Tôhoku Math. J.
48 (1996), 561–573

# ISO-LENGTH-SPECTRAL PROBLEM FOR COMPLETE HYPERBOLIC SURFACES OF FINITE TYPE

KENTARO ITO

(Received July 10, 1995, revised December 21, 1995)

**Abstract.** We consider complete hyperbolic surfaces with punctures and holes. The aim of this paper is to show that there exist pairs of hyperbolic surfaces of any genus not less than 56 which are iso-length-spectral but not isometric, for arbitrarily fixed numbers of punctures and holes.

**1. Introduction.** In this paper, a *hyperbolic surface* is a complete orientable 2-dimensional Riemannian manifold with constant curvature $-1$. A hyperbolic surface of genus $g$ with $m$ punctures and $n$ holes and with no boundary is said to be of type $(g, m, n)$. Such surfaces are said to be of *finite type*. The *length spectrum* $\mathrm{Lsp}(M)$ of a hyperbolic surface $M$ of finite type is the collection of the lengths of closed geodesics on $M$ with multiplicities. As a set, $\mathrm{Lsp}(M)$ is discrete in $\boldsymbol{R}$, and each multiplicity is finite. Two hyperbolic surfaces $M_1$ and $M_2$ of finite type are said to be iso-length-spectral, or simply, *isospectral*, if $\mathrm{Lsp}(M_1)=\mathrm{Lsp}(M_2)$.

The following question is classical: "Does isospectral imply isometric?", and we refer to this problem as the iso-length-spectral problem. In the case of closed hyperbolic surfaces, the answer is negative. The first counterexamples were given by Vignéras [V] by arithmetic methods. Later, Sunada [S] found a more general approach to isospectral manifolds, and using this technique Buser [B1] and Brooks-Tse [BT] showed the existence of counterexamples for any genus $\geq 4$. However the problem is unsolved in the case of genus 2 and 3. On the other hand, Wolpert [W] showed that the answer is affirmative for generic hyperbolic surfaces, that is, the set of hyperbolic surfaces whose geometry is not uniquely determined by its length spectra is contained in a real proper subvariety in the Teichmüller space. For further information we refer the reader to [B2]. In this paper, we consider the case of non-compact hyperbolic surfaces of finite type. We denote by $\mathcal{M}(g, m, n)$ the moduli space of hyperbolic surfaces of type $(g, m, n)$, that is,

$$\mathcal{M}(g, m, n)=\{M : \text{hyperbolic surface of type } (g, m, n)\}/\sim .$$

Here $M\sim M'$ means that $M$ is isometric to $M'$. We denote the equivalence class of $M$ by $[M]$. We define the subset $\mathcal{N}(g, m, n)$ of $\mathcal{M}(g, m, n)$ as follows: an element $[M]$ of $\mathcal{M}(g, m, n)$ is contained in $\mathcal{N}(g, m, n)$ if there exists another element $[M']$ such that

1991 *Mathematics Subject Classification.* Primary 58G25.

$\mathrm{Lsp}(M) = \mathrm{Lsp}(M')$. The aim of this paper is to show the following:

THEOREM 1.1. *Let $g$, $m$ and $n$ be nonnegative integers. There exists a constant $c \in \mathbf{N}$ such that $\mathcal{N}(g, m, n)$ is nonempty for every $g \geq c$ and for any $m$ and $n$. Furhter, $c$ is not greater than* 56.

REMARK. On the other hand, it is known (cf. [H], [BS]) that $\mathcal{N}(1, 1, 0) = \mathcal{N}(1, 0, 1) = \mathcal{N}(0, m, n) = \varnothing$, where $m + n = 3$.

In the proof of the theorem, Sunada's construction for isospectral manifold and the Cayley graph play important roles, as we explaine in Section 2. Section 3 is devoted to the proof of the theorem.

ACKNOWLEDGMENTS. The author wishes to express his heartfelt gratitude to Professor Masahiko Taniguchi, Professor Hiroshige Shiga and Professor Toshiyuki Sugawa for their advice and encouragement. He is also grateful to the referee for careful reading of the previous versions of this paper and for valuable suggestions for improvement.

## 2. Sunada's Theorem and the Cayley Graph.

DEFINITION 2.1. Let $G$ be a finite group with two subgroups $H_1$ and $H_2$. We call $(G, H_1, H_2)$ a *Sunada triple* if the following conditions hold:
(a) $H_1$ is not conjugate to $H_2$ in $G$.
(b) For every conjugacy class $\{g\}_G$ in $G$ of $g \in G$,

$$\sharp(\{g\}_G \cap H_1) = \sharp(\{g\}_G \cap H_2) .$$

The Sunada triple in the next lemma is used in [BT] to construct isospectral pairs of compact hyperbolic surfaces of genus 4. This is also the only example of Sunada triples used in this paper.

LEMMA 2.2 (cf. [BT], [B2]). *Let $G = SL(3, 2)$, the group of all $3 \times 3$ matrices with coefficients in $\mathbf{Z}_2$ and with determinant* 1. *We consider two subgroups*

$$H_1 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \quad \text{and} \quad H_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\}$$

*with cardinalities $\sharp H_i = 24$ and with indices $[G : H_i] = 7$ $(i = 1, 2)$. Then $(G, H_1, H_2)$ is a Sunada triple. Further, let $A$, $B$ and $C$ be the following elements of $G$*:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

*Then the following holds*:

(a) $A^4 = B^3 = C^7 = I$ (*I is the unit matrix*), $C = ABA^{-1}B^{-1}$.

(b) *The powers* $I, C, \dots, C^6$ *form a set of right coset representatives for* $H_i \backslash G$ $(i = 1, 2)$.

(c) *A and B generate G*.

Now we describe the fundamental theorem to construct isospectral surfaces.

THEOREM 2.3 (Sunada [S], Buser [B2]). *Let* $(G, H_1, H_2)$ *be a Sunada triple and M a hyperbolic surface of finite type. Suppose that G acts on M as orientation-preserving isometries and the actions of* $H_1$ *and* $H_2$ *are free. Then the quotient surfaces* $H_1 \backslash M$ *and* $H_2 \backslash M$ *are isospectral*.

For a given finite group $G$, we will construct a hyperbolic surface, by combinatorial method, on which $G$ acts as orientation-preserving isometries. This method is explained in [B2]. To this end, we need the following:

DEFINITION 2.4. A graph consists of finitely many vertices and finitely many oriented edges joining two vertices.

(I) Let $G$ be a finite group with generators $A_1, \dots, A_n$ which are not necessarily pairwise distinct and may contain the unit element. The *Cayley graph* $\mathscr{G} = \mathscr{G}(G : A_1, \dots, A_n)$ of $G$ with respect to the generators $A_1, \dots, A_n$ is defined as follows:

(a) There exists a bijective map $w$ from $G$ to the vertex set of $\mathscr{G}$.

(b) For any pair $(g, g') \in G \times G$, let $A_{n_1}, \dots, A_{n_k}$ be the subsequence in the sequence $A_1, \dots, A_n$ of all generators $A$ satisfying $g' = gA$. Then the two vertices $w(g)$ and $w(g')$ are joined by exactly $k$ edges oriented from $w(g)$ to $w(g')$. Each of these edges is said to be of *type* $A_{n_i}$ for $i = 1, \dots, k$.

(II) For each subgroup $H$ of $G$, $H \backslash G$ denotes the right quotient set and $[g]$ denotes the right coset of $g \in G$. We define the *quotient graph* $H \backslash \mathscr{G} = H \backslash \mathscr{G}(G : A_1, \dots, A_n)$ as follows:

(a′) There exists a bijective map $\bar{w}$ from $H \backslash G$ to the vertex set of $H \backslash \mathscr{G}$.

(b′) For any pair $([g], [g']) \in (H \backslash G) \times (H \backslash G)$, let $A_{n_1}, \dots, A_{n_k}$ be the subsequence in the sequence $A_1, \dots, A_n$ of all generators $A$ satisfying $[g'] = [gA]$. Then the two vertices $\bar{w}(g)$ and $\bar{w}(g')$ are joined by exactly $k$ edges oriented from $\bar{w}(g)$ to $\bar{w}(g')$. Each of these edges are also said to be of *type* $A_{n_i}$ for $i = 1, \dots, k$.

EXAMPLE 2.5 (cf. [B2]). *Let* $(G, H_1, H_2)$ and $A, B, C \in G$ be as in Lemma 2.2. The quotient graphs $H_i \backslash \mathscr{G}(G : A, B)$ $(i = 1, 2)$ are given in Figure 2.1. We put the suffix $k$ for the vertex corresponding to the right coset $[C^k]$.

Let $\Delta = \{(x, y) \in \boldsymbol{R}^2 \mid x^2 + y^2 < 1\}$ be the unit disk with the hyperbolic metric $ds^2 = 4(dx^2 + dy^2)/(1 - (x^2 + y^2))^2$. A geodesic in $\Delta$ is a half-circle meeting $\partial \Delta = \{(x, y) \in \boldsymbol{R}^2 \mid x^2 + y^2 = 1\}$ perpendicularly at both ends.

Let $c$ be a geodesic arc or a closed geodesic on a hyperbolic surface. We denote by

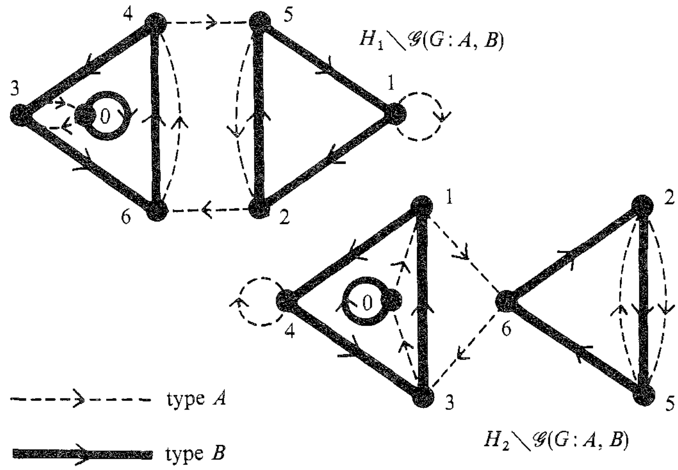


FIGURE 2.1.

$L(c)$ the hyperbolic length of $c$.

DEFINITION 2.6. A *geodesic polygon* $P$ is a simply connected closed subset of $\Delta$ whose relative boundary consists of finitely many geodesic arcs. A *vertex* of $P$ is either an intersection point in $\Delta$ of boundary geodesics or a connected component of $\bar{P}\cap\partial\Delta$, where $\bar{P}$ is the closure of $P$ in $\boldsymbol{R}^2$. Moreover, a vertex is said to be an *i-vertex*, *p-vertex* and *h-vertex* if it is contained in $\Delta$, contained in $\partial\Delta$ as a point, and contained in $\partial\Delta$ as an interval, respectively. A boundary geodesic arc joining two vertices of $P$ is called a *side* of $P$.

Let $G$ be a finite group. Now we will explain how to paste some copies of a geodesic polygon with respect to a Cayley graph of $G$ to obtain a hyperbolic surface on which $G$ acts as orientation-preserving isometries. First, we take a *fundamental polygon* $P$ of $G$, which is a geodesic polygon with $2n$ sides equipped with the following properties.

(1) No sides connect two vertices contained in $\partial\Delta$.

(2) The sides of $P$ have a division into pairs $\bigcup_{i=1}^{n}\{e_i, \bar{e}_i\}$ satisfying $L(e_i)=L(\bar{e}_i)$ for every $i$. Moreover, if $L(e_i)=L(\bar{e}_i)=\infty$, the end points of $e_i$ and $\bar{e}_i$ contained in $\partial\Delta$ are contained in the same vertex of $P$.

(3) There is a map $\Psi:\{1,\dots,n\}\to G$, $i\mapsto\Psi(i)=:A_i$, such that $\{A_i\}_{i=1}^{n}$ generate $G$.

(4) If $L(e_i)<\infty$, we parameterize $e_i$ and $\bar{e}_i$ by arc length with positive orientation as parts of the boundary of $P$: $e_i(s)$, $\bar{e}_i(s)$, $s\in[0, L(e_i)]$. If $L(e_i)=\infty$, one of the end points of $e_i$ and $\bar{e}_i$ are contained in $\Delta$ from (1). Therefore we adopt these points as start points and parameterize $e_i$ and $\bar{e}_i$ by arc length: $e_i(s)$, $\bar{e}_i(s)$, $s\in\boldsymbol{R}_+\cup\{0\}$.

Next we prepare $\sharp G$ copies $\{P_g\}_{g\in G}$ of the fundamental polygon $P$ of $G$ taken as above. The sides $e_i$ and $\bar{e}_i$ of $P_g$ are denoted by $e_i[P_g]$ and $\bar{e}_i[P_g]$, respectively. We glue the copies together according to the Cayley graph $\mathscr{G}=\mathscr{G}(G:A_1,\dots,A_n)$, that is, if

$g'=gA_i$ ($g, g' \in G$), then $P_g$ is pasted to $P_{g'}$ via the following identification of sides ($1 \le i \le n$):

$$\begin{cases} e_i[P_g](s)=\bar{e}_i[P_{g'}](L(e_i)-s) & (s\in[0, L(e_i)])\,, \quad \text{if} \quad L(e_i)<\infty\,. \\ e_i[P_g](s)=\bar{e}_i[P_{g'}](s) & (s\in \boldsymbol{R}_+\cup\{0\})\,, \quad \text{if} \quad L(e_i)=\infty\,. \end{cases}$$

Note that we paste the copies together preserving each orientation. The resulting surface is denoted by $(P, \mathscr{G})$. Since $A_1, \dots, A_n$ generate $G$, the Cayley graph is connected and hence the surface $(P, \mathscr{G})$ is connected.

DEFINITION 2.7. A point of $M=(P, \mathscr{G})$ is called a *vertex* of $M$ if it corresponds to an $i$-vertex of some $P_g$. Further, the *angle* at a vertex of $M$ is defined as the sum of the angles which come together at the vertex. A *singular point* of $M$ is a vertex of $M$ whose angle is different from $2\pi$.

The surface $M=(P, \mathscr{G})$ has a smooth hyperbolic structure ($M$ is said to be *smooth*, for short) if and only if $M$ has no singular point. If $M=(P, \mathscr{G})$ is smooth, $G$ naturally acts on $M$ as orientation-preserving isometries; indeed, for each $g\in G$, the natural isometry $P_h \to P_{gh}$ for every $h\in G$ can be extended to an isometry on $M$.

Similarly, for every subgroup $H$ of $G$, we can obtain a surface $M'=(P, H\backslash\mathscr{G})$ by gluing together $\sharp(H\backslash G)$ copies $\{P_{[g]}\}_{[g]\in H\backslash G}$ of $P$ with respect to the quotient graph $H\backslash\mathscr{G}=H\backslash\mathscr{G}(G: A_1, \dots, A_n)$. Now the following lemma is immediately.

LEMMA 2.8. *Let $G$ be a finite group and $H$ a subgroup of $G$. Suppose that $M=(P, \mathscr{G})$ and $M'=(P, H\backslash\mathscr{G})$ are smooth for a suitable fundamental polygon $P$ of $G$. Then $M'$ is isometric to the quotient surface $H\backslash M$.*

Let $(G, H_1, H_2)$ be a Sunada triple. Suppose that $M=(P, \mathscr{G})$, $M_1=(P, H_1\backslash\mathscr{G})$ and $M_2=(P, H_2\backslash\mathscr{G})$ are smooth for a suitable fundamental polygon $P$ of $G$. Then $M_1$ and $M_2$ are isospectral by the above lemma and Theorem 2.3.

**3. The Proof of Theorem 1.1.** We take $(G, H_1, H_2)$ as in Lemma 2.2. Let $h, k, p, q, s$ and $t$ be integers such that $h\in N$, $k, p, s\in N\cup\{0\}$ and $q, t\in\{0, \dots, 6\}$. For every $h, k, p, q, s$ and $t$ as above, a fundamental polygon $P$ of $G$ satisfying the following conditions can be constructed:

(i) $M=(P, \mathscr{G})$, $M_1=(P, H_1\backslash\mathscr{G})$ and $M_2=(P, H_2\backslash\mathscr{G})$ are smooth, and hence $M_1$ and $M_2$ are isospectral.

(ii) $M_1$ and $M_2$ are of type $(g, m, n)$; where

$$\begin{cases} g=7h+3(k+q+t)+1\,, \\ m=7p+q\,, \\ n=7s+t\,. \end{cases} \tag{3.1}$$

(iii) $M_1$ is not isometric to $M_2$.

In (3.1), $p, q, s$ and $t$ are uniquely determined for every pair $m, n\in N\cup\{0\}$. Since $7h+3k$ exhaust all integers greater than 18 and since $3(q+t)+1\le 37$, every triple $(g, m, n)$

($g \geq 56$; $m, n \in N \cup \{0\}$) can be written as in the form (3.1). Therefore the assertion of Theorem 1.1 is proved. In the following, we construct a fundamental polygon $P$ of $G$ satisfying the condition (i), (ii) and (iii) above.

3.1. Fundamental construction of $P$. By abuse of notation, the length of a geodesic $c$ is also denoted by $c$ whenever there is no confusion. Put $N=4h+k+p+q+s+t+1$. Take arbitrarily $x \in \boldsymbol{R}_+$, which will be determined later in the argument. Let $\mathscr{D}$ be a geodesic triangle whose two edges have the same length $x$ with inner angle $2\pi/N$ (see Figure 3.1). Let $y$ be the remaining edge of $\mathscr{D}$ and $\varphi$ a included angle between $x$ and $y$. Note that $y$ and $\varphi$ go to $\infty$ and 0, respectively, when $x$ goes to $\infty$. Further we construct geodesic polygons $\mathscr{D}^{(1)}$, $\mathscr{D}^{(2)}$, $\mathscr{D}'$ and $\mathscr{D}^{(3)}$ as in Figure 3.1. By elementary arguments in hyperbolic geometry, it is easily seen that $\theta_1$, $\theta_2$, $\theta_3$ and $a$ are uniquely determined by $y$ (and hence by $x$). For example, $\sinh a = \tanh(y/2)$ and $\cos\theta_3 = \tanh^2(y/2)$. Therefore $\theta_1$, $\theta_2$ and $\theta_3$ go to 0 when $x$ goes to $\infty$.

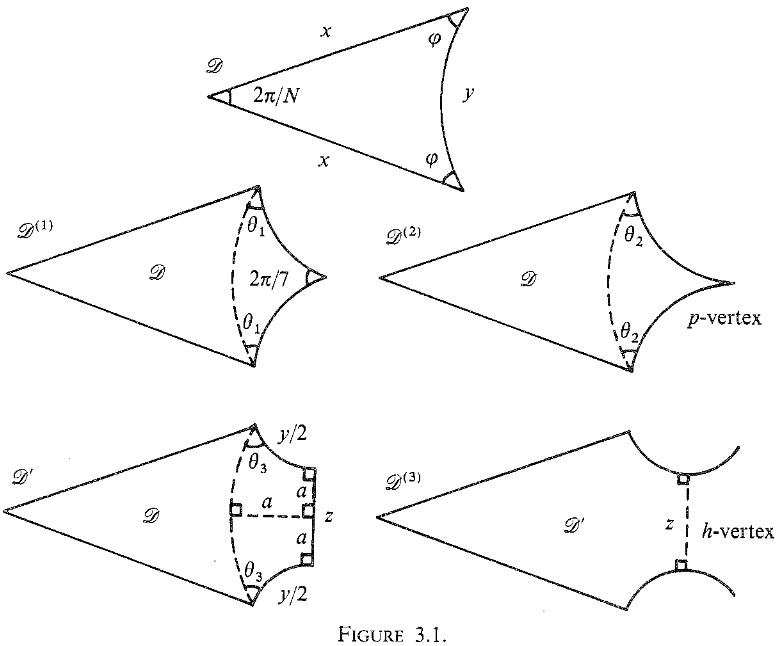


FIGURE 3.1.

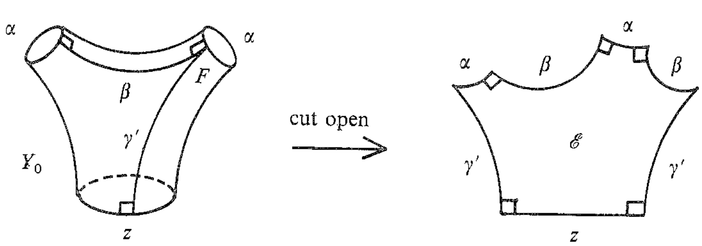


FIGURE 3.2.

Now take arbitrary $\alpha \in \boldsymbol{R}_+$ to obtain a $Y$-piece $Y_0$ whose boundary geodesics have lengths $\alpha$, $\alpha$ and $z$ (see Figure 3.2). Here a *Y-piece* is a connected hyperbolic planar surface whose boundary consists of three closed geodesics. Let $\beta$ be a unique simple geodesic arc perpendicular to both $\alpha$, and let $F$ be one of the end points of $\beta$. We take a simple geodesic arc $\gamma'$ which connect $F$ and $z$ and are perpendicular to $z$. We cut $Y_0$ open along $\beta$ and $\gamma'$ to obtain a geodesic heptagon $\mathscr{E}$ (see Figure 3.2). We obtain a geodesic octagon $\mathscr{D}^{(4)}$ by pasting $\mathscr{E}$ and $\mathscr{D}'$ together along $z$.

Now we prepare the following $N$ geodesic polygons;

$$\begin{aligned} 4h \text{ copies of } \mathscr{D} &: \mathscr{D}_1, \ldots, \mathscr{D}_{4h}, \\ k \text{ copies of } \mathscr{D}^{(1)} &: \mathscr{D}_1^{(1)}, \ldots, \mathscr{D}_k^{(1)}, \\ p+q \text{ copies of } \mathscr{D}^{(2)} &: \mathscr{D}_1^{(2)}, \ldots, \mathscr{D}_{p+q}^{(2)}, \\ s+t \text{ copies of } \mathscr{D}^{(3)} &: \mathscr{D}_1^{(3)}, \ldots, \mathscr{D}_{s+t}^{(3)}, \end{aligned}$$

and $\mathscr{D}^{(4)}$. Paste them together along $x$ in this order to obtain a geodesic polygon $P$ (see Figure 3.3). We name the sides of $P$ in the negative direction of the boundary of $P$:

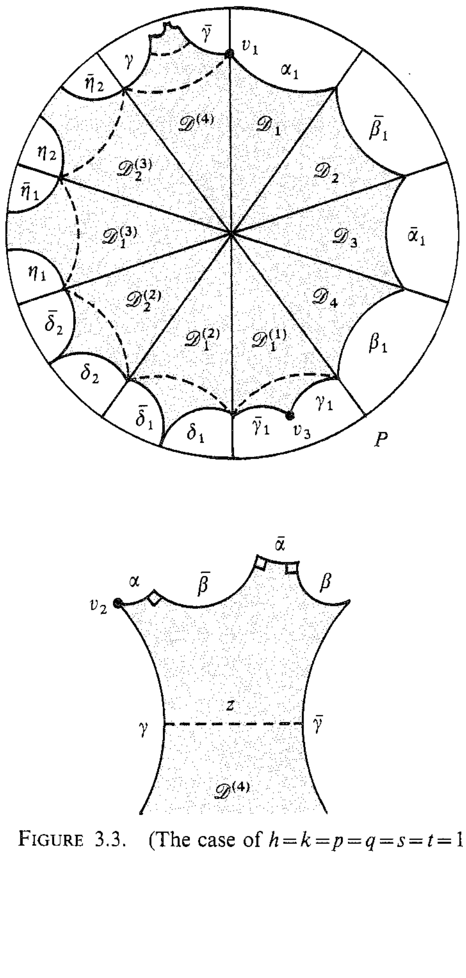


FIGURE 3.3. (The case of $h=k=p=q=s=t=1$.)

$$\alpha_1, \bar{\beta}_1, \bar{\alpha}_1, \beta_1, \ldots, \alpha_h, \bar{\beta}_h, \bar{\alpha}_h, \beta_h,$$
$$\gamma_1, \bar{\gamma}_1, \ldots, \gamma_k, \bar{\gamma}_k,$$
$$\delta_1, \bar{\delta}_1, \ldots, \delta_{p+q}, \bar{\delta}_{p+q},$$
$$\eta_1, \bar{\eta}_1, \ldots, \eta_{s+t}, \bar{\eta}_{s+t},$$
$$\gamma, \alpha, \bar{\beta}, \bar{\alpha}, \beta, \bar{\gamma}.$$

Now we obtain a division of the sides of $P$ into pairs $\{\alpha_1, \bar{\alpha}_1\} \cup \cdots \cup \{\alpha, \bar{\alpha}\} \cup \{\beta, \bar{\beta}\} \cup \{\gamma, \bar{\gamma}\}$ satisfying $L(\alpha_1)=L(\bar{\alpha}_1)$ and so on.

3.2. Correspondence between the sides of $P$ and the generators of $G$. We need the following lemma, which is immediately proved by induction on $l$.

LEMMA 3.1. *For any $l \in \boldsymbol{N}$ and for any $b \in \boldsymbol{Z}$ such that $b \not\equiv 0 \pmod 7$, there exist $\mu_1, \ldots, \mu_l \in \boldsymbol{Z}$ which satisfy*
(1) $\mu_i \not\equiv 0 \pmod 7 \quad (1 \le i \le l)$,
(2) $\sum_{i=1}^{l} \mu_i \equiv b \pmod 7$.

Let $\mu_1, \ldots, \mu_{k+q+t}$ be integers satisfying the conditions in the above lemma for the case $l=k+q+t$ and $b=-1$. For each side of $P$, we assign an element of $G$ as follows:

$$\alpha_1 \mapsto I,\ \beta_1 \mapsto I, \ldots,\ \alpha_h \mapsto I,\ \beta_h \mapsto I,\ \gamma_1 \mapsto C^{\mu_1}, \ldots,\ \gamma_k \mapsto C^{\mu_k},$$
$$\delta_1 \mapsto I, \ldots,\ \delta_p \mapsto I,\ \delta_{p+1} \mapsto C^{\mu_{k+1}}, \ldots,\ \delta_{p+q} \mapsto C^{\mu_{k+q}},$$
$$\eta_1 \mapsto I, \ldots,\ \eta_s \mapsto I,\ \eta_{s+1} \mapsto C^{\mu_{k+q+1}}, \ldots,\ \eta_{s+t} \mapsto C^{\mu_{k+q+t}},$$
$$\gamma \mapsto C,\ \alpha \mapsto A,\ \beta \mapsto B.$$

Here $I$ is the unit matrix and $A$, $B$ and $C$ are elements of $G$ as in Lemma 2.2. We remark that all assigned elements of $G$ generate $G$ because $A$ and $B$ alone generate $G$. Now we obtain a fundamental polygon $P$ of $G$.

3.3. $M$ is smooth. We obtain the surface $M$ by gluing together $\sharp G$ copies $\{P_g\}_{g \in G}$ of $P$ with respect to the Cayley graph $\mathscr{G}=\mathscr{G}(G: I, \ldots, I, C^{\mu_1}, \ldots, C^{\mu_{k+q+t}}, C, A, B)$. Here we show that $M$ has no singular point. Take the vertex $v_1$ of $P$ between $\alpha_1$ and $\bar{\gamma}$, $v_2$ between $\alpha$ and $\gamma$, and $v_{2+i}$ between $\gamma_i$ and $\bar{\gamma}_i$ for $1 \le i \le k$, respectively (see Figure 3.3). The vertex in $P_g$ corresponding to $v_i$ is denoted by $v_i(g)$ $(1 \le i \le k+2)$. We now define an equivalence relation in the set of all $i$-vertices of $\{P_g\}_{g \in G}$: two $i$-vertices are equivalent if they are identified in $M$. The equivalence class of an $i$-vertex $v$ is denote by $[v]$. Obviously, the equivalence classes are in one-to-one correspondence with the vertices of $M$.

LEMMA 3.2. *The set of all i-vartices of $\{P_g\}_{g \in G}$ has the following decomposition*:

$$\left(\bigcup_{g \in G} [v_1(g)]\right) \cup \left(\bigcup_{g \in G} [v_2(g)]\right) \cup \left(\bigcup_{\substack{i=3, \cdots, k+2 \\ g\langle C\rangle \in G/\langle C\rangle}} [v_i(g)]\right) \tag{3.2}$$

*where $\langle C \rangle$ is the cyclic group generated by $C$.*

PROOF. First we determine the equivalence class $[v_1(g)]$ for some $g \in G$. The right-hand side $\alpha_1$ of the $i$-vertex $v_1(g)$ of the copy $P_g$ is glued to the side $\bar{\alpha}_1$ of $P_{gI} = P_g$. Therefore the $i$-vertex $v_1(g)$ is equivalent to the $i$-vertex of $P_g$ between $\bar{\alpha}_1$ and $\beta_1$. We continue this procedure to obtain the $i$-vertices equivalent to $v_1(g)$ (see Figure 3.4). This procedure finishes when the side $\gamma$ of $P_{gC^{\mu_{k+q+t}}}$ is glued to the side $\bar{\gamma}$ of $P_g$; note that

$$\left( \prod_{j=1}^{k+q+t} C^{\mu_j} \right) C = I$$

holds. Similarly the equivalence classes $[v_i(g)]$, $2 \le i \le k+2$, $g \in G$ are determined in view of the facts that $ABA^{-1}B^{-1} = C$ and that $C^{\mu_j}$ has order 7 for every $j \in \{1, \ldots, k\}$. Note

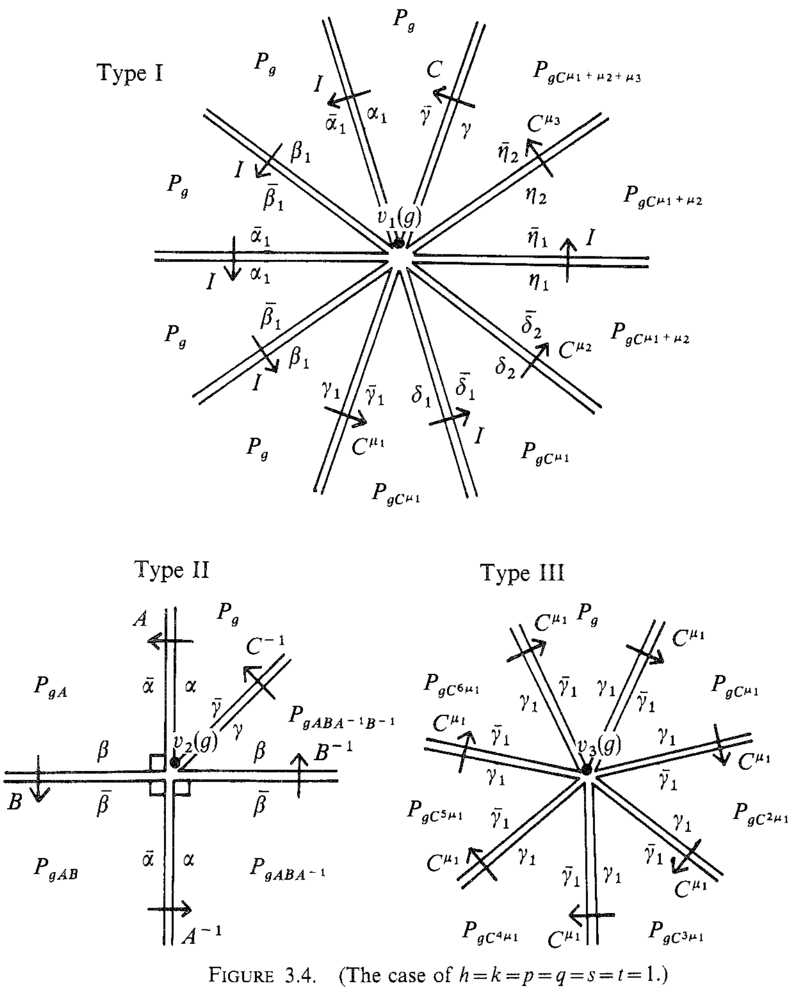


FIGURE 3.4. (The case of $h = k = p = q = s = t = 1$.)

that $\sharp[v_1(g)]=N$, $\sharp[v_2(g)]=5$ and $\sharp[v_i(g)]=7$ $(i=3,\dots,k+2)$ for any $g\in G$. One can easily see that $v_i(g)$ is equivalent to $v_j(g')$ if and only if $i=j\in\{1,2\}$ and $g=g'$, or $i=j\in\{3,\dots,k+2\}$ and $g'\in g\langle C\rangle$. Therefore the equivalence classes appearing in (3.2) are pairwise disjoint.

On the other hand, an easy calculation reveals that these equivalence classes exhaust the set of all $i$-vertices of $\{P_g\}_{g\in G}$, because the latter set has cardinality $\sharp G(N+k+5)$. □

The equivalent classes $[v_1(g)]$ $(g\in G)$ are said to be of type I, $[v_2(g)]$ are of type II and $[v_i(g)]$ $(3\le i\le k+2)$ are of type III, respectively. A vertex of $M$ corresponding to an equivalence class of type I is also said to be of type I, and so on. We show that the angles at the vertices of $M$ are always equal to $2\pi$. It is true for a vertex of types II and III. By a suitable choice of $x$, it is also true for a vertex of type I, as can be seen from the following observation. The angle is greater than $2\pi$ when $x$ goes to 0 because $N\ge 5$. On the other hand, $\varphi$, $\theta_1$, $\theta_2$ and $\theta_3$ go to 0 and hence the angle goes to 0 when $x$ goes to $\infty$.

3.4. $M_1$ and $M_2$ are smooth. Recall that we obtain the quotient surfaces $M_i=H_i\diagdown M$ $(i=1,2)$ by pasting together seven copies $\{P_{[g]}\}_{[g]\in H_i\diagdown G}=\{P_{[C^j]}\}_{j=0}^{6}$ of $P$ with respect to the quotient graph $H_i\diagdown\mathscr{G}$. In this and the next subsections, we only deal with the surface $M_1$, but the same argument works for $M_2$. Here we prove that for any $\eta\in H_1\diagdown\{I\}$ the action of $\eta$ on $M$ has no fixed point. Indeed, otherwise a fixed point must be a vertex of $M$. Therefore it suffices to show that $H_1\cap G_v=\{I\}$ for every vertex $v$ of $M$, where $G_v$ is the stabilizer of $v$. By Lemma 3.2 every vertex $v$ corresponds to some equivalence class $[v_i(g)]$. Under this correspondence, $\tau\in G_v$ if and only if $v_i(\tau g)$ is equivalent to $v_i(g)$. Therefore, $\tau=I$ and hence $G_v=\{I\}$, if $v$ is of type I or II. If $v$ is of type III, $\tau\in G_v$ if and only if $\tau g\in g\langle C\rangle$. In this case, $G_v=g\langle C\rangle g^{-1}$. Because the cardinality of $\langle C\rangle$ is 7, so is $g\langle C\rangle g^{-1}$. Since $\sharp G_v=7$ and since $\sharp H_1=24$, we obtain $H_1\cap G_v=\{\mathrm{I}\}$.

3.5. Types of $M_1$ and $M_2$. We determine the number of punctures of $M_1$. Observe that each $p$-vertex of $\mathscr{D}_i^{(2)}\subset P_{[C^j]}$, $1\le i\le p$, $0\le j\le 6$, form a puncture by itself. On the other hand, the seven $p$-vertices of $\{\mathscr{D}_i^{(2)}\subset P_{[C^j]}\}_{j=0}^{6}$ form a puncture for every $p+1\le i\le p+q$. Therefore $M_1$ has $7p+q$ punctures. Similarly, $M_1$ has $7s+t$ holes. Then an elementary calculation of the Euler number shows that the genus of $M_1$ is equal to $7h+3(k+q+t)+1$.

3.6. $M_1$ and $M_2$ are not isometric. Now we explain how some modification, if necessary, on $P$ makes $M_1$ and $M_2$ non-isometric. We consider the quotient surface $M_0=G\diagdown M$ with singular points which we obtain by pasting together the sides of $P$ with respect to the quotient graph $G\diagdown\mathscr{G}$. We remark that $\alpha$ and $\bar{\alpha}$ (resp. $\beta$ and $\bar{\beta}$) yield a simple closed geodesic, which we also denote by $\alpha$ (resp. $\beta$). One can see, from $[G:H_i]=7$ $(i=1,2)$, that $M_1$ and $M_2$ are seven sheeted covering surfaces of $M_0$. There are $k$ singular points on $M_0$, all of whose angles are $2\pi/7$. A curve $c$ on $M_0$ traversing singular points are also said to be a geodesic on $M_0$ if $c$ is a piecewise geodesic on the surface $M_0 \setminus \{\text{singular points of } M_0\}$ and if the both two angles made by two geodesic segments of $c$ at each singular point are equal to $\pi/7$. Note that every geodesic on $M_0$ in this sense can be lifted to a geodesic on $M_1$ and $M_2$.

From now on, $i$ stands for 1 or 2. Here we explain how the pattern of linkage of the lifts of $\alpha$ on $M_i$ can be seen from the pattern of linkage of the edges of type $B$ on the graph $H_i \setminus \mathscr{G}$. First observe that each edge $\tilde{\alpha}$ of $P_{[g]}$ on $M_i$ is a lift of $\alpha$. Let $V([g])$ be a vertex of $P_{[g]}$ between $\tilde{\alpha}$ and $\bar{\beta}$. Then vertices $\bar{w}([g])$ and $\bar{w}([g'])$ are joined by an edge of type $B$ oriented from $\bar{w}([g])$ to $\bar{w}([g'])$ on the graph $H_i \setminus \mathscr{G}$ if and only if the edge $\beta$ of $P_{[g]}$ and the edge $\bar{\beta}$ *of* $P_{[g']}$ are glued together on $M_i$, that is, the vertices $V([g])$ and $V([g'])$ are joined by the edge $\tilde{\alpha}$ (of $P_{[g]}$) on $M_i$.

From the above observation and from Figure 2.1, one can see that the seven lifts of $\alpha$ on $M_i$ yield three simple closed geodesics $\tilde{\alpha}_i$, $\tilde{\alpha}'_i$ and $\tilde{\alpha}''_i$. Here we assume that $L(\tilde{\alpha}_i) = \alpha$ and $L(\tilde{\alpha}'_i) = L(\tilde{\alpha}''_i) = 3\alpha$. If we take $\alpha$ sufficiently small, $\alpha$ is a unique prime closed geodesic on $M_0$ which has length $\leq 3\alpha$. Then $\tilde{\alpha}_i$, $\tilde{\alpha}'_i$ and $\tilde{\alpha}''_i$ are characterized as the prime closed geodesic of length $\leq 3\alpha$ on $M_i$. Similarly, the lifts of $\beta$ on $M_i$ correspond to the edges of type $A$ of the graph $H_i \setminus \mathscr{G}$. We look at the two lifts of $\beta$ on $M_i$ which connect $\tilde{\alpha}_i$ with the union $\tilde{\alpha}'_i \cup \tilde{\alpha}''_i$. They are common perpendiculars between $\tilde{\alpha}_i$ and $\tilde{\alpha}'_i \cup \tilde{\alpha}''_i$. On $M_1$ the two perpendiculars form a closed geodesic. On the other hand, it is not the case on $M_2$. Therefore, $M_1$ and $M_2$ are not isometric, if the following condition $(*)$ holds:

$(*)$ The closed geodesic $\beta$ is a unique geodesic arc of length $\beta$ on $M_0$ which is perpendicular to $\alpha$ at both ends.

Indeed, we have the next lemma.

LEMMA 3.3. *A suitable Fenchel-Nielsen deformation of* $M_0$ *satisfies the condition* $(*)$.

Here a Fenchel-Nielsen deformation is defined as follows.

DEFINITION 3.4. Let $S$ be a hyperbolic surface which may have singular points. Let $c$ be a simple closed geodesic on $M_0$ traversing no singular point. The *Fenchel-Nielsen deformation* of $S$ with respect to $c$ is defined as follows:

First cut $S$ along $c$ to obtain a (possibly disconnected) surface with geodesic boundary. Each of the two sides of the cut is equipped with the orientation induced by that of $S$. Next rotate one side by length $t$ relative to the other side in the negative direction. Then glue the sides in their new position to obtain a new hyperbolic surface $S_t$.

PROOF OF LEMMA 3.3. We consider the Fenchel-Nielsen deformation $(M_0)_t$ of $M_0$ with respect to $z$. Let $\mathscr{A}_t$ be the set of all geodesics on $(M_0)_t$ which are perpendicular to $\alpha$ at both ends. Let $\mathscr{B}_t$ be a subset of $\mathscr{A}_t$ all of whose elements intersect $z$, and $\mathscr{C}_t$ a subset of $\mathscr{A}_t$ all of whose elements have length $\beta$. The assertion of Lemma 3.3 follows if there exists a real number $t$ such that $\mathscr{C}_t = \{\beta\}$. Since one can easily see that $\mathscr{C}_t \setminus \{\beta\} \subset \mathscr{B}_t$, the condition $\mathscr{C}_t = \{\beta\}$ is equivalent to the condition $\mathscr{B}_t \cap \mathscr{C}_t = \varnothing$.

Now take a branched Riemannian covering map $\psi_0 : \Delta \to M_0$. Fix a lift $\alpha_0$ on $\Delta$

of $\alpha$. For a lift $\tilde{z}$ of $z$, we decompose $\Delta$ into a disjoint union $\Delta_1 \sqcup \tilde{z} \sqcup \Delta_2$, where the half space $\Delta_1$ contains $\alpha_0$. We define a bijective map $\varphi_{\tilde{z},t}: \Delta \to \Delta$ ($t \in \boldsymbol{R}$) as follows: $\varphi_{\tilde{z},t}(\xi) = \xi$ for $\xi \in \Delta_1$, and $\varphi_{\tilde{z},t}(\xi)$ for $\xi \in \Delta_2$ is the image of $\xi$ of when we slide $\Delta_2$ relative to $\Delta_1$ by length $t$. The sliding-direction is defined in the same manner as in Definition 3.4. The map $\varphi_{\tilde{z},t}$ is not yet defined on $\tilde{z}$, but one can define it suitably as the following arguments work well. Next, for any $\xi \in \Delta$, let $\{\tilde{z}_1, \dots, \tilde{z}_j\}$ be the list of all lifts of $z$ which separate $\alpha_0$ from $\xi$. Moreover, we assume that $\tilde{z}_1$ separates $\alpha_0$ from $\tilde{z}_2, \dots, \tilde{z}_j, \xi$, while $\tilde{z}_2$ separates $\alpha_0$ and $\tilde{z}_1$ from $\tilde{z}_3, \dots, \tilde{z}_j, \xi$, and so on. Needless to say, the list may be empty. Now we define a map on $\Delta$ as $\varphi_t(\xi) := \varphi_{\tilde{z}_1,t} \circ \cdots \circ \varphi_{\tilde{z}_j,t}(\xi)$. Then $\varphi_t: \Delta \to \Delta$ is a bijective map. With the trivial map $f_t: M_0 \to (M_0)_t$, we now obtain a branched Riemannian covering map $\psi_t := f_t \circ \psi_0 \circ (\varphi_t)^{-1}: \Delta \to (M_0)_t$ so that we have the following commutative diagram:

$$\begin{array}{ccc} \Delta & \xrightarrow{\varphi_t} & \Delta \\ {\scriptstyle \psi_0}\downarrow & & \downarrow{\scriptstyle \psi_t} \\ M_0 & \xrightarrow[f_t]{} & (M_0)_t \, . \end{array}$$

With the above preparation, we define a bijective map $\Phi_t: \mathscr{B}_0 \to \mathscr{B}_t$ as follows: For an element $l \in \mathscr{B}_0$, we take a lift $\tilde{l}$ with the starting point on $\alpha_0$. Then $\tilde{l}$ is a unique common perpendicular between $\alpha_0$ and another lift $\alpha'_0$ of $\alpha$. Note that $\tilde{l}$ intersects distinct lifts of $z$ even in number. Let $\tilde{l}_t$ be the unique common perpendicular between $\alpha_0$ and $\varphi_t(\alpha'_0)$. Then we define $\Phi_t(l) := \psi_t(\tilde{l}_t)$. Observe that this map is well-defined (i.e., the definition does not depend on the choice of the lift $\tilde{l}$) and that this map is bijective. It is easily seen that the length $L(\Phi_t(l))$ depends real analytically on $t$. Because $L(\Phi_t(l)) \to +\infty$ when $t \to \pm\infty$, it is not a constant. Moreover, we remark that the subset $\{l \in \mathscr{B}_t \mid L(l) \leq R\}$ of $\mathscr{B}_t$ is a finite set for any constant $R \in \boldsymbol{R}_+$, in view of the discreteness of the covering transformation group of $\psi_t$. Therefore, $\mathscr{C}_t = \{\beta\}$ holds for a generic real number $t$; where "generic" means that the set of all real numbers $t$ such that $\mathscr{C}_t \setminus \{\beta\} \neq \varnothing$ is a discrete subset of $\boldsymbol{R}$. $\square$

Now we take a real number $t$ satisfying $|t| < z$ and $\mathscr{C}_t = \{\beta\}$. We modify the construction of $P$ as follows: Paste $\mathscr{E}$ and $\mathscr{D}'$ along $z$ after sliding them by length $t$ in a suitable direction. This new fundamental polygon $P$ satisfies the conditions (i), (ii) and (iii) above. Therefore we complete the proof of Theorem 1.1.

## References

[BT] R. Brooks and R. Tse, Isospectral surfaces of small genus, Nagoya Math. J. 107 (1987), 13–24.

[B1] P. Buser, Isospectral Riemann surfaces, Ann. Inst. Fourier (Grenoble) 36 (1986), 167–192.

[B2] P. Buser, Geometry and spectra of compact Riemann surfaces, Birkhäuser-Verlarg, Boston-Basel-Berlin, 1992.

[BS] P. BUSER AND K.-D. SEMMLER, The geometry and spectrum of the one holed torus, Comment. Math. Helv. 63 (1988), 259–274.

[H] A. HAAS, Length spectra as moduli for hyperbolic surfaces, Duke Math. J. 52 (1985), 923–934.

[S] T. SUNADA, Riemannian coverings and isospectral manifolds, Ann. of Math. (2) 121 (1985), 169–186.

[V] M.-F. VIGNERÁS, Variétés riemaniennes isospectrales et non isométriques, Ann. of Math. (2) 112 (1980), 21–32.

[W] S. A. WOLPERT, The length spectra as moduli for compact Riemann surfaces, Ann. of Math. (2) 109 (1979), 323–351.

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
OH-OKAYAMA, MEGURO
TOKYO 152
JAPAN

*E-mail address*: itoken@math.titech.ac.jp