Weyl Groups and Artin Groups Associated to Weighted Projective Lines

(joint work with Yuuki Shiraishi and Kentaro Wada)

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Aim

Want to understand interesting correspondences among

1. Complex (Algebraic) Geometry.
2. Symplectic Geometry.
3. Representation Theory.

McKay correspondence, strange duality, ...

Three kinds of triangulated categories from different origins:

1. Derived category of coherent sheaves on an algebraic stack.
2. Derived category of Fukaya category (of Lagrangian submanifolds).
3. Derived category of finite dimesional modules over a finite dimensional algebra.

Equivalences among them = Homological Mirror Symmetry

On the other hand,

there are three different constructions of Frobenius structures:

1. Gromov–Witten theory.
2. Deformation theory.
3. Invariant theory of Weyl groups.

Isomorphisms among these = Classical Mirror Symmetry

Frobenius structure: a “flat family” of commutative Frobenius algebras over a complex manifold

These Mirror Symmetries should be related
via space of Bridgeland's stability conditions.

In order to make this idea some precise statements,
we first study basic properties of Weyl groups and Artin groups
associated to weighted projective lines.

Weighted Projective Lines

$A = (a_1, \ldots, a_r)$: a tuple of positive integers ($r \geq 3$).
Set

$$\mu_A := 2 + \sum_{k=1}^{r} (a_r - 1)\,, \quad \chi_A := 2 + \sum_{k=1}^{r} \left(\frac{1}{a_r} - 1 \right).$$

$\Lambda = (\lambda_4, \ldots, \lambda_r)$: a tuple of pairwise distinct points on $\mathbb{P}^1(\mathbb{C}) \setminus \{\infty, 0, 1\}$.

Set $R_{A,\Lambda} := \mathbb{C}[X_1, \dots, X_r] / I_\Lambda$ where I_Λ is an ideal generated by

$$X_i^{a_i} - X_2^{a_2} + \lambda_i X_1^{a_1}, \quad i = 3, \dots, r \quad (\lambda_3 := 1).$$

Denote by L_A an abelian group defined as the quotient

$$L_A := \left(\bigoplus_{i=1}^{r} \mathbb{Z}\vec{X}_i \right) \Big/ \left(a_i\vec{X}_i - a_j\vec{X}_j ; 1 \leq i < j \leq r \right).$$

Definition 1

Let r, A and Λ be as above. Define a stack $\mathbb{P}^1_{A,\Lambda}$ by

$$\mathbb{P}^1_{A,\Lambda} := \left[\left(\mathrm{Spec}(R_{A,\Lambda}) \backslash \{0\} \right) / \mathrm{Spec}(\mathbb{C}L_A) \right],$$

which is called the *weighted projective line* of type (A, Λ).

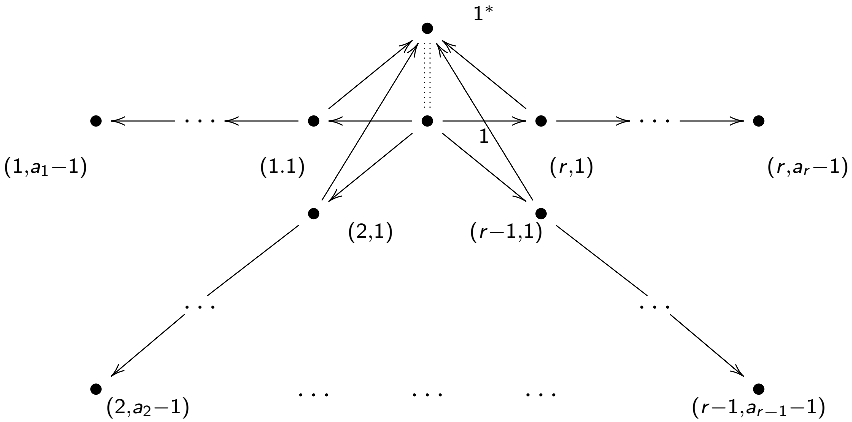
Theorem 2 (Geigle–Lenzing '87)

The category $D^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda})$ admits a full strongly exceptional collection. In particular, there are triangulated equivalences $D^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda}) \cong D^b(\mathbb{C}C_{A,\Lambda}) \cong D^b(\mathbb{C}\widetilde{\mathbb{T}}_{A,\Lambda})$ where $C_{A,\Lambda}$ is the Ringel's canonical algebra of type (A,Λ) and $\widetilde{\mathbb{T}}_{A,\Lambda}$ is a bound quiver in the next slide.

Theorem 3 (T '08)

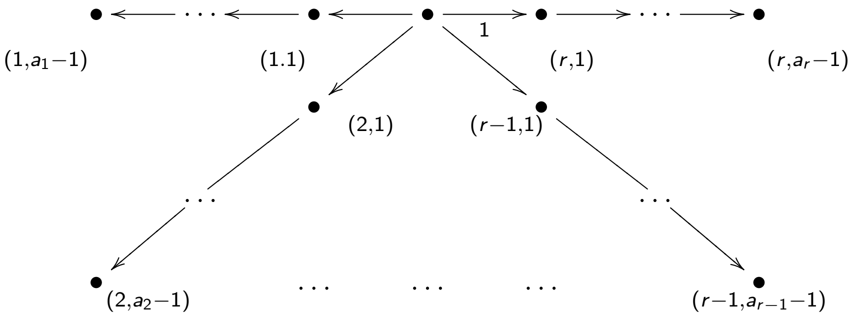
Assume that $A = (a_1, a_2, 2)$. We have a triangulated equivalence $D^b\mathrm{Fuk}^{\to}(f_A) \cong D^b(\mathbb{C}C_{A,\Lambda}) \cong D^b(\mathbb{C}\widetilde{\mathbb{T}}_A)$, where $f_A := x_1^{a_1} + x_2^{a_2} + x_3^2 - cx_1x_2x_3$ for some $c \in \mathbb{C} \setminus \{0\}$.

The bound quiver $\widetilde{\mathbb{T}}_{A,\Lambda}$



There are two relations from 1 to 1^* depending on Λ.

The star quiver $\mathbb{T}_A$



Root lattice

Denote by $K_0(\widetilde{\mathbb{T}}_{A,\Lambda})$ the *Grothendieck group* of $D^b(\mathbb{C}\widetilde{\mathbb{T}}_{A,\Lambda})$;

$$K_0(\widetilde{\mathbb{T}}_{A,\Lambda}) = \bigoplus_{v \in \widetilde{\mathbb{T}}_{A,\Lambda}} \mathbb{Z}\widetilde{\alpha}_v.$$

The *Euler form* $\chi : K_0(\widetilde{\mathbb{T}}_{A,\Lambda}) \times K_0(\widetilde{\mathbb{T}}_{A,\Lambda}) \longrightarrow \mathbb{Z}$ is defined by

$$\chi([\mathcal{M}],[\mathcal{N}]) := \sum_{p \in \mathbb{Z}} (-1)^p \dim_{\mathbb{C}} \mathrm{Hom}_{D^b(\mathbb{C}\widetilde{\mathbb{T}}_{A,\Lambda})}(\mathcal{M},\mathcal{N}[p]).$$

The *Cartan form* $I : K_0(\widetilde{\mathbb{T}}_{A,\Lambda}) \times K_0(\widetilde{\mathbb{T}}_{A,\Lambda}) \longrightarrow \mathbb{Z}$ is defined by

$$I(\gamma,\gamma') := \chi(\gamma,\gamma') + \chi(\gamma',\gamma).$$

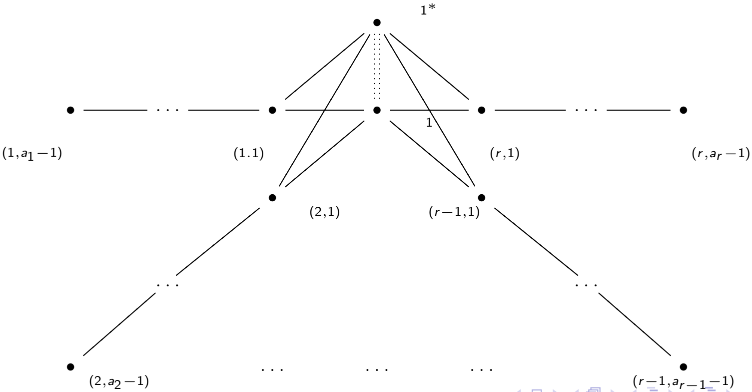
Define similarly for $\mathbb{T}_A$.

Coxeter–Dynkin diagram $\widetilde{T}_A$

$$I(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = 0 \iff \circ_v \qquad \circ_{v'}$$

$$I(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = -1 \iff \circ_v \text{———} \circ_{v'}$$

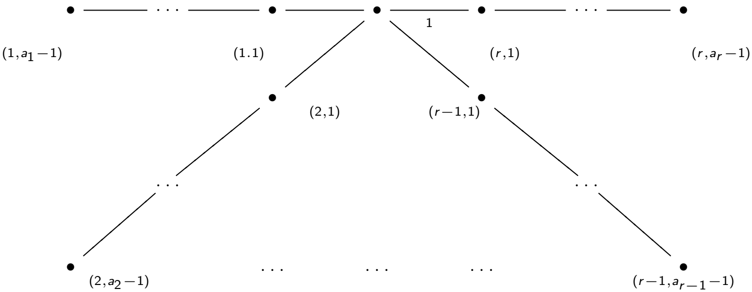
$$I(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = +2 \iff \circ_v \cdots\cdots \circ_{v'}$$



Coxeter–Dynkin diagram T_A

$$I(\alpha_v, \alpha_{v'}) = 0 \iff \circ_v \qquad \circ_{v'}$$

$$I(\alpha_v, \alpha_{v'}) = -1 \iff \circ_v \text{ ——— } \circ_{v'}$$



$\chi_A \neq 0 \iff$ the Cartan form I on $K_0(T_A)$ is non-degenerate.

Weyl groups

Define the *simple reflection* $\widetilde{r}_v$ on $K_0(\widetilde{\mathbb{T}}_{A,\Lambda})_{\mathbb{Q}}$ by

$$\widetilde{r}_v(\widetilde{\lambda}) := \widetilde{\lambda} - I(\widetilde{\lambda}, \widetilde{\alpha}_v)\widetilde{\alpha}_v, \quad \widetilde{\lambda} \in K_0(\widetilde{\mathbb{T}}_{A,\Lambda})_{\mathbb{Q}}.$$

Definition 4

The subgroup of $\mathrm{GL}(K_0(\widetilde{\mathbb{T}}_{A,\Lambda})_{\mathbb{Q}})$ generated by simple reflections is called the *Weyl group* associated to $\widetilde{\mathbb{T}}_{A,\Lambda}$ and is denoted by $W(\widetilde{T}_A)$ (since it depends only on the Coxeter–Dynkin diagram $\widetilde{T}_A$).

Remark 5

Actually, $W(\widetilde{T}_A)$ depends only on the (generalized) root system associated to $D^b(\mathbb{C}\widetilde{\mathbb{T}}_{A,\Lambda})$ ($\cong D^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda})$).

Define similarly the Weyl group $W(T_A)$ associated to $\mathbb{T}_A$.

Definition 6

1. If $\chi_A = 0$, then $W(\widetilde{T}_A)$ is called the *elliptic Weyl group*.
2. If $\chi_A < 0$, then $W(\widetilde{T}_A)$ is called the *cuspidal Weyl group*.

If $\chi_A > 0$, then $W(\widetilde{T}_A)$ is isomorphic to an *affine Weyl group*. Indeed, we have the following:

Theorem 7 (Shiraishi–T–Wada, in preparation)

There is a split-exact sequence

$$\{1\} \longrightarrow K_0(\mathbb{T}_A)/\mathrm{rad}(I) \xrightarrow{\widetilde{t}} W(\widetilde{T}_A) \xrightarrow{p} W(T_A) \longrightarrow \{1\},$$

where $\widetilde{t}$ *and* p *are defined by*

$$\widetilde{t}(\widetilde{\alpha}_v)(\widetilde{\lambda}) := \widetilde{t}_v(\widetilde{\lambda}) := \widetilde{\lambda} - I(\widetilde{\lambda}, \widetilde{\alpha}_v)(\widetilde{\alpha}_{1^*} - \widetilde{\alpha}_1),$$

$$p(\widetilde{r}_1) = p(\widetilde{r}_{1^*}) = r_1, \quad p(\widetilde{r}_v) = r_v, \quad v \in T_A.$$

Remark 8

The element $\widetilde{\alpha}_{1^*} - \widetilde{\alpha}_1 \in K_0(\widetilde{\mathbb{T}}_{A,\Lambda})$ belongs to the radical of the Cartan form I.

Lemma 9 (Key Lemma)

1. *We have* $\widetilde{t}_1 = \widetilde{r}_1 \widetilde{r}_{1^*}$, $\widetilde{t}_{(i,1)} = \widetilde{r}_{(i,1)} \widetilde{t}_1 \widetilde{r}_{(i,1)} \widetilde{t}_1^{-1}$ *and* $\widetilde{t}_{(i,j)} = \widetilde{r}_{(i,j)} \widetilde{t}_{(i,j-1)} \widetilde{r}_{(i,j)} \widetilde{t}_{(i,j-1)}^{-1}$.
2. *The elements* $\widetilde{t}_v$, $v \in T_A$ *commute with each other.*
3. *Let* N *be the smallest normal subgroup of* $W(\widetilde{T}_A)$ *containing* $\widetilde{t}_1$. *We have* $N = \mathrm{Ker}(p)$ *and* $\widetilde{t}_{(i,j)} \in N$ *for all* i, j.

If $\chi_A = 0$, then the group $W(T_A) \ltimes K_0(\mathbb{T}_A)$ is a central extension of the elliptic Weyl group $W(\widetilde{T}_A)$, which is called the hyperbolic extension of $W(\widetilde{T}_A)$ (Saito–Takebayashi).

Weyl groups as generalized Coxeter groups

Generalizing a result by Saito–Takebayashi for $\chi_A = 0$, we have

Theorem 10 (STW)

Let $W'(\widetilde{T}_A)$ be a group described by the generators $\{\widetilde{w}_v \mid v \in \widetilde{T}_A\}$ and the generalized Coxeter relations. Then we have $W'(\widetilde{T}_A) \cong W(T_A) \ltimes K_0(\mathbb{T}_A)$.

$$\widetilde{w}_v^2 = 1 \quad \text{for all} \quad v \in \widetilde{T}_A, \qquad \textbf{(W0)}$$

$$(\widetilde{w}_v \widetilde{w}_{v'})^2 = 1 \quad \text{if} \quad I(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = 0, \qquad \textbf{(W1.0)}$$

$$(\widetilde{w}_v \widetilde{w}_{v'})^3 = 1 \quad \text{if} \quad I(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = -1, \qquad \textbf{(W1.1)}$$

$$\widetilde{w}_{(i,1)} \widetilde{u}_1 \widetilde{w}_{(i,1)} \widetilde{u}_1 = \widetilde{u}_1 \widetilde{w}_{(i,1)} \widetilde{u}_1 \widetilde{w}_{(i,1)}, \qquad \textbf{(W2)}$$

$$\widetilde{w}_{(i,1)} \widetilde{u}_{(j,1)} = \widetilde{u}_{(j,1)} \widetilde{w}_{(i,1)}, \ \widetilde{w}_{(j,1)} \widetilde{u}_{(i,1)} = \widetilde{u}_{(i,1)} \widetilde{w}_{(j,1)} \ \text{if } i \neq j. \qquad \textbf{(W3)}$$

where $\widetilde{u}_1 := \widetilde{w}_1 \widetilde{w}_{1^*}$ and $\widetilde{u}_{(i,1)} := \widetilde{w}_{(i,1)} \widetilde{u}_1 \widetilde{w}_{(i,1)} \widetilde{u}_1^{-1}$.

The correspondence $(\widetilde{w}_v, \widetilde{u}_v) \Longleftrightarrow (\widetilde{r}_v, \widetilde{t}_v)$ gives the isomorphism.

Remark 11 (cf. Yamada '00)

Under **W0**, the relations **W2** and **W3** are equivalent to those introduced by Saito–Takebayashi ('97) for $\chi_A = 0$.

Orbit space

From now on, we assume that $\chi_A \neq 0$ to simplify some definitions. ($\chi_A \neq 0 \iff$ the Cartan matrix for T_A is non-degenerate.)

To T_A, one can associate a Kac–Moody Lie algebra and hence one can define a set of roots in a standard way.

Consider (the interior of) the complexified *Tits cone*

$$\mathcal{E}(T_A) := \{h \in \mathfrak{h}(T_A) \mid \langle \alpha, \mathrm{Im}(h) \rangle > 0 \text{ for } \alpha \in \Delta(T_A)^+_{im}\},$$

where $\Delta(T_A)^+_{im}$ is the subset of $\Delta(T_A)^+$ consisting of positive imaginary roots.

The group $W(\widetilde{T}_A) \cong W(T_A) \ltimes K_0(\mathbb{T}_A)$ acts properly discontinuously on $\mathcal{E}(\mathbb{T}_A)$.

Consider the complex manifold of dimension μ_A:

$$\widetilde{M}_A := \mathcal{E}(T_A)/W(\widetilde{T}_A) \times \begin{cases} \mathbb{C} & \chi_A > 0 \\ \mathbb{H} & \chi_A < 0 \end{cases}$$

where $\mathbb{H}$ is the complex upper half plane.

Conjecture 12

The space of Bridgeland's stability conditions on $D^b\mathrm{coh}(\mathbb{P}^1_{A,\Lambda})$ is isomorphic to $\widetilde{M}_A$.

Conjecture 13

There is a Frobenius structure on $\widetilde{M}_A$ isomorphic to the one constructed from the Gromov–Witten theory for $\mathbb{P}^1_{A,\Lambda}$.

Theorem 14 (Satake–T '08)

"Conjecture 13 for $\chi_A = 0$" is true.

Theorem 15 (Ishibashi–Shiraishi–T '12)

Conjecture 13 is true if $\chi_A > 0$.

Theorem 16 (Shiraishi–T, in preparation)

Conjecture 13 is true if the "property (P)" holds.

A PDF file is available at
http://frompde.sissa.it/workshop2013/talks/16Mon/Takahashi.pdf

Remark 17

One can check the "property (P)" easily for $\chi_A \geq 0$.
(Saito '90 for $\chi_A = 0$, Dubrovin–Zhang '98 for $\chi_A > 0$)

Fundamental groups of regular orbit spaces

Set

$$\widetilde{M}_A^{reg} := \mathcal{E}(T_A)^{reg} / W(\widetilde{T}_A) \times \begin{cases} \mathbb{C} & \chi_A > 0 \\ \mathbb{H} & \chi_A < 0 \end{cases}$$

where

$$\mathcal{E}(T_A)^{reg} := \mathcal{E}(T_A) \setminus \{\text{reflection hyperplanes}\}.$$

Want to understand the universal covering of $\widetilde{M}_A^{reg}$.

The universal covering of $\widetilde{M}_A^{reg}$ should be the space of Bridgeland's stability conditions on some triangulated category associated to the derived preprojective algebra (2-CY completion) of $\mathbb{C}\widetilde{\mathbb{T}}_{A,\Lambda}$.

The fundamental group of $\widetilde{M}_A^{reg}$ should determine the autoequivalence group of the derived category.

Artin groups

Definition 18 (cf. Yamada '00 when $\chi_A = 0$)

The *Artin group* $G(\widetilde{T}_A)$ is a group defined by the generators $\{\widetilde{g}_v \mid v \in \widetilde{T}_A\}$ and relations:

$$\widetilde{g}_v \widetilde{g}_{v'} = \widetilde{g}_{v'} \widetilde{g}_v \quad \text{if} \quad I(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = 0, \qquad \textbf{(A1.0)}$$

$$\widetilde{g}_v \widetilde{g}_{v'} \widetilde{g}_v = \widetilde{g}_{v'} \widetilde{g}_v \widetilde{g}_{v'} \quad \text{if} \quad I(\widetilde{\alpha}_v, \widetilde{\alpha}_{v'}) = -1, \qquad \textbf{(A1.1)}$$

$$\widetilde{g}_{(i,1)} \widetilde{s}_1 \widetilde{g}_{(i,1)} \widetilde{s}_1 = \widetilde{s}_1 \widetilde{g}_{(i,1)} \widetilde{s}_1 \widetilde{g}_{(i,1)}, \qquad \textbf{(A2)}$$

$$\widetilde{g}_{(i,1)} \widetilde{s}_{(j,1)} = \widetilde{s}_{(j,1)} \widetilde{g}_{(i,1)}, \ \widetilde{g}_{(j,1)} \widetilde{s}_{(i,1)} = \widetilde{s}_{(i,1)} \widetilde{g}_{(j,1)} \text{ if } i \neq j. \qquad \textbf{(A3)}$$

where $\widetilde{s}_1 := \widetilde{g}_1 \widetilde{g}_{1^*}$ and $\widetilde{s}_{(i,1)} := \widetilde{g}_{(i,1)} \widetilde{s}_1 \widetilde{g}_{(i,1)} \widetilde{s}_1^{-1}$.

Proposition 19

The correspondence $\widetilde{g}_v \mapsto \widetilde{w}_v$ *for* $v \in \widetilde{T}_A$ *induces an isomorphism*

$$G(\widetilde{T}_A) \Big/ \langle \widetilde{g}_v^2 \mid v \in \widetilde{T}_A \rangle \cong W(\widetilde{T}_A).$$

Remark 20

The elements $\widetilde{s}_{(i,j)}$ defined inductively by

$$\widetilde{s}_{(i,j)} := \widetilde{g}_{(i,j)} \widetilde{s}_{(i,j-1)} \widetilde{g}_{(i,j)} \widetilde{s}_{(i,j-1)}^{-1},$$

are mapped to $\widetilde{t}_{(i,j)}$. $(\widetilde{t}_v(\widetilde{\lambda}) := \widetilde{\lambda} - I(\widetilde{\lambda}, \widetilde{\alpha}_v)(\widetilde{\alpha}_{1^*} - \widetilde{\alpha}_1))$

Artin groups as fundamental groups

Generalizing Yamada's result for $\chi_A = 0$, we have

Theorem 21 (STW)

If $\chi_A \neq 0$, then there is an isomorphism

$$G(\widetilde{T}_A) \cong \pi_1\left(\widetilde{M}_A^{reg}, *\right) \cong \pi_1\left(\mathcal{E}(T_A)^{reg}/W(\widetilde{T}_A), *\right).$$

Key: Van der Lek's description of $\pi_1\left(\mathcal{E}(T_A)^{reg}/W(\widetilde{T}_A), *\right)$.

Lemma 22 (Van der Lek '83)

The group $\pi_1\left(\mathcal{E}(T_A)^{reg}/W(\widetilde{T}_A),*\right)$ *can be described by the generators* $\{g_v, s_v \mid v \in T_A\}$ *and the following relations*:

$$g_v g_{v'} = g_{v'} g_v \quad \text{if} \quad I(\alpha_v, \alpha_{v'}) = 0, \qquad \textbf{(A'1.0)}$$

$$g_v g_{v'} g_v = g_{v'} g_v g_{v'} \quad \text{if} \quad I(\alpha_v, \alpha_{v'}) = -1, \qquad \textbf{(A'1.1)}$$

$$g_v s_{v'} = s_{v'} g_v \quad \text{if} \quad I(\alpha_v, \alpha_{v'}) = 0, \qquad \textbf{(A'2)}$$

$$g_v s_{v'} g_v = s_{v'} s_v \quad \text{if} \quad I(\alpha_v, \alpha_{v'}) = -1. \qquad \textbf{(A'3)}$$

Roughly speaking, the correspondence $(\widetilde{g}_v, \widetilde{s}_v) \Longleftrightarrow (g_v, s_v)$ gives the isomorphism $G(\widetilde{T}_A) \cong \pi_1\left(\mathcal{E}(T_A)^{reg}/W(\widetilde{T}_A),*\right)$.

Thank you very much!

Happy 65th birthday Prof. Yamagata!