Periodicity of selfinjective algebras of polynomial growth

Jerzy Białkowski

(Nagoya, November 2013)

joint work with Karin Erdmann and Andrzej Skowroński
$K$ – algebraically closed field
algebra – finite dimensional $K$-algebra
$A$ an algebra
mod $A$ category of finitely dimensional right $A$-modules
$M$ module in mod $A$, exact sequence in mod $A$

$$0 \rightarrow \Omega_A(M) \rightarrow P_A(M) \rightarrow M \rightarrow 0$$

syzygy module of $M$ projective cover

$M$ is periodic if $\Omega^n_A(M) \cong M$ for some $n \geq 1$

$A$ selfinjective and $M$ periodic, then the Ext-algebra of $M$

$$\text{Ext}_A^*(M, M) = \bigoplus_{i \geq 0} \text{Ext}_A^i(M, M)$$

is a graded noetherian algebra (Schultz, 1986)
\[ A^e = A^{\text{op}} \otimes_K A \text{ enveloping algebra of } A \]

mod \( A^e = \text{bimod } A \) category of finite dimensional \( A \)-\( A \)-bimodules

A projective in mod \( A^e \iff A \text{ is a semisimple algebra} \)

A is a \textbf{periodic algebra} if \( A \) is a periodic module in mod \( A^e \)
(periodic \( A \)-\( A \)-bimodule)

A periodic algebra \( \Rightarrow \) A selfinjective algebra

A periodic algebra \( \Rightarrow \) mod \( A \) is a \textbf{periodic category}
(\begin{align*}
\text{every module in mod } A \text{ without projective direct summands is periodic}
\end{align*})

In fact, we have

\[ \Omega^n_{A^e}(A) \cong A \text{ in mod } A^e \Rightarrow \Omega^n_A(M) \cong M \text{ for any module } M \text{ in mod } A \]
without projective direct summands
$HH^*(A) = \text{Ext}^*_{A^e}(A, A)$ the Hochschild cohomology algebra of $A$

A selfinjective $\Rightarrow A^e$ selfinjective $\Rightarrow \text{Ext}^i_{A^e}(A, A) \cong \text{Hom}_{A^e}(\Omega^i_{A^e}(A), A)$ for $i \geq 0$

A periodic algebra $\Rightarrow HH^*(A)$ a graded noetherian algebra

$D^b(\text{mod } A)$ derived category of bounded complexes over $\text{mod } A$

$A, \Lambda$ algebras

$A$ and $\Lambda$ are derived equivalent if $D^b(\text{mod } A)$ and $D^b(\text{mod } \Lambda)$ are equivalent as triangulated categories

$A$ and $\Lambda$ are derived equivalent $\xrightarrow{\text{Happel, Rickard}} HH^*(A) \cong HH^*(\Lambda)$ as graded $K$-algebras

**Theorem (Rickard)**

Let $A$ and $\Lambda$ be derived equivalent algebras.

A is a periodic algebra $\iff \Lambda$ is a periodic algebra

Moreover, if $A$ and $\Lambda$ are periodic, then their periods coincide.
**PROBLEM**

Determine the periodic algebras (up to Morita equivalence, derived equivalence)

**PERIODICITY CONJECTURE**

Assume $A$ is an algebra for which all simple modules in $\text{mod} \ A$ are periodic. Then $A$ is a periodic algebra.

**Theorem (Green–Snashall–Solberg, 2003)**

Let $A$ be an algebra such that every simple module in $\text{mod} \ A$ is periodic. Then $A$ is a selfinjective algebra and $\Omega^d_{A^e}(A) \cong 1 A_\sigma$ for a positive integer $d$ and a $K$-algebra automorphism $\sigma$ of $A$.

Is then $\sigma$ of finite order?
Selfinjective Nakayama algebras

\[ N^m_n(K) = K\Delta_n/I_{m,n}, \quad m \geq 2, \ n \geq 1 \]

- \( K\Delta_n \) path algebra of \( \Delta_n \)
- \( I_{m,n} \) ideal generated by all compositions of \( m \) consecutive arrows in \( \Delta_n \)

\( N^m_n(K) \) symmetric algebra \( \iff \) \( n|m+1 \)
\( N^m_n(K) \) a periodic algebra (Erdmann–Holm, 1999)

period of \( N^m_n(K) = \begin{cases} 
  n & \text{char } K = 2, m = 2, n \text{ odd} \\
  \frac{2 \text{lcm}(m,n)}{m} & \text{otherwise}
\end{cases} \)

Hence every **Brauer tree algebra** is a periodic algebra
Brauer tree algebras are derived equivalent to symmetric Nakayama algebras (Rickard, 1989)
Tame and wild algebras

Theorem (Drozd, 1979)

Every algebra $A$ is either tame or wild, and not both.

A and algebra

$A$ is **wild** if there is a $K\langle x, y \rangle$-bimodule $M$ such that:
- $M$ is a finite rank free left $K\langle x, y \rangle$-module
- the functor $- \otimes_{K\langle x, y \rangle} \ker \rightarrow \ker \otimes A$ preserves indecomposability and respects isomorphism classes.

A wild algebra $\Rightarrow$ for any algebra $\Lambda$ over $K$ there is an exact functor $F : \ker \rightarrow \ker A$ which preserves indecomposability and respects isomorphism classes.

$A$ is **tame** if, for any dimension $d$, there is a finite number of $K[x]$-$A$-bimodules $M_i$, $1 \leq i \leq n_d$, such that:
- $M_i$, $1 \leq i \leq n_d$, are finite rank free left $K[x]$-modules,
- $K[x]/(x - \lambda) \otimes_{K[x]} M_i$, $\lambda \in K$, $i \in \{1, \ldots, n_d\}$, exhaust all but finitely many isomorphism classes of indecomposable modules of dimension $d$ in $\ker A$.

$\mu_A(d)$ the least number of $K[x]$-$A$-bimodules satisfying the above condition for $d$. 
Hierarchy of algebras

finite type

∀ \( d \geq 1 \) \( \mu_A(d) = 0 \)

domestic type

∃ \( m \geq 1 \) ∀ \( d \geq 1 \) \( \mu_A(d) \leq m \)

polynomial growth

∃ \( m \geq 1 \) ∀ \( d \geq 1 \) \( \mu_A(d) \leq d^m \)

tame type

∀ \( d \geq 1 \) \( \mu_A(d) < \infty \)

wild

wild

wild

wild
Preprojective algebras of Dynkin type

Δ Dynkin graph of type \( \mathbb{A}_n(n \geq 1), \mathbb{D}_n(n \geq 4), \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8 \)

\( Q_\Delta \) double quiver of \( \Delta \)

\( P(\Delta) = KQ_\Delta/I_\Delta \) preprojective algebra of type \( \Delta \)

\( KQ_\Delta \) the path algebra of \( Q_\Delta \)

\( I_\Delta \) the ideal of \( KQ_\Delta \) generated by the sums \( \sum_{a,ia=v} a\bar{a} \) for all 2-cycles at the vertices \( v \) of \( Q_\Delta \)

\( P(\Delta) \) finite dimensional selfinjective \( K \)-algebra

\( \Omega^6_{P(\Delta)e}(P(\Delta)) \cong P(\Delta) \) for \( \Delta \neq \mathbb{A}_1 \)


For \( \Delta \neq \mathbb{A}_n(n \leq 5) \) and \( \mathbb{D}_4 \), the algebras \( P(\Delta) \) are \textit{wild}
\[ Q_{A_n} : \]
\[
\begin{array}{c}
0 \\ \bar{a}_0
\end{array} \quad \begin{array}{c}
1 \\ \bar{a}_1
\end{array} \quad \begin{array}{c}
2 \\ \bar{a}_2
\end{array} \quad \cdots \quad \begin{array}{c}
n-2 \\ \bar{a}_{n-2}
\end{array} \quad \begin{array}{c}
n-1 \\ \bar{a}_{n-1}
\end{array}
\]

\[ (n \geq 1) \]

\[ Q_{D_n} : \]
\[
\begin{array}{c}
\bar{a}_0 \\ \bar{a}_1
\end{array} \quad \begin{array}{c}
2 \\ \bar{a}_2
\end{array} \quad \begin{array}{c}
3 \\ \bar{a}_3
\end{array} \quad \cdots \quad \begin{array}{c}
n-2 \\ \bar{a}_{n-2}
\end{array} \quad \begin{array}{c}
n-1 \\ \bar{a}_{n-1}
\end{array}
\]

\[ (n \geq 4) \]

\[ Q_{E_6} : \]
\[
\begin{array}{c}
1 \\ \bar{a}_1
\end{array} \quad \begin{array}{c}
2 \\ \bar{a}_2
\end{array} \quad \begin{array}{c}
3 \\ \bar{a}_3
\end{array} \quad \begin{array}{c}
4 \\ \bar{a}_4
\end{array} \quad \begin{array}{c}
5 \\ \bar{a}_5
\end{array}
\]

\[ Q_{E_7} : \]
\[
\begin{array}{c}
1 \\ \bar{a}_1
\end{array} \quad \begin{array}{c}
2 \\ \bar{a}_2
\end{array} \quad \begin{array}{c}
3 \\ \bar{a}_3
\end{array} \quad \begin{array}{c}
4 \\ \bar{a}_4
\end{array} \quad \begin{array}{c}
5 \\ \bar{a}_5
\end{array} \quad \begin{array}{c}
6 \\ \bar{a}_6
\end{array}
\]

\[ Q_{E_8} : \]
\[
\begin{array}{c}
1 \\ \bar{a}_1
\end{array} \quad \begin{array}{c}
2 \\ \bar{a}_2
\end{array} \quad \begin{array}{c}
3 \\ \bar{a}_3
\end{array} \quad \begin{array}{c}
4 \\ \bar{a}_4
\end{array} \quad \begin{array}{c}
5 \\ \bar{a}_5
\end{array} \quad \begin{array}{c}
6 \\ \bar{a}_6
\end{array} \quad \begin{array}{c}
7 \\ \bar{a}_7
\end{array}
\]
Algebras of quaternion type

A is of **quaternion type** if
- $A$ is indecomposable and symmetric
- $A$ is tame of infinite type
- every indecomposable nonprojective module in $\text{mod } A$ is periodic with period dividing 4
- the Cartan matrix $C_A$ is nonsingular

**Erdmann** proved (1988) that every algebra of quaternion type is Morita equivalent to an algebra in 12 families of algebras listed below. Moreover, $A$ is of **pure quaternion type**, if $A$ is not of polynomial growth.

**Theorem (Erdmann-Skowroński, 2006)**

*Let $A$ be an algebra of pure quaternion type. Then $A$ is a periodic algebra of period 4.*

The derived equivalence classification of algebras of pure quaternion type established by **Holm (1999)** is applied.

The algebras of pure quaternion type are **tame**.
Periodicity of selfinjective algebras of polynomial growth

**$Q^k(c)$:**

\[
\alpha \quad \bullet \quad \beta
\]

\[
\alpha^2 = (\beta \alpha)^{k-1} \beta + c(\alpha \beta)^k \\
\beta^2 = (\alpha \beta)^{k-1} \alpha \\
(\alpha \beta)^k = (\beta \alpha)^k, (\alpha \beta)^k \alpha = 0 \\
k \geq 2
\]

**$Q^k(c,d)$:**

\[
\alpha \quad \bullet \quad \beta
\]

\[
\alpha^2 = (\beta \alpha)^{k-1} \beta + c(\alpha \beta)^k \\
\beta^2 = (\alpha \beta)^{k-1} \alpha + d(\alpha \beta)^k \\
(\alpha \beta)^k = (\beta \alpha)^k, (\alpha \beta)^k \alpha = 0 \\
(\beta \alpha)^k \beta = 0 \\
k \geq 2, c, d \in K, (c,d) \neq (0,0)
\]

**$Q(2A)^k(c)$:**

\[
\alpha \quad \beta \quad \gamma
\]

\[
\gamma \beta \gamma = (\gamma \alpha \beta)^{k-1} \gamma \alpha \\
\beta \gamma \beta = (\alpha \beta \gamma)^{k-1} \alpha \beta \\
\alpha^2 \beta = 0 \\
k \geq 2, c \in K
\]

**$Q(2B)^{k,s}(a, c)$:**

\[
\alpha \quad \bullet \quad \eta
\]

\[
\gamma \beta = \eta^{s-1}, \\
\beta \eta = (\alpha \beta \gamma)^{k-1} \alpha \beta \\
\eta \gamma = (\gamma \alpha \beta)^{k-1} \gamma \alpha \\
\alpha^2 = a(\beta \gamma \alpha)^{k-1} \beta \gamma + c(\beta \gamma \alpha)^k \\
\alpha^2 \beta = 0, \gamma \alpha^2 = 0 \\
k \geq 1, s \geq 3, a \in K^*, c \in K
\]

**$Q(2B)^{s}(a, c)$:**

\[
\alpha \quad \bullet \quad \eta
\]

\[
\alpha \beta = \beta \eta, \eta \gamma = \gamma \alpha, \beta \gamma = \alpha^2 \\
\gamma \beta = \eta^2 + a \eta^{s-1} + c \eta^s \\
\alpha^2 + 1 = 0, \eta^{s+1} = 0 \\
\gamma \alpha^{s-1} = 0, \alpha^{s-1} \beta = 0 \\
s \geq 4, a \in K^*, c \in K
\]

**$Q(2B)^{t}(a, c)$:**

\[
\alpha \quad \bullet \quad \eta
\]

\[
\alpha \beta = \beta \eta, \eta \gamma = \gamma \alpha, \beta \gamma = \alpha^2 \\
\gamma \beta = a \eta^{t-1} + c \eta^t \\
\alpha^4 = 0, \eta^{t+1} = 0, \gamma \alpha^2 = 0 \\
\alpha^2 \beta = 0 \\
t \geq 3, a \in K^*, c \in K \\
(t = 3 \Rightarrow a \neq 1, t > 3 \Rightarrow a = 1)
Periodicity of selfinjective algebras of polynomial growth

$Q(3A)_1^k, s(d)$:

$Q(3A)_2^k$:

$Q(3B)^{k, s}$:

$Q(3C)^{k, s}$:

$Q(3D)^{k, s, t}$:

$Q(3K)^{a, b, c}$:
Theorem (Dugas, 2010)

Let $A$ be a nonsimple, indecomposable, selfinjective algebra of finite type. Then $A$ is a periodic algebra.

Special cases:
- Erdmann-Holm-Snashall (1999, 2002) Dynkin type $\tilde{\mathbb{A}}_n$
- Brenner-Butler-King (2002): the trivial extension algebras $T(B) = B \ltimes D(B)$ of titled algebras $B$ of Dynkin type
- Erdmann-Skowroński (2008): almost all standard selfinjective algebras of finite type

Basic, nonsimple, indecomposable, selfinjective algebras of finite type

Periodicity of standard selfinjective algebras of finite type follows from the periodicity of preprojective algebras of Dynkin type (Brenner-Butler-King, Dugas)

Periodicity of nonstandard selfinjective algebras of finite type follows from the periodicity of a family of Brauer tree algebras (Dugas)
Λ basic, indecomposable algebra

$1_\Lambda = e_1 + \cdots + e_n$,
e$_1, \ldots, e_n$ pairwise orthogonal primitive idempotents of Λ

G a finite subgroup of Aut$_K(\Lambda)$ acting freely on the chosen set

e$_1, \ldots, e_n$ of primitive idempotents

Then we have a finite Galois covering $\Lambda \rightarrow \Lambda/G$ where $\Lambda/G$ is the orbit algebra of $\Lambda$ with respect to $G$

In fact, we have an isomorphism of $K$-algebras

$\Lambda/G \cong \Lambda^G = \{ \lambda \in \Lambda \mid g(\lambda) = \lambda \text{ for all } g \in G \}$ invariant algebra

(Auslander-Reiten-Smalø, 1989)

**Theorem (Dugas, 2010)**

*Let $A$ and $\Lambda$ be basic indecomposable algebras related by a finite Galois covering $\Lambda \rightarrow A = \Lambda/G$. Then

$\Lambda$ is a periodic algebra $\iff A$ is a periodic algebra*
Selfinjective algebras of polynomial growth

Theorem (Skowroński, 1989, 2006)

Let $A$ be a basic, indecomposable, selfinjective algebra over an algebraically closed field $K$. Then

1. $A$ is representation-infinite domestic $\iff$ $A$ is socle equivalent to an orbit algebra $\hat{B}/G$, where $B$ is a tilted algebra of Euclidean type and $G$ is an admissible infinite cyclic group of automorphisms of $\hat{B}$.

2. $A$ is nondomestic of polynomial growth $\iff$ $A$ is socle equivalent to an orbit algebra $\hat{B}/G$, where $B$ is a tubular algebra and $G$ is an admissible infinite cyclic group of automorphisms of $\hat{B}$.

Two selfinjective algebras $A$ and $\Lambda$ are **socle equivalent** if the quotient algebras $A/\text{soc}(A)$ and $\Lambda/\text{soc}(\Lambda)$ are isomorphic.
Riedtmann’s classification of selfinjective algebras of finite type can be presented as follows

**Theorem**

Let $A$ be a basic, nonsimple, indecomposable, selfinjective algebra over an algebraically closed field. Then $A$ is of finite type $\iff$ $A$ is socle equivalent to an orbit algebra $\hat{B}/G$, where $B$ is a tilted algebra of Dynkin type and $G$ is an admissible infinite cyclic group of automorphisms of $\hat{B}$. 
THEOREM (Białkowski-Erdmann-Skowroński, 2013)

Let $A$ be a basic, indecomposable, representation-infinite selfinjective algebra of polynomial growth. The following statements are equivalent.

1. All simple modules in $\text{mod} \ A$ are periodic.
2. $A$ is a periodic algebra.
3. $A$ is socle equivalent to an orbit algebra $\hat{B}/G$, where $B$ is a tubular algebra and $G$ is an admissible infinite cyclic group of automorphisms of the repetitive category $\hat{B}$ of $B$.

2 $\Rightarrow$ 1 Known
1 $\Rightarrow$ 3 A socle equivalent to $\hat{B}/G$, $B$ tilted algebra of Euclidean type, then $\Gamma_A$ admits an acyclic component $C$ (with the stable part $C^s = \mathbb{Z}\Delta$ for a Euclidean quiver $\Delta$) containing a simple module $S$. But then $S$ is not periodic, because for simple modules the periodicity is equivalent to $\tau_A$-periodicity.
3 $\Rightarrow$ 2 New (difficult) part.

We will present the main ingredients of our proof of this implication.
Tubular algebras

A **tubular algebra** \( B \) is a tubular (branch) extension of a tame concealed algebra of one of the tubular types \((2, 2, 2, 2), (3, 3, 3), (2, 4, 4), \) or \((2, 3, 6)\) (*Ringel, 1984*)

- \( \text{gl. dim } B = 2 \)
- \( \text{rk } K_0(B) = 6, 8, 9, \) or \( 10 \)
- \( B \) is triangular nondomestic of polynomial growth
- The Auslander-Reiten quiver \( \Gamma_B \) of \( B \) is of the form

![Diagram of Auslander-Reiten quiver](image)

- \( \mathcal{P}^B \) preprojective component of Euclidean type
- \( \mathcal{P}_1(K) \)-family of ray tubes
- \( \bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q^B \) \( \mathcal{P}_1(K) \)-family of stable tubes
- \( \mathcal{T}_\infty^B \) \( \mathcal{P}_1(K) \)-family of coray tubes
- \( \mathcal{Q}^B \) preinjective component of Euclidean type
Canonical tubular algebras

\( \Lambda(2, 2, 2, 2, \lambda), \lambda \in K \setminus \{0, 1\} \), given by the quiver

\[
\begin{array}{c}
\bullet \\
\alpha_1 & \beta_1 \\
\alpha_2 & \beta_2 \\
\gamma_1 & \delta_1 \\
\delta_2 & \gamma_2 \\
\bullet
\end{array}
\]

and the relations \( \alpha_2 \alpha_1 + \beta_2 \beta_1 + \gamma_2 \gamma_1 = 0, \alpha_2 \alpha_1 + \lambda \beta_2 \beta_1 + \delta_2 \delta_1 = 0. \)

\( \Lambda(p, q, r), (p, q, r) \in \{(3, 3, 3), (2, 4, 4), (2, 3, 6)\} \), given by the quiver

\[
\begin{array}{c}
\bullet \\
\alpha_1 & \alpha_2 & \cdots & \alpha_{p-1} \\
\beta_1 & \beta_2 & \cdots & \beta_{q-1} & \beta_q \\
\gamma_1 & \gamma_2 & \cdots & \gamma_{r-1} & \gamma_r \\
\bullet
\end{array}
\]

and the relation \( \alpha_p \cdots \alpha_2 \alpha_1 + \beta_q \cdots \beta_2 \beta_1 + \gamma_r \cdots \gamma_2 \gamma_1 = 0. \)
Theorem (Ringel, 1984)

Let $B$ be a basic, indecomposable algebra. Then

\[ B \text{ is a tubular algebra} \iff B = \text{End}_\Lambda(T) \text{ for a canonical tubular algebra } \Lambda \text{ and a (multiplicity-free) tilting module } T \text{ in the additive category of } \mathcal{P}_0^\Lambda \cup \mathcal{T}_0^\Lambda \cup \bigcup_{q \in \mathbb{Q}^+} T_q^\Lambda \]

Hence, any two tubular algebras $B$ and $C$ of the same tubular type $(p, q, r) \in \{(3, 3, 3), (2, 4, 4), (2, 3, 6)\}$ are derived equivalent.

Similarly, every two tubular algebras $B$ and $C$ of tubular type $(2, 2, 2, 2)$ given by the same canonical tubular algebra $\Lambda(2, 2, 2, 2, \lambda)$ are derived equivalent.
$B$ tubular algebra, $G$ admissible infinite cyclic group of automorphisms of $\hat{B}$, $A = \hat{B}/G$

Then $\Gamma_A = \Gamma_{\hat{B}}/G$ and has the following clock structure

where $\ast$ denote projective modules, $r \geq 3$, $Q_i^{i-1} = Q \cap (i-1, i)$ for any $i \in \{1, \ldots, r\}$, and

1. for each $i \in \{0, 1, \ldots, r-1\}$, $T_i^A$ is a $P_1(K)$-family of quasi-tubes (the stable parts are stable tubes);

2. for each $q \in Q_i^{i-1}$, $i \in \{1, \ldots, r\}$, $T_q^A$ is a $P_1(K)$-family of stable tubes;

3. all $P_1(K)$-families $T_q^A$, $q \in Q \cap [0, r]$, have the same tubular type $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$, or $(2, 3, 6)$. 


For an algebra $B$ and a positive integer $r$, we have the $r$-fold trivial extension algebra of $B$

$$T(B)^{(r)} = \widehat{B}/(\nu^r_B) = \left\{ \begin{pmatrix} b_1 & 0 & 0 \\ f_2 & b_2 & 0 \\ 0 & f_3 & b_3 \\ \vdots & \vdots & \vdots \\ 0 & f_{r-1} & b_{r-1} \\ 0 & f_1 & b_1 \end{pmatrix} : b_1, \ldots, b_{r-1} \in B, \ f_1, \ldots, f_{r-1} \in D(B) \right\}$$

$T(B)^{(1)} \cong T(B) = B \times D(B)$ the trivial extension algebra of $B$ by the injective cogenerator $D(B) = \text{Hom}_B(B, K)$

$T(B)^{(r)}$ is a symmetric algebra $\iff r = 1$
Standard selfinjective algebra of tubular type: a selfinjective algebra of the form $A = \hat{B}/H$, where $B$ is a tubular algebra and $H$ is an admissible infinite cyclic group of automorphisms of $\hat{B}$

Then $A$ admits a simply connected Galois covering $\hat{B} \to \hat{B}/H = A$

Theorem

Let $A$ be a basic, indecomposable algebra. The following statements are equivalent.

1. $A$ is a standard selfinjective algebra of tubular type.
2. $A$ is isomorphic to an orbit algebra $T(B)^{(r)}/G$, where $B$ is a tubular algebra, $r$ a positive integer, and $G$ an admissible finite automorphism group of $T(B)^{(r)}$.

Białkowski-Skowroński (2002): tubular types $(2, 2, 2, 2), (3, 3, 3), (2, 4, 4)$

Lenzing-Skowroński (2000): tubular type $(2, 3, 6)$

Note that $T(B)^{(r)}/G \cong (T(B)^{(r)})^G$ invariant algebra
Theorem (Rickard, 1989)

Let $B$ and $C$ be derived equivalent algebras. Then the trivial extension algebras $T(B)$ and $T(C)$ are derived equivalent.

We may consider the following scheme of finite Galois coverings

\[
\begin{array}{c}
T(B)^{(r)} \\
\downarrow \\
T(B)^{(r)}/G \\
\downarrow \\
T(B) \sim_{\text{der}} T(C) \\
\downarrow \\
T(C)^{(s)}/H
\end{array}
\]

where $B$ and $C$ are derived equivalent tubular algebras, $r$, $s$ positive integers, $G$, $H$ admissible finite automorphism groups of $T(B)^{(r)}$ and $T(C)^{(s)}$, respectively.

Then

$T(B)^{(r)}/G$ is a periodic algebra $\iff$ $T(C)^{(s)}/H$ is a periodic algebra
Nonstandard nondomestic selfinjective algebras of polynomial growth
(socle deformations of standard selfinjective algebras of tubular type)

Occur only in characteristic 2 and 3

Λ nonstandard nondomestic selfinjective algebra of polynomial growth

Then there exists a unique standard selfinjective algebra Λ′ of tubular type such that

1. \( \dim_K \Lambda = \dim_K \Lambda' \)
2. Λ and Λ′ are socle equivalent (but Λ \( \not\cong \) Λ′)
3. Λ′ is a geometric degeneration of an Λ (belongs to the closure \( \overline{GL_K(d)}\Lambda \) in the affine variety of \( K \)-algebras of dimension \( d = \dim_K \Lambda = \dim_K \Lambda' \))

Λ′ the **standard form** of Λ

The pairs Λ and Λ′ are described by the tables

(Białkowski-Skowroński, 2004)
### Periodicity of selfinjective algebras of polynomial growth

<table>
<thead>
<tr>
<th>characteristic</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>tubular type</td>
<td>(3,3,3)</td>
<td>(2,2,2,2)</td>
</tr>
<tr>
<td>nonstandard algebras</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Lambda_1 )</td>
<td>( \alpha ) &amp; ( \gamma ) &amp; ( \beta )</td>
<td></td>
</tr>
<tr>
<td>( \alpha^2 = \gamma \beta ), ( \beta \alpha \gamma = \beta \alpha^2 \gamma ), ( \beta \alpha^2 = 0 ), ( \alpha^2 \gamma = 0 )</td>
<td></td>
<td></td>
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<tr>
<td>( \Lambda_2 )</td>
<td>( \alpha ) &amp; ( \gamma ) &amp; ( \beta )</td>
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<tr>
<td>( \alpha^2 \gamma = 0 ), ( \beta \alpha^2 = 0 ), ( \gamma \beta \gamma = 0 ), ( \beta \gamma \beta = 0 ), ( \beta \gamma = \beta \alpha \gamma ), ( \alpha^3 = \gamma \beta )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Lambda_3(\lambda) ), ( \lambda \in K \setminus {0,1} )</td>
<td>( \alpha ) &amp; ( \gamma ) &amp; ( \beta )</td>
<td>( \gamma \beta \gamma = \gamma \alpha = \beta \gamma ), ( \sigma \beta = \alpha \sigma )</td>
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<tr>
<td>( \alpha^2 = \gamma \beta )</td>
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<tr>
<td>standard algebras</td>
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<tr>
<td>( \Lambda_1' )</td>
<td>( \alpha ) &amp; ( \gamma ) &amp; ( \beta )</td>
<td>( \alpha ) &amp; ( \gamma ) &amp; ( \beta )</td>
</tr>
<tr>
<td>( \alpha^2 = \gamma \beta ), ( \beta \alpha \gamma = 0 )</td>
<td>( \alpha^2 = \gamma \beta ), ( \beta \alpha \gamma = 0 )</td>
<td>( \alpha^2 \gamma = 0 ), ( \beta \alpha^2 = 0 ), ( \gamma \beta \gamma = 0 ), ( \beta \gamma \beta = 0 ), ( \beta \gamma = \beta \alpha \gamma ), ( \alpha^3 = \gamma \beta )</td>
</tr>
<tr>
<td>( \Lambda_2' )</td>
<td>( \alpha ) &amp; ( \gamma ) &amp; ( \beta )</td>
<td>( \alpha ) &amp; ( \gamma ) &amp; ( \beta )</td>
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<td>( \alpha ) &amp; ( \gamma ) &amp; ( \beta )</td>
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<td>( \alpha ) &amp; ( \gamma ) &amp; ( \beta )</td>
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<tr>
<td>( \alpha ) &amp; ( \gamma ) &amp; ( \beta )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>characteristic</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>---------------</td>
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<tr>
<td>tubular type</td>
<td>(2,4,4)</td>
<td></td>
</tr>
</tbody>
</table>

**Nonstandard algebras**

- $\Lambda_4$
  - $\delta \beta \delta = \alpha \gamma$
  - $(\beta \delta)^3 \beta = 0$
  - $\gamma \beta \alpha \gamma = 0$
  - $\alpha \gamma \beta \alpha = 0$
  - $\gamma \beta \alpha = \gamma \beta \delta \beta \alpha$

- $\Lambda_5$
  - $\alpha^2 = \gamma \beta$, $\alpha^3 = \delta \sigma$
  - $\beta \delta = 0$, $\sigma \gamma = 0$
  - $\alpha \delta = 0$, $\sigma \alpha = 0$
  - $\gamma \beta \gamma = 0$, $\beta \gamma \beta = 0$
  - $\beta \gamma = \beta \alpha \gamma$

- $\Lambda_6$
  - $\alpha \delta \gamma \delta = 0$
  - $\gamma \delta \gamma \beta = 0$
  - $\beta \alpha \beta = 0$
  - $\alpha \beta = \alpha \delta \gamma \beta$
  - $\beta \alpha = \delta \gamma \delta \gamma$

- $\Lambda_7$
  - $\beta \delta = \beta \alpha \delta$, $\alpha \sigma = 0$
  - $\alpha \delta = \sigma \gamma$, $\gamma \beta \alpha = 0$
  - $\alpha^2 = \delta \beta$, $\gamma \beta \delta = 0$
  - $\beta \delta \beta = 0$, $\delta \beta \delta = 0$

- $\Lambda_8$
  - $\delta \beta = \delta \alpha \beta$, $\sigma \alpha = 0$
  - $\delta \alpha = \gamma \sigma$, $\alpha \beta \gamma = 0$
  - $\alpha^2 = \beta \delta$, $\delta \beta \gamma = 0$
  - $\beta \delta \beta = 0$, $\delta \beta \delta = 0$

**Standard algebras**

- $\Lambda'_4$
  - $\delta \beta \delta = \alpha \gamma$
  - $(\beta \delta)^3 \beta = 0$
  - $\gamma \beta \alpha = 0$

- $\Lambda'_5$
  - $\alpha^2 = \gamma \beta$, $\alpha^3 = \delta \sigma$
  - $\beta \delta = 0$, $\sigma \gamma = 0$
  - $\alpha \delta = 0$, $\sigma \alpha = 0$
  - $\beta \gamma = 0$

- $\Lambda'_6$
  - $\alpha \delta \gamma \delta = 0$
  - $\gamma \delta \gamma \beta = 0$
  - $\alpha \beta = 0$
  - $\beta \alpha = \delta \gamma \delta \gamma$

- $\Lambda'_7$
  - $\beta \delta = 0$, $\alpha \sigma = 0$
  - $\alpha \delta = \sigma \gamma$, $\gamma \beta \alpha = 0$
  - $\alpha^2 = \delta \beta$

- $\Lambda'_8$
  - $\delta \beta = 0$, $\sigma \alpha = 0$
  - $\delta \alpha = \gamma \sigma$, $\alpha \beta \gamma = 0$
  - $\alpha^2 = \beta \delta$
Theorem (Białykowski-Holm-Skowroński, 2003)

1. \(\Lambda_1\) and \(\Lambda_2\) are derived equivalent (\(\text{char } K = 3\))
2. \(\Lambda_1'\) and \(\Lambda_2'\) are derived equivalent (\(\text{char } K \text{ arbitrary}\))
3. \(\Lambda_4, \Lambda_5, \Lambda_6, \Lambda_7, \Lambda_8\) are derived equivalent (\(\text{char } K = 2\))
4. \(\Lambda_4', \Lambda_5', \Lambda_6', \Lambda_7', \Lambda_8'\) are derived equivalent (\(\text{char } K \text{ arbitrary}\))

- \(\Lambda_i\) and \(\Lambda_i'\), \(i \in \{1, \ldots, 8\}\) are symmetric algebras
- \(\Lambda_9\) (\(\text{char } K = 2\)) is weakly symmetric but not symmetric
- \(\Lambda_9'\) (\(\text{char } K = 2\)) is symmetric
- \(\Lambda_9'\) (\(\text{char } K \neq 2\)) is weakly symmetric but not symmetric
- \(\Lambda_{10}\) and \(\Lambda_{10}'\) are not weakly symmetric
**THEOREM**

- \( \Lambda_1 (\operatorname{char} K = 3) \) periodic algebra of period 6
- \( \Lambda'_1 (\operatorname{char} K \text{ arbitrary}) \) periodic algebra of period 6
- \( \Lambda_3(\lambda) (\operatorname{char} K = 2) \) periodic algebra of period 4
- \( \Lambda'_3(\lambda) (\operatorname{char} K \text{ arbitrary}) \) periodic algebra of period 4
- \( \Lambda_6 (\operatorname{char} K = 2) \) periodic algebra of period 8
- \( \Lambda'_6 (\operatorname{char} K \text{ arbitrary}) \) periodic algebra of period 8
- \( \Lambda_9 (\operatorname{char} K = 2) \) periodic algebra of period 6
- \( \Lambda'_9 \) periodic algebra of period \( \begin{cases} 3 & \text{char } K = 2 \\ 6 & \text{char } K \neq 2 \end{cases} \)
- \( \Lambda_{10} (\operatorname{char} K = 2) \) periodic algebra of period 6
- \( \Lambda'_{10} \) periodic algebra of period \( \begin{cases} 3 & \text{char } K = 2 \\ 6 & \text{char } K \neq 2 \end{cases} \)

Hard work
Λ basic, indecomposable, finite dimensional algebra, K algebraically closed

\[ 1_A = e_1 + \cdots + e_n, \ e_1, \ldots, e_n \text{ pairwise orthogonal primitive idempotents of } A \]

\[ e_i \otimes e_j, \ i, j \in \{1, \ldots, n\}, \text{ pairwise orthogonal primitive idempotents of } A^e = A^{\text{op}} \otimes_K A \]

\[ 1_{A^e} = \sum_{1 \leq i, j \leq n} e_i \otimes e_j \]

\[ P(i, j) = (e_i \otimes e_j)A^e = Ae_i \otimes e_j A, \ i, j \in \{1, \ldots, n\}, \text{ complete set of pairwise nonisomorphic indecomposable projective modules in } \text{mod} \ A^e = \text{bimod} \ A \]

\[ S_i = e_i A / e_i \text{ rad } A, \ i \in \{1, \ldots, n\}, \text{ complete set of pairwise nonisomorphic simple modules in } \text{mod} \ A \]

The following theorem describes the terms of a minimal projective bimodule resolution of A.

**Theorem (Happel, 1989)**

A admits a minimal projective resolution in mod A^e of the form

\[
\cdots \rightarrow P_r \xrightarrow{d_r} P_{r-1} \rightarrow \cdots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \rightarrow 0,
\]

where

\[
P_r = \bigoplus_{0 \leq i, j \leq n} P(i, j)^{\dim_K \text{Ext}^r_A(S_i, S_j)}.
\]
Theorem (Białkowski–Erdmann–Skowroński, 2013)

Let $A$ and $\Lambda$ be representation-infinite periodic algebras of polynomial growth such that $\Lambda$ is a nonstandard algebra and $A$ a standard algebra. Then $A$ and $\Lambda$ are not derived equivalent.

Symmetric algebras case Holm-Skowroński (2011), using Külshammer ideals

A similar result holds for representation-finite selfinjective algebras Asashiba (1999)

Holm-Skowroński (2006): different proof using Külshammer ideals