

Hence we have

$$\operatorname{ASpec} \mathcal{A}_\Gamma = \operatorname{ASupp} M_\Gamma = \operatorname{ASupp} S = \{\overline{S}\}.$$

Example 3.6. Let Γ be the colored quiver

$$\begin{tikzcd} \bullet \arrow[d, "1"'] \arrow[loop right, "2"] \\ \bullet \arrow[loop right, "3"] \end{tikzcd}.$$

Then $\operatorname{ASpec}\mathcal{A}_\Gamma$ consists of two elements, and every element is maximal.

Example 3.7. Let Γ be the colored quiver

$$\bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \xrightarrow{3} \cdots.$$

Then $\operatorname{ASpec}\mathcal{A}_\Gamma = \{\overline{M_\Gamma}, \overline{K}\}$, and we have $\overline{M_\Gamma} < \overline{K}$.

We introduce a notation of *bold arrow*. The precise definition is given in [Kan13, Notation 7.20]. It is explained by using an example here.

Example 3.8. Let Γ be the colored quiver

$$\begin{tikzcd} v \arrow[d, "1"] \\ w \end{tikzcd}.$$

Then the diagram

$$\Gamma \Longrightarrow \Gamma \Longrightarrow \Gamma$$

represents the colored quiver

$$\begin{tikzcd} v_a \arrow[r, "2"] \arrow[d, "1"'] \arrow[dr, "3"] & v_b \arrow[r, "6"] \arrow[d, "1"] \arrow[dr, "7"] & v_c \arrow[d, "1"] \\ w_a \arrow[r, "5"'] \arrow[ur, "4"] & w_b \arrow[r, "9"'] \arrow[ur, "8"] & w_c \end{tikzcd}.$$

Lemma 3.9. *Let Γ be a colored quiver. Let $\widetilde{\Gamma}$ a colored quiver*

$$\Gamma \Longrightarrow \Gamma \Longrightarrow \Gamma \Longrightarrow \cdots.$$

Then $\operatorname{ASpec}\mathcal{A}_{\widetilde{\Gamma}} = \operatorname{ASpec}\mathcal{A}_\Gamma \cup \{\overline{M_\Gamma}\}$, where $\overline{M_\Gamma}$ is smaller than any element in $\operatorname{ASpec}\mathcal{A}_\Gamma$.

References

[Con20] B. Conrad, Grothendieck duality and base change, Lecture Notes in Mathematics, 1750, *Springer-Verlag, Berlin*, 2000, vi+296 pp.

[Hoc69] M. Hochster, Prime ideal structure in commutative rings, *Trans. Amer. Math. Soc.* **142** (1969), 43–60.

[Kan13] R. Kanda, Specialization orders on atom spectra of Grothendieck categories, arXiv:1308.3928v2, written in 2013, 39 pp.

[Spe72] T. P. Speed, On the order of prime ideals, *Algebra Universalis* **2** (1972), 85–87.

Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya-shi, Aichi-ken, 464-8602, Japan

E-mail address: `kanda.ryo@a.mbox.nagoya-u.ac.jp`