Cyclic homology of truncated quiver algebras and notes on the no loops conjecture for Hochschild homology

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Section 1

Introduction

Notation

- K: a commutative ring
- $\mathbb{N}$: the set of all natural numbers containing 0
- Δ: a finite quiver
- Δ_n: the set of all paths of length $n (n \geq 0)$ in Δ
- R_Δ: the arrow ideal of $K\Delta$
- G_n: a cyclic group of order n generated by t_n
- $A^{\otimes n} := A \otimes_K A \otimes_K \cdots \otimes_K A$ for a K-algebra A (the n-fold tensor product of A)
- $A^e := A \otimes_K A^{\mathrm{op}}$: the enveloping algebra of A

Hochschild homology groups

Definition

For K-algebra A, *the Hochschild complex* **is the following complex:**

$$C(A) : \cdots \xrightarrow{b} A^{\otimes n+1} \xrightarrow{b} A^{\otimes n} \xrightarrow{b} \cdots \xrightarrow{b} A^{\otimes 2} \xrightarrow{b} A \longrightarrow 0,$$

where the K-homomorphisms $b : A^{\otimes n+1} \to A^{\otimes n}$ is given by

$$b(x_0 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i (x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n) + (-1)^n (x_n x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1}).$$

$HH_n(A) := H_n(C(A))$: *the n-th Hochschild homology group of A* **($n \geq 0$).**

Hochschild homology groups

If a unital K-algebra A is projective as a module over K, then there is an isomorphism

$$HH_n(A) \cong \mathrm{Tor}_n^{A^e}(A, A).$$

For a unital K-algebra A, the bar resolution $C^{\mathrm{bar}}(A)$ of A is given by

$$C^{\mathrm{bar}}(A) : \cdots \xrightarrow{b'} A^{\otimes n+1} \xrightarrow{b'} A^{\otimes n} \xrightarrow{b'} \cdots \xrightarrow{b'} A^{\otimes 2} \xrightarrow{b'} A \longrightarrow 0,$$

where the differentials b' are given by

$$b'(x_0 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i (x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n).$$

Cyclic group action

Let n be a positive integer. The action of G_n on the module $A^{\otimes n}$ is given by

$$t_n \cdot (x_1 \otimes x_2, \otimes \cdots \otimes x_n) := (-1)^{n-1}(x_n \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1}),$$

where $x_i \in A$.
Let $D := 1 - t_n$ and $N := 1 + t_n + t_n^2 + \cdots + t_n^{n-1}$ as elements of the group ring $\mathbb{Z}G_n$. Then

$$D : A^{\otimes n} \longrightarrow A^{\otimes n}$$
$$\text{and} \quad N : A^{\otimes n} \longrightarrow A^{\otimes n}$$

are K-homomorphisms.

Cyclic homology groups

Definition

For a K-algebra A, the following is a first quadrant bicomplex $CC(A)$ of A:

$$
\begin{array}{ccccccccc}
 & \downarrow b & & \downarrow -b' & & \downarrow b & \\
CC(A)_{*,2}: & A^{\otimes 3} & \xleftarrow{D} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{D} \\
 & \downarrow b & & \downarrow -b' & & \downarrow b & \\
CC(A)_{*,1}: & A^{\otimes 2} & \xleftarrow{D} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{D} \\
 & \downarrow b & & \downarrow -b' & & \downarrow b & \\
CC(A)_{*,0}: & A & \xleftarrow{D} & A & \xleftarrow{N} & A & \xleftarrow{D} \\
 & CC(A)_{0,*} & & CC(A)_{1,*} & & CC(A)_{2,*} &
\end{array}
$$

$HC_n(A) := H_n(\mathrm{Tot}(CC(A)))$: *the n-th cyclic homology group of $A (n \geq 0)$*

Cyclic homology is developed as a non-commutative variant of the de Rham cohomology by Connes (Loday, 1992). Aside from this, cyclic homology is developed by Tsygan, Loday and Quillen. Cyclic homology is closely related to Hochschild homology, because of the definition itself and also the periodicity exact sequence by Connes:

$$\cdots \to HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} \cdots .$$

It is well known that cyclic homology is Morita invariant. Moreover, the following theorem is shown by König, Liu, Zhou.

Theorem(König, Liu and Zhou, 2011)

Let R and T be two finite dimensional algebras over an algebraically closed field which are stably equivalent of Morita type. Then,

(1) For $n > 0$ odd, $\dim HC_n(R) = \dim HC_n(T)$.

(2) Suppose that R and T have no semisimple direct summands. Then for any $n \geq 0$ even, the following statements are equivalent:

  (i) the Auslander-Reiten conjecture holds for this stable equivalence of Morita type;

  (ii) $\dim HC_n(R) = \dim HC_n(T)$.

The cyclic homology of algebras is investigated for classes of various algebras, for example,

- 2-nilpotent algebras (Cibils, 1990)
- truncated quiver algebras over a field of characteristic zero (Taillefer, 2001)
- quadratic monomial algebras (Sköldberg, 2001)
- monomial algebras over a field of characteristic zero (Han, 2006)

etc.
We want to know the influence of a ground ring of an algebra on its cyclic homology. In particular, we are going to focus on the module structure of cyclic homology.

Aim

Aim

(1) **Find the dimension formula of the cyclic homology of truncated quiver algebras $K\Delta/R_{\Delta}^{m}$ over a field K of positive characteristic.**

(2) **Show the m-truncated cycles version of the no loops conjecture for a class of algebras over an algebraically closed field.**

Section 2

Hochschild homology and cyclic homology of truncated quiver algebras

Truncated quiver algebra $K\Delta/R_\Delta^m$

Following the notation by Sköldberg(1999), we recall characteristics of truncated quiver algebras. By adjoining an element $\perp$, we will consider the following set:

$$\hat{\Delta} = \{\perp\} \cup \bigcup_{i=0}^{\infty} \Delta_i.$$

This set is a semigroup with the multiplication defined by

$$\delta \cdot \gamma = \begin{cases} \delta\gamma & \text{if} \quad t(\delta) = s(\gamma), \\ \perp & \text{otherwise}, \end{cases} \qquad \delta, \gamma \in \bigcup_{i=0}^{\infty} \Delta_i,$$

and

$$\perp \cdot \gamma = \gamma \cdot \perp = \perp, \quad \gamma \in \hat{\Delta}.$$

$K\hat{\Delta}$ is a semigroup algebra, and the path algebra $K\Delta$ is isomorphic to $K\hat{\Delta}/(\perp)$.

- **$K\Delta$ is $\hat{\Delta}$-graded, that is, $K\Delta = \bigoplus_{\gamma\in\hat{\Delta}}(K\Delta)_\gamma$, where $(K\Delta)_\gamma = K\gamma$ for $\gamma \in \bigcup_{i=0}^{\infty}\Delta_i$ and $(K\Delta)_\perp = 0$.**
- **$K\Delta$ is $\mathbb{N}$-graded, that is, $K\Delta = \bigoplus_{i=0}^{\infty} K\Delta_i$.**
- **R_Δ^m is $\hat{\Delta}$-graded and $\mathbb{N}$-graded.**

Thus the truncated quiver algebra $K\Delta/R_\Delta^m$ is a $\hat{\Delta}$-graded and an $\mathbb{N}$-graded algebra.

Theorem(Sköldberg, 1999)

The following is a projective $\hat{\Delta}$-graded resolution of a truncated quiver algebra $A = K\Delta/R_{\Delta}^{m}$ as a left A^e-module:

$$P_A : \cdots \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0.$$

Here each terms are $\hat{\Delta}$-graded modules defined by

$$P_i = A \otimes_{K\Delta_0} K\Gamma^{(i)} \otimes_{K\Delta_0} A \quad \text{for } i \geq 0,$$
$$P_0 = A \otimes_{K\Delta_0} K\Delta_0 \otimes_{K\Delta_0} A \cong A \otimes_{K\Delta_0} A,$$

where $\Gamma^{(i)}$ is given by

$$\Gamma^{(i)} = \begin{cases} \Delta_{cm} & \text{if } i = 2c \ (c \geq 0), \\ \Delta_{cm+1} & \text{if } i = 2c+1 \ (c \geq 0), \end{cases}$$

Theorem(Sköldberg, 1999)

and the differentials are defined by

$$d_{2c}(\alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \beta) = \sum_{j=0}^{m-1} \alpha\alpha_1 \cdots \alpha_j \otimes \alpha_{1+j} \cdots \alpha_{(c-1)m+1+j} \otimes \alpha_{(c-1)m+2+j} \cdots \alpha_{cm}\beta,$$

$$d_{2c+1}(\alpha \otimes \alpha_1 \cdots \alpha_{cm+1} \otimes \beta) = \alpha\alpha_1 \otimes \alpha_2 \cdots \alpha_{cm+1} \otimes \beta - \alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \alpha_{cm+1}\beta,$$

where $\alpha_i \in \Delta_1$ for $i \geq 1$ and $\alpha, \beta \in A$ and ε is the multiplication.

Cycle and Basic cycle

A path is called a *cycle* if its source coincides with its target. A cycle γ is called a *basic cycle* if there does not exist a cycle δ such that $\gamma = \delta^j$ for some positive integer $j(\geq 2)$.

- **Δ_n^{c}: the set of all cycles of length n.**
- **Δ_n^{b}: the set of all basic cycles of length n.**

Cycle and Basic cycle

Let a path $\alpha_1 \cdots \alpha_{n-1}\alpha_n$ be a cycle, where α_i is an arrow in Δ. Then the action of G_n on Δ_n^{c} is given by

$$t_n \cdot (\alpha_1 \cdots \alpha_{n-1}\alpha_n) := \alpha_n \alpha_1 \cdots \alpha_{n-1}.$$

Similarly, G_n acts on Δ_n^{b}.

- **a_n: the cardinal number of the set of all G_n-orbits on Δ_n^{c}.**
- **b_n: the cardinal number of the set of all G_n-orbits on Δ_n^{b}.**

Hcohschild homology of truncated quiver algebras

L denotes the complex $A \otimes_{K\Delta_0^e} K\Gamma^{(*)}$, which is isomorphic to $A \otimes_{A^e} P_A$ by the following isomorphism φ:

$$\varphi : A \otimes_{A^e} P_A \xrightarrow{\sim} A \otimes_{A^e} A^e \otimes_{K\Delta_0^e} K\Gamma^{(*)} \xrightarrow{\sim} A \otimes_{K\Delta_0^e} K\Gamma^{(*)}.$$

Since the complex L is decomposed into the subcomplexes $L_{\overline{\gamma}}$

$$L \cong \bigoplus_{q=0}^{\infty} \bigoplus_{\overline{\gamma} \in \Delta_q^c / G_q} L_{\overline{\gamma}},$$

the p-th Hochschild homology group $HH_p(A)$ is $\mathbb{N}$-graded, so we have

$$HH_p(A) = \bigoplus_{q=0}^{\infty} HH_{p,q}(A) \quad \text{for } p \geq 0.$$

Hochschild homology of truncated quiver algebras

Theorem (Sköldberg, 1999)

Let K be a commutative ring, $A = K\Delta/R_{\Delta}^{m}$ and $q = cm + e$ for $0 \leq e \leq m-1$. Then the degree q part of the p-th Hochschild homology $HH_{p,q}(A)$ is the following as a K-module.

$$
\begin{cases}
K^{a_q} & \text{if } 1 \leq e \leq m-1 \text{ and } 2c \leq p \leq 2c+1, \\
\bigoplus_{r|q} \left(K^{\gcd(m,r)-1} \oplus \mathrm{Ker}\left(\cdot \frac{m}{\gcd(m,r)} : K \longrightarrow K \right) \right)^{b_r} & \text{if } e = 0 \text{ and } 0 < 2c-1 = p, \\
\bigoplus_{r|q} \left(K^{\gcd(m,r)-1} \oplus \mathrm{Coker}\left(\cdot \frac{m}{\gcd(m,r)} : K \longrightarrow K \right) \right)^{b_r} & \text{if } e = 0 \text{ and } 0 < 2c = p, \\
K^{\#\Delta_0} & \text{if } p = q = 0, \\
0 & \text{otherwise.}
\end{cases}
$$

Cyclic homology of truncated quiver algebras

Theorem (Taillefer, 2001)

Let K be a field of charactaristic zero. The cyclic homology group of a truncated quiver algebra A is given by

$$\dim_K HC_{2c}(A) = \#\Delta_0 + \sum_{e=1}^{m-1} a_{cm+e},$$

$$\dim_K HC_{2c+1}(A) = \sum_{\substack{r>0 \\ \text{s.t. } r|(c+1)m}} (\gcd(m,r)-1)b_r.$$

Section 3

Main result

Main result

Theorem (Itagaki, Sanada)

Let K be a field of characteristic ζ and $A = K\Delta/R_{\Delta}^{m}$. Then the dimension formula of the cyclic homology of A is given by, for $c \geq 0$,

$$\dim_K HC_{2c}(A) = \#\Delta_0 + \sum_{e=1}^{m-1} a_{cm+e} + \sum_{c'=0}^{c-1} \sum_{e=1}^{m-1} \sum_{\substack{r>0 \\ \text{s.t. } r\zeta \mid c'm+e}} b_r + \sum_{c'=1}^{c} \sum_{\substack{r>0 \\ \text{s.t. } r \mid c'm, \\ \gcd(m,r)\zeta \mid m}} b_r + \sum_{c'=1}^{c} \sum_{\substack{r>0 \\ \text{s.t. } r\zeta \mid \gcd(m,r)c'}} (\gcd(m,r)-1) b_r,$$

Main result

Theorem(Itagaki, Sanada)

$$\dim_K HC_{2c+1}(A) = \sum_{\substack{r>0 \\ \text{s.t. } r|(c+1)m}} (\gcd(m,r)-1)b_r + \sum_{c'=0}^{c}\sum_{e=1}^{m-1} \sum_{\substack{r>0 \\ \text{s.t. } r\zeta|c'm+e}} b_r + \sum_{c'=1}^{c+1} \sum_{\substack{r>0 \\ \text{s.t. } r|c'm, \\ \gcd(m,r)\zeta|m}} b_r + \sum_{c'=1}^{c} \sum_{\substack{r>0 \\ \text{s.t. } r\zeta|\gcd(m,r)c'}} (\gcd(m,r)-1)b_r.$$

Cibils' projective resolution

Lemma (Cibils, 1989)

Let I be an admissible ideal of $K\Delta$. E denotes the subalgebra of $A = K\Delta/I$ generated by Δ_0. The following is a projective resolution of A as left A^e-module:

$$Q_A : \cdots \to Q_n \xrightarrow{d_n} Q_{n-1} \to \cdots \to Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} A \to 0$$

where $Q_n = A \otimes_E (\mathrm{rad}\, A)^{\otimes_E^n} \otimes_E A$ and differntials are given by

$$d_n(x_0 \otimes \cdots \otimes x_{n+1}) = \sum_{i=0}^{n} (-1)^i (x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1})$$

for $x_0, x_{n+1} \in A$, $x_1, \ldots, x_n \in \mathrm{rad}\, A$.

Cibils' mixed complex

Let K be a field. For a K-algebra A with a decomposition $A = E \oplus r$, where E is a separable subalgebra and r is a two-sided ideal, Cibils(1990) gives the following mixed complex

$(A \otimes_{E^e} r^{\otimes n}_E, b, B)$:

$b : A \otimes_{E^e} r^{\otimes n+1}_E \to A \otimes_{E^e} r^{\otimes n}_E$ is the Hochschild boundary and

$B : A \otimes_{E^e} r^{\otimes n}_E \to A \otimes_{E^e} r^{\otimes n+1}_E$ is given by the formula

$$\begin{aligned} & B(x_0 \otimes_{E^e} x_1 \otimes \cdots \otimes x_n) \\ & \quad = \sum_{i=0}^{n} (-1)^{in} (1 \otimes_{E^e} x_i \otimes \cdots \otimes (x_0)_r \otimes \cdots \otimes x_{i-1}), \end{aligned}$$

where $x_0 = (x_0)_E + (x_0)_r \in E \oplus r$.

The cyclic homology groups of the mixed complex and the cyclic homology groups of A are isomorphic.

The cyclic homology groups of Cibils' mixed complex is isomorphic to the total homology groups of the following first quadrant bicomplex:

$$
\begin{array}{ccccccc}
\downarrow b & & \downarrow b & & \downarrow b & & \\
A \otimes_{E^e} r_E^{\otimes 2} & \xleftarrow{B} & A \otimes_{E^e} r & \xleftarrow{B} & A & \longleftarrow & \\
\downarrow b & & \downarrow b & & \downarrow & & \\
A \otimes_{E^e} r & \xleftarrow{B} & A & \longleftarrow & 0 & \longleftarrow & \\
\downarrow b & & \downarrow & & \downarrow & & \\
A & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow &
\end{array}
$$

Denote the above bicomplex by $\overline{C}_E(A)$. For a truncated quiver algebra $A = K\Delta/R_\Delta^m$, there exists the decomposition $K\Delta/R_\Delta^m = K\Delta_0 \oplus \operatorname{rad} A$.

We consider the spectral sequence associated with the filtration $F\mathrm{Tot}(\overline{C}_E(A))$ given by

$$(F^p\mathrm{Tot}(\overline{C}_E(A)))_n = \bigoplus_{i\le p} \overline{C}_E(A)_{i,n-i}.$$

The cyclic homology groups $HC_n(A)$ is given by

$$HC_n(A) \cong H_n(\mathrm{Tot}(\overline{C}_E(A))) \cong \bigoplus_{p+q=n} E^2_{p,q}.$$

E^1-page of the spectral sequence is drawn by the following illustration:

$$\begin{array}{ccccccc}
HH_2(A) & \xleftarrow{B} & HH_1(A) & \xleftarrow{B} & HH_0(A) & \xleftarrow{0} \\
HH_1(A) & \xleftarrow{B} & HH_0(A) & \xleftarrow{0} & 0 & \xleftarrow{0} \\
HH_0(A) & \xleftarrow{0} & 0 & \xleftarrow{0} & 0 & \xleftarrow{0}
\end{array}$$

$B : HH_n(A) \to HH_{n+1}(A)$

Ames, Cagliero and Tirao (2009) give chain maps $\iota : P_A \to Q_A$, and $\pi : Q_A \to P_A$ which induce isomorphism $HH_n(A) \cong HH_n(A)$ for $n \geq 0$. By means of these chain maps, we consider $B : HH_n(A) \to HH_{n+1}(A)$ as follows:

Sköldberg:

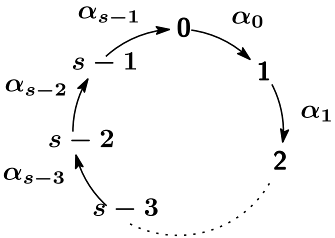
$$\begin{array}{ccc}
A \otimes_{K\Delta_0^e} K\Gamma^{(n)} & & A \otimes_{K\Delta_0^e} K\Gamma^{(n+1)} \\
\downarrow \wr & & \uparrow \wr \\
A \otimes_{A^e} P_n & & A \otimes_{A^e} P_{n+1} \\
\downarrow \mathrm{id}_A \otimes \iota & & \uparrow \mathrm{id}_A \otimes \pi \\
A \otimes_{A^e} Q_n & & A \otimes_{A^e} Q_{n+1} \\
\downarrow \wr & & \uparrow \wr \\
A \otimes_{K\Delta_0^e} \mathrm{rad}\, A^{\otimes^n_{K\Delta_0}} & \xrightarrow{B} & A \otimes_{K\Delta_0^e} \mathrm{rad}\, A^{\otimes^{n+1}_{K\Delta_0}}
\end{array}$$

Cibils:

Section 4

Example

Let K be a field of characteristic ζ. We consider the cyclic homoology of a truncated quiver algebra $A = K\Delta/R_{\Delta}^{m}$, where Δ is the following quiver:



The truncated quiver algebra A is called a truncated cycle algebra. Note that a_r and b_r is given by

$$a_r = \begin{cases} 1 & \text{if} \quad s|r, \\ 0 & \text{otherwise}, \end{cases} \qquad b_r = \begin{cases} 1 & \text{if} \quad s = r, \\ 0 & \text{otherwise}. \end{cases}$$

For $x \in \mathbb{R}$, we denote the largest integer i satisfying $i \leq x$ by $[x]$. The dimension formula of the cyclic homology of A is given by

$$\begin{aligned}
&\dim_K HC_{2c}(A) \\
&= s + \left[\frac{(c+1)m-1}{s}\right] - \left[\frac{cm}{s}\right] + \sum_{c'=0}^{c-1}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right] - \left[\frac{c'm}{s\zeta}\right]\right) \\
&\quad + \left(\left[\frac{m}{\gcd(m,s)\zeta}\right] - \left[\frac{m-1}{\gcd(m,s)\zeta}\right]\right)\sum_{c'=1}^{c}\left(\left[\frac{c'm}{s}\right] - \left[\frac{c'm-1}{s}\right]\right) \\
&\quad + (\gcd(m,s)-1)\left[\frac{\gcd(m,s)c}{s\zeta}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\dim_K HC_{2c+1}(A) \\
&= (\gcd(m,s)-1)\left(\left[\frac{(c+1)m}{s}\right]-\left[\frac{(c+1)m-1}{s}\right]+\left[\frac{\gcd(m,s)c}{s\zeta}\right]\right) \\
&\quad+\left(\left[\frac{m}{\gcd(m,s)\zeta}\right]-\left[\frac{m-1}{\gcd(m,s)\zeta}\right]\right)\sum_{c'=1}^{c+1}\left(\left[\frac{c'm}{s}\right]-\left[\frac{c'm-1}{s}\right]\right) \\
&\quad+\sum_{c'=0}^{c}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right]-\left[\frac{c'm}{s\zeta}\right]\right).
\end{aligned}$$

When $\zeta = 0$, the dimension of the cyclic homology of A is as follows:

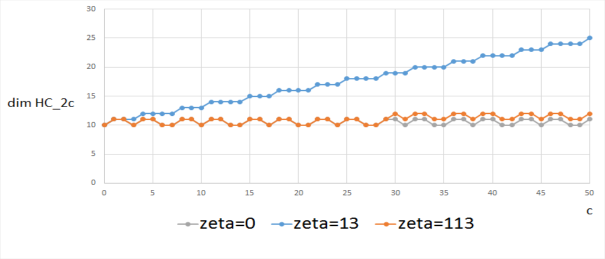
$$\dim_K HC_{2c}(A) = s + \left[\frac{(c+1)m-1}{s}\right] - \left[\frac{cm}{s}\right],$$

$$\dim_K HC_{2c+1}(A)$$

$$= (\gcd(m,s)-1)\left(\left[\frac{(c+1)m}{s}\right] - \left[\frac{(c+1)m-1}{s}\right]\right).$$

A change of $\dim_K HC_{2c}(A)$ by ζ

We fix $s = 7$, $m = 26$. The graphs of $\dim_K HC_{2c}(A)$ of the three cases that $\zeta = 0$, $\zeta = 13$ and $\zeta = 113$ are as follows:



Section 5

Application of the chain map and truncated quiver algebras

The no loops conjecture is as follows:

The no loops conjecture

If A is a finite dimensional algebra of finite global dimension then $\mathrm{Ext}^1_A(S,S)=0$ for all simple A-modules S, or equivalently, A has no loops in its quiver.

It is shown that this conjecture holds for a large class of the finite dimensional algebra including all those which the ground field is an algebraically closed field by Igusa (1990), and this is derived from an earlier result of Lenzing (1969). Bergh, Han and Madsen show the 2-truncated cycles version of this conjecture.

The $\mathbf{2}$-truncated cycles version of the no loops conjecture

If A is a finite dimensional algebra of finite global dimension then A has no 2-truncated cycles in its quiver

Definition (m-truncated cycle)

Let Δ be a quiver with a finite number of vertices and $I(\subset R_\Delta^2)$ an ideal in the path algebra $K\Delta$. A finite sequence $\alpha_1, \dots, \alpha_u$ of arrows in $K\Delta/I$ which satisfies the equations $t(\alpha_i) = s(\alpha_{i+1})(i = 1, \dots, u-1)$ and $t(\alpha_u) = s(\alpha_1)$ is *an m-truncated cycle* **for an integer $m \geq 2$ if**

$$\alpha_i \cdots \alpha_{i+m-1} = 0 \quad \text{and} \quad \alpha_i \cdots \alpha_{i+m-2} \neq 0 \quad \text{in } K\Delta/I$$

for all i, where the indices are modulo u.

Definition (Hochschild homology dimension)

The Hochschild homology dimension **of an algebra A, denoted by $\mathrm{HHdim}\, A$, is defined by**

$$\mathrm{HHdim}\, A = \sup\{n \geq 0 | HH_n(A) \neq 0\}.$$

Theorem (Bergh, Han, Madsen, 2012)

Let K be a commutative ring, Δ a quiver with a finite number of vertices and I an ideal contained in R_Δ^2. Suppose $K\Delta/I$ contains a 2-truncated cycle $x_1, \ldots, x_u$. Then for every $n \geq 1$ with $un \equiv u \pmod 2$, the element

$$(x_1 \otimes \cdots \otimes x_u)^{\otimes n}$$

represents a nonzero element in $HH_{un-1}(K\Delta/I)$. In particular, $\operatorname{HHdim} K\Delta/I = \infty$.

By the above theorem, they show the 2-truncated cycles version of the no loops conjecture.

Corollary (Bergh, Han, Madsen, 2012)

Let K be a field, Δ a finite quiver and I an admissible ideal in $K\Delta$. If the algebra $K\Delta/I$ has finite global dimension, then it contains no 2-truncated cycles.

Moreover, they propose the m-truncated cycles version of the no loops conjecture.

The m-truncated cycles version of the no loops conjecture

If A is a finite dimensional algebra of finite global dimension then A has no m-truncated cycles in its quiver

They show that the above conjecture holds for monoial algebaras. We show that the conjecture holds for other algebras by the similar way as the proof of the 2-truncated cycles version.

Theorem (Itagaki, Sanada)

Let K be a field, Δ a finite quiver and I an ideal contained in R^m_Δ. Suppose that $K\Delta/I$ contains an m-truncated cycle $\alpha_1, \dots, \alpha_u$. Then for every $n \geq 1$ with $un \equiv 0 \pmod m$, the element

$$\begin{aligned}\alpha_{(c-1)m+2} \cdots \alpha_{cm} \otimes \alpha_1 \otimes \alpha_2 \cdots \alpha_m \\ \otimes \alpha_{m+1} \otimes \alpha_{m+2} \cdots \alpha_{2m} \otimes \alpha_{2m+1} \otimes \cdots \\ \otimes \alpha_{(c-2)m+2} \cdots \alpha_{(c-1)m} \otimes \alpha_{(c-1)m+1}\end{aligned}$$

represents a nonzero element in $HH_{2c-1}(K\Delta/I)$, where $c = un/m$. In particular, the Hochschild homology dimension $\mathrm{HHdim}\,(K\Delta/I) = \infty$.

For an algebra $A = K\Delta/I$ with $I \subset R_\Delta^m$, by the surjective map $A \to A' = K\Delta/R_\Delta^m$ and the chain map π which induce the map $\overline{\pi} : H_n(A \otimes_{A'^e} Q_{A'}) \to H_n(A' \otimes_{A'^e} P_{A'})$, we have the composition map as follows:

$$\begin{aligned} H_n(C(A)) &\to H_n(C(A')) \\ &\xrightarrow{\sim} H_n(A' \otimes_{K\Delta_0^e} (\mathrm{rad}\, A')^{\otimes^*_{K\Delta_0}}) \\ &\xrightarrow{\sim} H_n(A' \otimes_{A'^e} Q_{A'}) \\ &\xrightarrow{\overline{\pi}} H_n(A \otimes_{A'^e} P_{A'}) \\ &\xrightarrow{\sim} H_n(L) = HH_n(A'). \end{aligned}$$

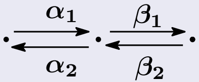
We already find the basis of $H_n(L)$ in order to compute the dimension formula of the cyclic homology of truncated quiver algebras. Therefore, it is enough to find an element of $H_n(C(A))$ whose image of the above composition map coincides with a member of the basis of $H_n(L)$.

Corollary (Itagaki, Sanada)

Let K be an algebraically closed field, Δ a finite quiver and I an admissible ideal in $K\Delta$ with $I \subset R_{\Delta}^{m}$. If the algebra $K\Delta/I$ has finite global dimension, then it contains no m-truncated cycles.

Example

Let A be an algebra given by the quiver:



with relations:

$$(\alpha_i\alpha_{i+1})^3 = (\beta_i\beta_{i+1})^2 = 0,\ \alpha_1\beta_1\beta_2\alpha_2 = (\alpha_1\alpha_2)^2,$$

where the indices are modulo 2 and i is a positive integer modulo 2. Then, A has the 4-truncated cycle

$$\beta_1, \beta_2.$$

Therefore the global dimension of A is infinite.

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