Selfinjective algebras of finite representation type with maximal almost split sequences

Marta Błaszkiewicz

joint with Andrzej Skowroński

Nagoya, November 2013

Abstract

Main aim

We investigate the structure of finite dimensional selfinjective algebras of finite representation type over an arbitrary field having almost split sequences of modules whose middle term has the maximal possible number of indecomposable direct summands.

Notation

- algebra - a basic indecomposable finite dimensional associative K-algebra with an identity over a (fixed) field K

A an algebra

- $\mathrm{mod}\, A$ the category of finite dimensional (over K) right A-modules
- $D = \mathrm{Hom}_K(-, K)\colon\ \mathrm{mod}\, A \to \mathrm{mod}\, A^{\mathrm{op}}$ the standard duality on $\mathrm{mod}\, A$
- Γ_A the Auslander-Reiten quiver of A
- Γ_A^s the stable Auslander-Reiten quiver of A
- $\tau_A = D\mathrm{Tr}$, $\tau_A^{-1} = \mathrm{Tr}D$ the Auslander-Reiten translations

Notation

- A is ***selfinjective*** if $A \cong D(A)$ in $\operatorname{mod} A$ (projective modules = injective modules).
- A is of ***finite representation type*** if $\operatorname{mod} A$ admits only finitely many indecomposable modules up to isomorphism.
- A, A' selfinjective algebras
  A and A' are ***socle equivalent*** if the factor algebras $A/\operatorname{soc} A$ and $A'/\operatorname{soc} A'$ are isomorphic.

Notation

Dynkin quiver a valued quiver whose underlying graph is one of the Dynkin graphs

$$\mathbb{A}_n : \circ \frac{\qquad}{} \circ \frac{\qquad}{} \circ - \ldots - \circ \frac{\qquad}{} \circ \qquad (n \text{ vertices}), n \geqslant 1$$

$$\mathbb{B}_n : \circ \xrightarrow{(1,2)} \circ \frac{\qquad}{} \circ - \ldots - \circ \frac{\qquad}{} \circ \qquad (n \text{ vertices}), n \geqslant 2$$

$$\mathbb{C}_n : \circ \xrightarrow{(2,1)} \circ \frac{\qquad}{} \circ - \ldots - \circ \frac{\qquad}{} \circ \qquad (n \text{ vertices}), n \geqslant 3$$

$$\begin{tikzcd}[arrows=dash]
 & \circ \arrow[d] & & & & \\
\mathbb{D}_n : \circ \arrow[r] & \circ \arrow[r] & \circ - \ldots - \circ \arrow[r] & \circ & & (n \text{ vertices }), n \geqslant 4
\end{tikzcd}$$

Notation

$$\mathbb{E}_6 : \begin{tikzcd} & & \circ \arrow[d, dash] & & \\ \circ \arrow[r, dash] & \circ \arrow[r, dash] & \circ \arrow[r, dash] & \circ \arrow[r, dash] & \circ \end{tikzcd}$$

$$\mathbb{E}_7 : \begin{tikzcd} & & \circ \arrow[d, dash] & & & \\ \circ \arrow[r, dash] & \circ \arrow[r, dash] & \circ \arrow[r, dash] & \circ \arrow[r, dash] & \circ \arrow[r, dash] & \circ \end{tikzcd}$$

$$\mathbb{E}_8 : \begin{tikzcd} & & \circ \arrow[d, dash] & & & & \\ \circ \arrow[r, dash] & \circ \arrow[r, dash] & \circ \arrow[r, dash] & \circ \arrow[r, dash] & \circ \arrow[r, dash] & \circ \arrow[r, dash] & \circ \end{tikzcd}$$

$$\mathbb{F}_4 : \begin{tikzcd} \circ \arrow[r, dash] & \circ \arrow[r, dash, "(1{,}2)"] & \circ \arrow[r, dash] & \circ \end{tikzcd}$$

$$\mathbb{G}_2 : \begin{tikzcd} \circ \arrow[r, dash, "(1{,}3)"] & \circ \end{tikzcd}$$

Motivation

Theorem (Riedtmann, Waschbüsch)

Let A be a nonsimple selfinjective algebra over an algebraically closed field K. The following conditions are equivalent.

1. A is of finite representation type.
2. A is socle equivalent to an orbit algebra $\bar{A} = \widehat{B}/G$, where B is a tilted algebra of Dynkin type $\mathbb{A}_n (n \geqslant 1)$, $\mathbb{D}_n (n \geqslant 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$, and G is an admissible infinite cyclic group of automorphisms of $\widehat{B}$.

If K is of characteristic different from 2, then $A \cong \bar{A}$.

Theorem (Riedtmann, Todorov)

Theorem (Riedtmann, Todorov)

Let A be a nonsimple selfinjective algebra of finite representation type over an arbitrary field K. Then the stable Auslander-Reiten quiver Γ_A^s of A is isomorphic to the orbit quiver $\mathbb{Z}\Delta/G$, where Δ is a Dynkin quiver of type $\mathbb{A}_n (n \geqslant 1)$, $\mathbb{B}_n (n \geqslant 2)$, $\mathbb{C}_n (n \geqslant 3)$, $\mathbb{D}_n (n \geqslant 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$ or $\mathbb{G}_2$, and G is an admissible infinite cyclic group of automorphisms of the translation quiver $\mathbb{Z}\Delta$.

Maximal almost split sequences

A an algebra

X a nonprojective indecomposable module in $\operatorname{mod} A$

$0 \longrightarrow \tau_A X \longrightarrow Y \longrightarrow X \longrightarrow 0$ an almost split sequence in $\operatorname{mod} A$

$\tau_A X$ noninjective indecomposable module in $\operatorname{mod} A$

$$\alpha(X) = r$$

$Y = Y_1 \oplus \ldots \oplus Y_r, \qquad Y_1, \ldots, Y_r$ indecomposable modules in $\operatorname{mod} A$

Theorem (Bautista-Brenner)

- A an algebra of finite representation type $\Longrightarrow \alpha(X) \leqslant 4$
- $\alpha(X) = 4 \Longrightarrow Y_i$ projective-injective module in $\operatorname{mod} A$, for some $i \in \{1, \ldots, r\}$

Maximal almost split sequences

Definition

A an algebra of finite representation type, X a nonprojective indecomposable module in $\operatorname{mod} A$. An almost split sequence in $\operatorname{mod} A$

$$0 \longrightarrow \tau_A X \longrightarrow Y \longrightarrow X \longrightarrow 0$$

with $\alpha(X) = 4$ is said to be a ***maximal almost split sequence***.

A an algebra

P a projective-injective indecomposable module in $\operatorname{mod} A$

$$0 \longrightarrow \operatorname{rad} P \longrightarrow (\operatorname{rad} P/\operatorname{soc} P) \oplus P \longrightarrow P/\operatorname{soc} P \longrightarrow 0$$

an almost split sequence in $\operatorname{mod} A$

Maximal almost split sequences

Example

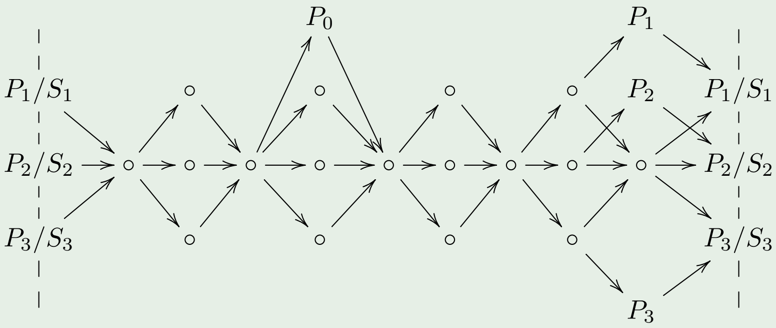
$Q = \overrightarrow{\mathbb{D}}_4$

$$\begin{tikzcd} 1 & \\ 2 & 0 \arrow[l] \arrow[ul] \arrow[dl] \\ 3 & \end{tikzcd}$$

Maximal almost split sequences

Example

$\Gamma_{T(KQ)}$



Tilted algebras of Dynkin type

H a hereditary algebra of Dynkin type (i.e. Q_H is a Dynkin quiver $\Leftrightarrow$ H is of finite representation type)
T a tilting module in $\operatorname{mod} H$

Then the endomorphism algebra $B = \operatorname{End}_H(T)$ is called a ***tilted algebra of Dynkin type.***

- B an algebra of finite representation type
- All modules $\operatorname{Hom}_H(T, I)$ in $\operatorname{mod} B$, where I is an indecomposable injective module in $\operatorname{mod} H$, form a section Δ_T of Γ_B. Moreover, $\Delta_T \cong Q_H^{\mathrm{op}}$.

Family $\mathcal{B}_{max}$

- K a field
- $\mathbb{M} = (F_i, {}_iM_j)_{1 \leqslant i,j \leqslant n}$ a K-species
  - F_i finite dimensional division K-algebra
  - ${}_iM_j$ F_i-F_j-bimodule, K acts centrally, $\dim_K {}_iM_j$ finite

Definition

$Q_{\mathbb{M}}$ quiver of $\mathbb{M}$

- $1, 2, \ldots, n$ vertices of $Q_{\mathbb{M}}$
- The arrows of $Q_{\mathbb{M}}$ are the valued arrows

$$i \xrightarrow{(d_{ij}, d'_{ij})} j \ ,$$

with $d_{ij} = \dim_{F_j} {}_iM_j$, $\quad d'_{ij} = \dim_{F_i} {}_iM_j$, $\quad$ provided ${}_iM_j \neq 0$

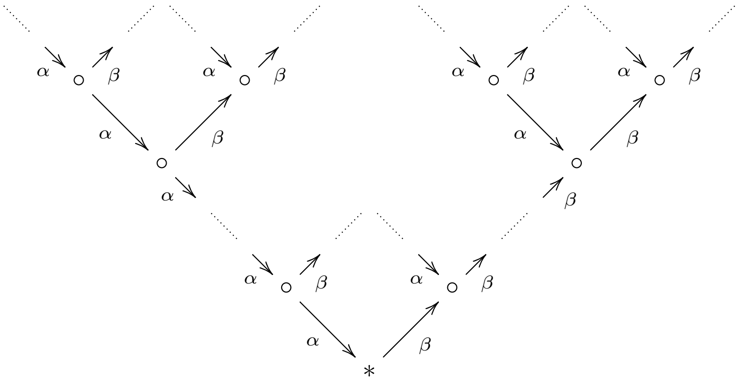
Family $\mathcal{B}_{max}$

$$R_{\mathbb{M}} = \prod_{i=1}^{n} F_i, \quad M_{\mathbb{M}} = \bigoplus_{i,j=1}^{n} {}_iM_j$$

- $T(\mathbb{M})$ the tensor algebra of the $R_{\mathbb{M}}$-$R_{\mathbb{M}}$-bimodule $M_{\mathbb{M}}$
- $Q_{\mathbb{M}}$ acyclic:
  - $T(\mathbb{M})$ finite dimensional hereditary K-algebra
  - $Q_{T(\mathbb{M})} = Q_{\mathbb{M}}$
- $i \longrightarrow j \;\equiv\; i \xrightarrow{(1,1)} j$

Family $\mathcal{B}_{max}$

Branch $\mathcal{L}$:

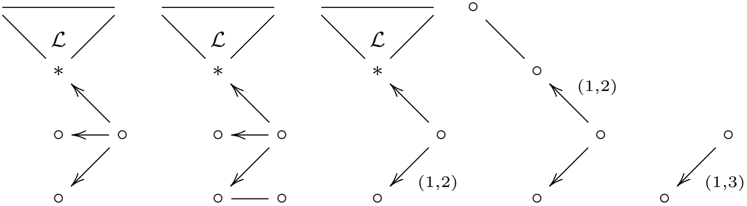


$|\mathcal{L}|$ the number of vertices of $\mathcal{L}$

Family $\mathcal{B}_{max}$

$$B(\mathbb{M}, \mathcal{L}) = T(\mathbb{M})/I(\mathbb{M}, \mathcal{L})$$

- $Q_{\mathbb{M}}$



$|\mathcal{L}| = 2, 3, 4$

- $I(\mathbb{M}, \mathcal{L})$ the ideal of $T(\mathbb{M})$ generated by ${}_iM_j \otimes_{R_{\mathbb{M}}} {}_jM_k$ for all paths $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ in $\mathcal{L}$

$B(\mathbb{M}, \mathcal{L})$ a tilted algebra of Dynkin type $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{B}_n$, $\mathbb{F}_4$, $\mathbb{G}_2$

Family $\mathcal{B}_{max}$

Proposition

Let B be a tilted algebra from the family $\mathcal{B}_{max}$ and G an admissible group of automorphisms of $\widehat{B}$. Then G is an infinite cyclic group generated by an automorphism $\varphi\nu_{\widehat{B}}^{m}$ for a positive integer m and a rigid automorphism φ of $\widehat{B}$.

Selfinjective algebras of Dynkin type

$A = \widehat{B}/G$

B a tilted algebra of Dynkin type Δ, G an admissible infinite cyclic automorphism group of $\widehat{B}$

- $\widehat{B}$ a locally representation-finite category
- $F_\lambda\colon \operatorname{mod} \widehat{B} \to \operatorname{mod} A$ is dense, preserves the almost split sequences
- $A = \widehat{B}/G$ is a selfinjective algebra of finite representation type
- $\Gamma_A = \Gamma_{\widehat{B}/G} = \Gamma_{\widehat{B}}/G$
- $\Gamma^s_A = \Gamma^s_{\widehat{B}/G} = \Gamma^s_{\widehat{B}}/G = \mathbb{Z}\Delta/G$

Definition

$A = \widehat{B}/G$ is said to be a ***selfinjective algebra of Dynkin type***.

Theorem 1

Let A be a selfinjective algebra of Dynkin type. The following statements are equivalent.

1. $\operatorname{mod} A$ admits a maximal almost split sequence.
2. A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}}^{m})$, where B is an algebra from $\mathcal{B}_{max}$, m a positive integer, $\nu_{\widehat{B}}$ the Nakayama automorphism of $\widehat{B}$, and φ a rigid automorphism of $\widehat{B}$.

Theorem 2

Let A be a selfinjective algebra over an algebraically closed field K. The following statements are equivalent.

1. A is of finite representation type and $\operatorname{mod} A$ admits a maximal almost split sequence.
2. A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}}^{m})$, where B is an algebra from $\mathcal{B}_{max}$, m a positive integer, $\nu_{\widehat{B}}$ the Nakayama automorphism of $\widehat{B}$, and φ a rigid automorphism of $\widehat{B}$.

A a nonstandard selfinjective algebra of finite representation type $\Rightarrow$ $\alpha(P/\operatorname{soc} P) \leqslant 3$ for any indecomposable projective module P in $\operatorname{mod} A$

A a selfinjective algebra of finite representation type

P a projective indecomposable module in $\operatorname{mod} A$

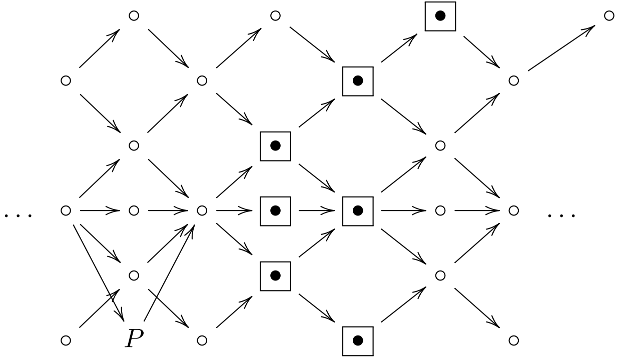
$\operatorname{rad} P/\operatorname{soc} P$ the direct sum of three indecomposable modules

- Δ_P the full valued subquiver of Γ_A given by the modules
  - $\tau_A^{-1}(P/\operatorname{soc} P)$
  - Z an indecomposable in $\operatorname{mod} A$ such that there is a sectional path in Γ_A from $P/\operatorname{soc} P$ to Z
- M_P the direct sum of all modules lying on Δ_P

Proposition

Δ_P is a Dynkin quiver without projective modules.

$\overrightarrow{\mathbb{E}}_7$



Theorem 3

Let A be a selfinjective algebra over a field K. The following statements are equivalent.

1. A is of finite representation type having an indecomposable projective module P in $\operatorname{mod} A$ with $\alpha(P/\operatorname{soc} P) = 4$ and $\operatorname{Hom}_A(M_P, \tau_A M_P) = 0$.
2. A is socle equivalent to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}}^m)$, where B is an algebra from $\mathcal{B}_{max}$, m a positive integer, $\nu_{\widehat{B}}$ the Nakayama automorphism of $\widehat{B}$, and φ a rigid automorphism of $\widehat{B}$.

Moreover, if K is an algebraically closed field, we may replace "socle equivalent" by "isomorphic".

Conjecture:

Let A be a selfinjective algebra of finite representation type and P an indecomposable projective module in $\operatorname{mod} A$ with $\alpha(P/\operatorname{soc} P) = 4$. Then $\operatorname{Hom}_A(M_P, \tau_A M_P) = 0$.

$(1) \Rightarrow (2)$

Let

- A a selfinjective algebra of finite representation type
- P an indecomposable projective module in $\operatorname{mod} A$, $\alpha(P/\operatorname{soc} P) = 4$
- Δ_P the associated full valued subquiver of Γ_A
- M_P the direct sum of indecomposable modules lying on Δ_P
- $I_P = r_A(M_P) = \{a \in A | M_P a = 0\}$ two-sided ideal of A
- $B_P = A/I_P$

$(1) \Rightarrow (2)$

Theorem

Assume $\mathrm{Hom}_A(M_P, \tau_A M_P) = 0$. *Then*

(i) B_P *is a tilted algebra of Dynkin type* Δ_P^{op}.

(ii) B_P *belongs to* $\mathcal{B}_{max}$.

(iii) *There is an idempotent* e_P *in* A *such that* $r_A(I_P) = e_P I_P$.

Then

- I_P is a deforming ideal of A (in the sense of Skowroński-Yamagata)
- A and $A[I_P]$ are socle equivalent
- $A[I_P] \cong \widehat{B_P}/(\psi\nu_{\widehat{B_P}})$, for some positive automorphism ψ of $\widehat{B_P}$

$\psi\nu_{\widehat{B_P}} = \varphi\nu_{\widehat{B_P}}^m$, for some rigid automorphism φ of $\widehat{B_P}$, and $m \geqslant 1$

$(1) \Rightarrow (2)$

A is socle equivalent to an orbit algebra $\widehat{B_P}/(\varphi\nu^m_{\widehat{B_P}})$,

B_P an algebra from $\mathcal{B}_{max}$, $m \geqslant 1$, φ a rigid automorphism of $\widehat{B_P}$

$(2) \Rightarrow (1)$

- $A' = \widehat{B}/(\varphi\nu^m_{\widehat{B}}) \Longrightarrow \operatorname{mod} A'$ admits a maximal almost split sequence
- P' the indecomposable projective direct summand of the middle term of this sequence
- A' a selfinjective algebra of finite type, $\operatorname{Hom}_{A'}(M_{P'}, \tau_{A'} M_{P'}) = 0$
- A, A' socle equivalent $\Longleftrightarrow \operatorname{mod}(A'/\operatorname{soc} A') \xrightarrow[\Phi]{\sim} \operatorname{mod}(A/\operatorname{soc} A)$
- $\Gamma_A \cong \Gamma_{A'}$
  - A a selfinjective algebra of finite type
  - $\Phi(M_{P'})$ the direct sum of nonprojective modules in $\operatorname{mod} A$
  - P an indecomposable projective module in $\operatorname{mod} A$ with $\alpha(P/\operatorname{soc} P) = 4$ such that $M_P = \Phi(M_{P'})$ and $\operatorname{Hom}_A(M_P, \tau_A M_P) = 0$

Example of Skowroński and Yamagata

Example

- $K = \mathbb{Z}_2(x)$, $L = K[y]/(y^2 - x)$
- $K \subseteq L$ finite field extension of a field K, $H^2(L, L) \neq 0$

$$0 \longrightarrow L \longrightarrow K[y]/(y^2 - x)^2 \longrightarrow L \longrightarrow 0$$

  the nonsplittable Hochschild extension of L by L
- $\alpha \colon L \times L \to L$ the corresponding nonsplitable 2-cocycle
- Q :

$$\begin{tikzcd} 1 & \\ 2 & 0 \arrow[l] \arrow[ul] \arrow[dl] \\ 3 & \end{tikzcd}$$

- $H = LQ$ the path algebra of Q over L

Example of Skowroński and Yamagata

Example

There exists a (nonsplittable) Hochschild extension of K-algebras

$$0 \longrightarrow D(H) \longrightarrow \widetilde{H} \longrightarrow H \longrightarrow 0$$

given by a 2-cocycle $\hat{\alpha}\colon H \times H \to D(H)$

- $\widetilde{H}$ is selfinjective K-algebra **(Yamagata)**
- $\widetilde{H}$ is symmetric K-algebra **(Ohnuki-Takeda-Yamagata)**
- $\widetilde{H}$ is socle equivalent to $T(H) = H \ltimes D(H) = H/(\nu_{\widehat{H}})$
- $\widetilde{H}$ is not isomorphic to an algebra of the form $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is a K-algebra, φ is a positive automorphism of $\widehat{B}$
- $\operatorname{mod} \widetilde{H}$ admits a maximal alomst split sequence

Thank you!