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Grafting and components of quasi-fuchsian projective structures

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Abstract

We give an expository account of our results in [Ito00] and [Itoa] on the bumping and self-bumping of components of quasi-fuchsian projective structures from the view point of [Itob] on continuity of grafting maps at boundary groups.

1. Introduction

We consider the space of projective structures $P(S)$ on a closed surface S of hyperbolic type and its open subset $Q(S)$ consists of projective structures with quasi-fuchsian holonomy. It is known that $Q(S)$ have infinitely many connected components. The aim of this note is to outline and explain how components of $Q(S)$ lies in $P(S)$, especially how these components bump or self-bump. Here we say that components $\mathcal{Q}$, $\mathcal{Q}'$ of $Q(S)$ *bump* if they have intersecting closures and that a component $\mathcal{Q}$ *self-bumps* if there is a point $\Sigma \in \partial\mathcal{Q}$ such that $U \cap \mathcal{Q}$ is disconnected for any sufficiently small neighborhood U of Σ. Studying how $Q(S)$ lies in $P(S)$ is closely related to studying how the quasi-fuchsian space $\mathcal{QF} = \mathcal{QF}(S)$ lies in the representation space $R(S)$, where $R(S)$ is the set of conjugacy classes of representations $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ with the algebraic topology and $\mathcal{QF} \subset R(S)$ is the subspace of faithful representations with quasi-fuchsian images.

Now let Γ be a geometrically finite Kleinian group with non-trivial space $AH(\Gamma)$ of conjugacy classes of discrete faithful representations $\Gamma \to \mathrm{PSL}_2(\mathbb{C})$. Then the bumping of components of the interior of $AH(\Gamma)$ are characterized by the topological data of the quotient manifold $\mathbb{H}^3/\Gamma$ by Anderson, Canary and McCullough [AC96], [ACM00]. In our setting, it is known that the quasi-fuchsian space $\mathcal{QF}$ is the interior of the space $AH(S)$ of discrete faithful representations and that $\mathcal{QF}$ consists of exactly one connected component. Nevertheless, McMullen [McM98] showed that $\mathcal{QF}$ self-bumps by using projective structures and ideas of Anderson and Canary [AC96]. In his argument, he used the fact that the local structure of the boundary of $\mathcal{QF}$ is equal to that of $Q(S) = hol^{-1}(\mathcal{QF})$ because the holonomy map $hol : P(S) \to R(S)$, assigning a projective structure to its holonomy representation, is a local homeomorphism (see [Hej75]). After that, Bromberg and Holt [BH01] showed that each component of the interior of $AH(\Gamma)$ self-bumps for more general Kleinian groups Γ without using projective structures. We refer the reader to a survey written by Canary [Can03] for further information on the bumping and self-bumping of deformation spaces of Kleinian groups.

In this note, we push ahead with the observation in [McM98] and studied the bumping and self-bumping of components of $Q(S) = hol^{-1}(\mathscr{QF})$. By Goldman's grafting theorem (Theorem C in [Gol87]), the set of components of $Q(S)$ is in one-to-one correspondence with the set $\mathscr{ML}_{\mathbb{N}}$ of integral measured laminations on S. Thus we obtain a decomposition $\bigsqcup_{\lambda \in \mathscr{ML}_{\mathbb{N}}} \mathscr{Q}_\lambda$ of $Q(S)$, where $\mathscr{Q}_\lambda$ is the connected component of $Q(S)$ associated to $\lambda \in \mathscr{ML}_{\mathbb{N}}$. Especially, the component $\mathscr{Q}_0$ for zero-lamination $0 \in \mathscr{ML}_{\mathbb{N}}$ consists of all quasi-fuchsian projective structures with injective developing map. We know that the map $hol|_{\mathscr{Q}_\lambda} : \mathscr{Q}_\lambda \to \mathscr{QF}$ is biholomorphic for each $\lambda \in \mathscr{ML}_{\mathbb{N}}$ and let $\Psi_\lambda : \mathscr{QF} \to \mathscr{Q}_\lambda$ denote the univalent local branch of hol^{-1}, which is called the *grafting map* for λ. In §3, we discuss conditions under which the map Ψ_λ is extended continuously to a boundary point of $\mathscr{QF}$. Recall that Bers' simultaneous uniformization gives a bijection $B : T(S) \times T(S) \to \mathscr{QF}$, where $T(S)$ denote the Teichmüller space of S. Suppose that a sequence $\rho_n = B(X_n, Y_n) \in \mathscr{QF}$ converges to $\rho_\infty \in \partial\mathscr{QF}$. Then we say that the convergence $\rho_n \to \rho_\infty$ is *standard* if there exists a compact subset K of $T(S)$ which contains all X_n or all Y_n; otherwise it is *exotic*. Then we have the following:

Theorem 1.1 ([Itob])**.** *For every $\lambda \in \mathscr{ML}_{\mathbb{N}}$, the grafting map $\Psi_\lambda : \mathscr{QF} \to P(S)$ takes every standardly convergent sequence to a convergent sequence in $\widehat{P}(S)$, where $\widehat{P}(S) = P(S) \cup \{\infty\}$ denote the one-point compactification of $P(S)$.*

On the other hand, in §4, we outline the following result, which is obtained by making use of exotically convergent sequences constructed by Anderson and Canary [AC96] and McMullen [McM98]:

Theorem 1.2 ([Ito00, Itoa])**.** (i) *Any two components of $Q(S)$ bump,*

(ii) *Every component of $Q(S)$ except for $\mathscr{Q}_0$ self-bumps, and*

(iii) *For any $n \in \mathbb{N}$, there exist n-components of $Q(S)$ which bump simultaneously.*

The same argument as in Theorem 1.2 reveals that the grafting map $\Psi_\lambda : \mathscr{QF} \to P(S)$ does not extend continuously to $\partial\mathscr{QF}$; see Theorem 5.1. Then Theorem 1.1 implies that only exotically convergent sequences cause this non-continuity and the bumping of distinct components of $Q(S)$.

2. Preliminaries

2.1. Quasi-fuchsian space

We let $R(S)$ denote the space of conjugacy classes $[\rho]$ of representations $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ with non-abelian image $\rho(\pi_1(S))$. (For simplicity, we denote $[\rho]$ by ρ if there is no confusion.) The space $R(S)$ is endowed with the algebraic topology and is known to be a complex manifold (see for example [MT98]). *Quasi-fuchsian space* $\mathscr{QF}$ is the subset of $R(S)$ of conjugacy classes of faithful representations whose images are quasi-fuchsian groups. Then $\mathscr{QF}$ is open, connected and contractible in $R(S)$. Let $\rho \in \mathscr{QF}$ with quasi-fuchsian image $\Gamma = \rho(\pi_1(S))$. Then the region of discontinuity Ω_Γ of Γ decomposes into two invariant components Ω_Γ^+ and Ω_Γ^-, and the representation ρ determines a pair $(\Omega_\Gamma^+/\Gamma, \Omega_\Gamma^-/\Gamma)$ in $T(S) \times T(\overline{S})$. Here $T(S)$ is the Teichmüller space of S, and $\overline{S}$ denotes S with orientation reversed. On the contrary, it was shown by Bers [Ber60] that each pair $(X, \overline{Y}) \in T(S) \times T(\overline{S})$ has its unique simultaneous uniformization $\rho = B(X, \overline{Y}) \in \mathscr{QF}$. Thus we have a parameterization

$$B : T(S) \times T(\overline{S}) \to \mathscr{QF}$$

of $\mathscr{QF}$. We define *vertical* and *horizontal Bers slices* in $\mathscr{QF}$ by $B_X = \{B(X, \overline{Y}) : \overline{Y} \in T(\overline{S})\}$ and $B_{\overline{Y}} = \{B(X, \overline{Y}) : X \in T(S)\}$. Bers showed that both B_X and $B_{\overline{Y}}$ are precompact in $R(S)$, whose frontiers are denoted by ∂B_X and $\partial B_{\overline{Y}}$.

2.2. Space of projective structures

A projective structure on S is a (G,X)-structure, where X is a Riemann sphere $\widehat{\mathbb{C}}$ and $G = \mathrm{PSL}_2(\mathbb{C})$ is the group of projective automorphisms of $\widehat{\mathbb{C}}$. We let $P(S)$ denote the space of marked projective structures on S. A projective structure $\Sigma \in P(S)$ determines its underlying conformal structure $\pi(\Sigma) \in T(S)$. It is known that $P(S)$ is a holomorphic affine bundle over $T(S)$ with the projection $\pi : P(S) \to T(S)$ and that each fiber $\pi^{-1}(X)$ for $X \in T(S)$ can be identified with the space of holomorphic quadratic differentials on X. As an usual (G,X)-structure, a projective structure $\Sigma \in P(S)$ determines a pair (f_Σ, ρ_Σ) of a developing map $f_\Sigma : \tilde{S} \to \widehat{\mathbb{C}}$ and a holonomy representation $\rho_\Sigma : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, which is uniquely determined up to $\mathrm{PSL}_2(\mathbb{C})$. We now define the *holonomy map*

$$hol : P(S) \to R(S)$$

by $\Sigma \mapsto [\rho_\Sigma]$. Hejhal [Hej75] showed that the map *hol* is a local homeomorphism and Earle [Ear81] and Hubbard [Hub81] independently showed that the map is holomorphic.

In this note, we are mainly concerned with the subset $Q(S) = hol^{-1}(\mathscr{QF})$ of $P(S)$. An element of $Q(S)$ is said to be *standard* if its developing map is injective; otherwise it is *exotic*. We denote by $\mathscr{Q}_0 \subset Q(S)$ the subset of standard projective structures. For a quasi-fuchsian representation $\rho = B(X, \overline{Y})$ with $\Gamma = \rho(\pi_1(S))$, the quotient surface $\Sigma = \Omega_\Gamma^+/\Gamma$ is regarded as a standard projective structure on S with bijective developing map $f_\Sigma : \widetilde{\Sigma} \to \Omega_\Gamma^+$, with holonomy representation $\rho_\Sigma = \rho$, and with underlying complex structure $X \in T(S)$. Let

$$\Psi_0 : \mathscr{QF} \to \mathscr{Q}_0$$

be the map defined by the correspondence $\rho \mapsto \Omega_\Gamma^+/\Gamma$ as above. Then the map Ψ_0 turns out to be a univalent local branch of hol^{-1} onto the connected component $\mathscr{Q}_0$ of $Q(S)$, which is called the *standard component*. It is known by Bers that every Bers slice $B_X \subset \mathscr{QF}$ is embedded by the map Ψ_0 into a bounded domain $\Psi_0(B_X)$ of the fiber $\pi^{-1}(X) \subset P(S)$.

2.3. Grafting

We let $\mathscr{ML}_{\mathbb{N}} = \mathscr{ML}_{\mathbb{N}}(S)$ denote the set of integral measured laminations, or the set of formal summation $\sum_{i=1}^{l} k_i c_i$ of homotopically distinct simple closed curves c_i on S with positive integer k_i weights. A *realization* $\widehat{\lambda}$ of $\lambda = \sum_{i=1}^{l} k_i c_i \in \mathscr{ML}_{\mathbb{N}}$ is a disjoint union of simple closed curves which realize each weighted simple closed curve $k_i c_i$ by k_i parallel disjoint simple closed curves homotopic to c_i. For two element $\lambda, \mu \in \mathscr{ML}_{\mathbb{N}}$, the geometric intersection number is denoted by $i(\lambda, \mu)$.

Let λ be non-zero element of $\mathscr{ML}_{\mathbb{N}}$. We now explain haw to obtain the grafting map

$$\mathrm{Gr}_\lambda : \mathscr{Q}_0 \to P(S),$$

which satisfies $hol \circ \mathrm{Gr}_\lambda \equiv hol$ on $\mathscr{Q}_0$. In this note, we shall give two equivalent definitions of grafting operation, the one given here is as usual, and the one given in §3.1 is due to Bromberg [Bro02]. We assume that λ is a simple closed curve c of weight one for simplicity and fix our notation as follows:

Notation 2.1. Let $\rho \in \mathscr{QF}$ with $\Gamma = \rho(\pi_1(S))$. Then we have projective structures $\Sigma = \Omega_\Gamma^+/\Gamma = \Psi_0(\rho)$ on S and $\Sigma^- = \Omega_\Gamma^-/\Gamma$ on $\bar{S}$. Suppose that $c^+ \subset \Sigma$ and $c^- \subset \Sigma^-$ are simple closed curves associated to $c \subset S$ and that $\gamma \in \Gamma \cong \pi_1(S)$ is a representative of the homotopy class of c. Let $\tilde{c}^+ \subset \Omega_\Gamma^+$ and $\tilde{c}^- \subset \Omega_\Gamma^-$ be the $\langle\gamma\rangle$-invariant lifts of $c^+ \subset \Sigma$ and $c^- \subset \Sigma^-$, respectively.

Definition 2.2 (Grafting I)**.** We adopt Notation 2.1. Let A_c be a cylinder $(\widehat{\mathbb{C}} - \tilde{c}^+)/\langle\gamma\rangle$ equipped with a projective structure induced from that of $\widehat{\mathbb{C}}$. Then the *grafting* $\mathrm{Gr}_c(\Sigma)$ is obtained by cutting Σ along c and inserting A_c at the cut locus without twisting.

For general $\lambda \in \mathscr{ML}_{\mathbb{N}}$, the grafting $\Sigma' = \mathrm{Gr}_\lambda(\Sigma)$ of Σ along λ is also defined by linearity. Then it is important to note that $\rho_\Sigma = \rho_{\Sigma'}$ is always satisfied and that the pull-back $\Lambda_{\Sigma'} := f_{\Sigma'}^{-1}(\Lambda_\Gamma)/\pi_1(\Sigma') \subset \Sigma'$ of the limit set Λ_Γ of the holonomy image $\Gamma = \rho_\Sigma(\pi_1(S)) = \rho_{\Sigma'}(\pi_1(S))$ is a realization of 2λ (see [Gol87]). Since $hol \circ \mathrm{Gr}_\lambda \equiv hol$ is satisfied on $\mathscr{Q}_0$, the grafting map Gr_λ takes $\mathscr{Q}_0$ biholomorphically onto the connected component $\mathscr{Q}_\lambda := \mathrm{Gr}_\lambda(\mathscr{Q}_0)$ of $Q(S)$. Thus we have a univalent local branch

$$\Psi_\lambda := \mathrm{Gr}_\lambda \circ \Psi_0 : \mathscr{QF} \to \mathscr{Q}_\lambda$$

of hol^{-1}. By abuse of terminology, we also call $\Psi_\lambda(\rho)$ the *grafting* of ρ along λ and Ψ_λ the *grafting map* for λ. By Goldman's grafting theorem [Gol87] below, we obtain the decomposition $\bigsqcup_{\lambda \in \mathscr{ML}_{\mathbb{N}}} \mathscr{Q}_\lambda$ of $Q(S)$ into its connected components.

Theorem 2.3 (Goldman [Gol87]). *For every $\rho \in \mathscr{QF}$, we have*

$$hol^{-1}(\rho) = \{\Psi_\lambda(\rho) : \lambda \in \mathscr{ML}_{\mathbb{N}}\}.$$

2.4. Sequences of quasi-fuchsian representations

Now we introduce the notion of standard and exotic convergence for a sequence $\rho_n \in \mathscr{QF}$ tending to a limit $\rho_\infty \in \partial\mathscr{QF}$.

Definition 2.4 (Standard and exotic convergence). Suppose that a sequence $\rho_n = B(X_n, \overline{Y_n}) \in \mathscr{QF}$ converges to $\rho_\infty \in \partial\mathscr{QF}$. Then the convergence $\rho_n \to \rho_\infty$ is said to be *standard* if (i) there exist compact set $K \subset T(S)$ such that $\{X_n\} \subset K$, or (ii) there exist compact set $\overline{K} \subset T(\overline{S})$ such that $\{\overline{Y_n}\} \subset \overline{K}$. Otherwise, we say that the convergence is *exotic*.

We let $\partial^+\mathscr{QF}$ and $\partial^-\mathscr{QF}$ denote the subsets of $\partial\mathscr{QF}$ of standard convergent limits of type (i) and (ii) respectively, and set $\partial^\pm\mathscr{QF} = \partial^+\mathscr{QF} \sqcup \partial^-\mathscr{QF}$. An element $\rho \in \partial\mathscr{QF}$ is called a *b-group* if the image $\Gamma = \rho(\pi_1(S))$ is a *b*-group, i.e., there exists exactly one simply connected invariant component of Ω_Γ. Then we remark that the set $\partial^\pm\mathscr{QF}$ is equals to the set of all *b*-groups in $\partial\mathscr{QF}$ and that the following hold (see [Itob]):

$$\partial^+\mathscr{QF} = \bigsqcup_{X \in T(S)} \partial B_X, \quad \partial^-\mathscr{QF} = \bigsqcup_{\overline{Y} \in T(\overline{S})} \partial B_{\overline{Y}}.$$

As we will explain in §2.5, there exists a sequence in $\mathscr{QF}$ which converges exotically into $\partial^\pm\mathscr{QF}$. On the other hand, the set $\partial\mathscr{QF} - \partial^\pm\mathscr{QF}$ is not empty, for instance, it contains a limit of a sequence which appears in Thurston's double limit theorem.

As a consequence of the following lemma, we see that the map $hol|_{\overline{\mathscr{Q}_0}} : \overline{\mathscr{Q}_0} \to \mathscr{QF} \sqcup \partial^+\mathscr{QF}$ is bijective, where $\overline{\mathscr{Q}_0}$ is the closure of $\mathscr{Q}_0$ in $P(S)$.

Lemma 2.5. *The map $\Psi_0 : \mathscr{QF} \to \mathscr{Q}_0$ takes every standardly convergent sequence $\rho_n \in \mathscr{QF}$ with $\lim \rho_n \in \partial^+\mathscr{QF}$ to a convergent sequence $\Sigma_n \in \mathscr{Q}_0$ with $\lim \Sigma_n \in \partial\mathscr{Q}_0$.*

Proof. Suppose that a sequence $\rho_n = B(X_n, \overline{Y_n}) \in \mathscr{QF}$ converges standardly to some $\rho_\infty \in \partial^+\mathscr{QF}$. Then we first show that $X_n \to X$ and that $\rho_\infty \in \partial B_X$ for some $X \in T(S)$. In fact, there exists a subsequence of $\{X_n\}$, denoted by the same symbol, which converges to some $X \in T(S)$. Now we take a new sequence $\rho'_n = B(X, \overline{Y_n})$ in B_X. Then the sequence $\{\rho'_n\}$ also converges to ρ_∞, since maximal dilatations of quasiconformal automorphisms of $\widehat{\mathbb{C}}$ conjugating ρ_n to ρ'_n tend to 1 as $n \to \infty$. This implies that $\rho_\infty \in \partial B_X$. Since $\partial B_{X_1} \cap \partial B_{X_2} = \emptyset$ if $X_1 \neq X_2$, $X_n \to X$ without passing to a subsequence. Therefore, any accumulation point $\Sigma_\infty \in \partial\mathscr{Q}_0$ of the precompact set $\{\Sigma_n \in \pi^{-1}(X_n) : n \in \mathbb{N}\}$ is contained in $\pi^{-1}(X)$. From the injectivity of the map $hol|_{\pi^{-1}(X)} : \pi^{-1}(X) \to R(S)$ (see [Kra71]), we see that Σ_∞ is uniquely determined by the condition $hol(\Sigma_\infty) = \rho_\infty$, and thus $\Sigma_n \to \Sigma_\infty$ without passing to a subsequence. □

We collect in Table 1 below the equivalent conditions with standard/exotic convergence of quasi-fuchsian representations, as a consequence of Lemma 2.5 and [Ito00, Proposition 3.4] (see also [McM98, Appendix A]). The situation in which we consider is as follows: suppose that a sequence $\rho_n \in \mathcal{QF}$ converges to $\rho_\infty \in \partial^+\mathcal{QF}$ and that the sequence $\Gamma_n = \rho_n(\pi_1(S))$ converges geometrically to a Kleinian group $\widehat{\Gamma}$, which contains the algebraic limit $\Gamma_\infty = \rho_\infty(\pi_1(S))$. Let Σ_∞ be the unique projective structure in $\partial\mathcal{Q}_0$ with holonomy ρ_∞ and let $\Phi : U \to P(S)$, $\rho_\infty \mapsto \Sigma_\infty$ be a univalent local branch of hol^{-1} which is defined on a neighborhood U of ρ_∞. Then the sequence $\Sigma_n = \Phi(\rho_n)$ converges to $\Sigma_\infty = \Phi(\rho_\infty)$. We denote by $\Omega^+_{\Gamma_\infty}$ the unique invariant component of the region of discontinuity Ω_{Γ_∞} of Γ_∞, which is equals to the image of the injective developing map $f_{\Sigma_\infty} : \widetilde{\Sigma}_\infty \to \widehat{\mathbb{C}}$. In this situation, all conditions in the same line in Table 1 are equivalent.

Table 1: Equivalent conditions with standard/exotic convergence.

$\rho_n \to \rho_\infty$: standard	$\rho_n \to \rho_\infty$: exotic
$\Omega^+_{\Gamma_\infty} \cap \Lambda(\widehat{\Gamma}) = \emptyset$	$\Omega^+_{\Gamma_\infty} \cap \Lambda(\widehat{\Gamma}) \neq \emptyset$
Σ_n are standard $(n \gg 0)$	Σ_n are exotic $(n \gg 0)$

2.5. ACM-sequences

We will explain a typical example of a sequence $\rho_n \in \mathcal{QF}$ converging exotically to $\rho_\infty \in \partial^+\mathcal{QF}$, which we call an ACM-sequence named after Anderson-Canary [AC96] and McMullen [McM98]. We remark that all the known such a sequence is basically obtained by their technique. We give here a brief survey and refer to [McM98] or [Ito00] for more details. Let c be a simple closed curve on S and let $\tau = \tau_c$ be the Dehn twist along c. Then an *ACM-sequence* in $\mathcal{QF}$ for c with a starting point $(X, \overline{Y}) \in T(S) \times T(\overline{S})$ is defined by

$$\rho_n = B(\tau^n X, \tau^{2n} \overline{Y}) \quad (n \in \mathbb{Z}),$$

which is known to converge to some $\rho_\infty \in \partial^+\mathcal{QF}$. For example, the convergence can be observed as follows: let us consider sequences $\eta_n = B(X, \tau^n \overline{Y})$ in B_X and $\eta'_n = B(\tau^{-n} X, \overline{Y})$ in $B_{\overline{Y}}$. Then $\rho_n = \eta_n \circ \tau_*^{-n}$ and $\eta'_n = \eta_n \circ \tau_*^n$ hold for all n, where τ_* is the group automorphism of $\pi_1(S)$ induced by τ. Since both sequences η_n and η'_n converge up to subsequence, the same argument in [KT90] (see also [Bro97]) reveals that the sequence ρ_n also converges up to subsequence. Moreover, we know that the sequences η_n, η'_n and ρ_n converge without passing to a subsequence from the Dehn filling construction; see [AC96] and [McM98]. Similarly, we obtain a convergent sequence

$$\rho_n = B(\tau^{kn} X, \tau^{(k+1)n} \overline{Y}) = \eta_n \circ \tau_*^{-kn} \quad (n \in \mathbb{Z})$$

for each $k \in \mathbb{Z}$, which converges standardly to its limit if and only if $k = 0, -1$ and whose limit is in $\partial^+ \mathscr{QF}$ if $k \geq 0$ and in $\partial^- \mathscr{QF}$ if $k \leq -1$.

We now define an ACM-sequence for general element $\lambda = \sum_{i=1}^{l} k_i c_i \in \mathscr{ML}_{\mathbb{N}}$, whose *support* is denoted by $\underline{\lambda} = \sqcup_i c_i$. The Dehn twist for λ is defined by $\tau_\lambda = \tau_{c_1}^{k_1} \circ \cdots \circ \tau_{c_l}^{k_l}$, and thus $\tau_{\underline{\lambda}} \circ \tau_\lambda = \tau_{c_1}^{k_1+1} \circ \cdots \circ \tau_{c_l}^{k_l+1}$. Then an ACM-sequence for λ is defined by

$$\rho_n = B(\tau_\lambda^n X, (\tau_{\underline{\lambda}} \circ \tau_\lambda)^n \overline{Y}) \quad (n \in \mathbb{Z}), \tag{2.1}$$

which converges exotically to some $\rho_\infty \in \partial^+ \mathscr{QF}$. (An ACM-sequence converging to some $\rho_\infty \in \partial^- \mathscr{QF}$ is also obtained by the same way, but we do not discuss such a sequence in this note.) We now recall some basic fact of the ACM-sequence $\{\rho_n\}$ as in (2,1): by passing to a subsequence if necessary, we may assume that the sequence $\Gamma_n = \rho_n(\pi_1(S))$ of quasi-fuchsian groups converges geometrically to a Kleinian group $\widehat{\Gamma}$, whose Kleinian manifold $N_{\widehat{\Gamma}}$ is homeomorphic to $S \times [-1,1] - \bigcup_i (c_i \times \{0\})$ and have conformal boundary $X \sqcup \overline{Y}$ up to marking. Then the algebraic limit $\Gamma_\infty = \rho_\infty(\pi_1(S))$ is a proper subgroup of $\widehat{\Gamma}$ which carried by an immersed surface $\varphi(S) \subset N_{\widehat{\Gamma}}$. Here the immersion $\varphi : S \to N_{\widehat{\Gamma}}$ up to homotopy is obtained from the identity map $S \to S \times \{-1\}$ by adding annulus which wraps around $c_i \times \{0\}$ for k_i-times for every i (see Figure 1). Then one can see that Γ_∞ is a b-group and that $\rho_\infty \in \partial^+ \mathscr{QF}$. On the contrary, an ACM-sequence as in (2.1) is obtained from a Kleinian manifold $N_{\widehat{\Gamma}}$ and an immersion $\varphi : S \to N_{\widehat{\Gamma}}$ as above by simultaneous $(1,n)$-Dehn filling at every rank-two cusps of $N_{\widehat{\Gamma}}$.

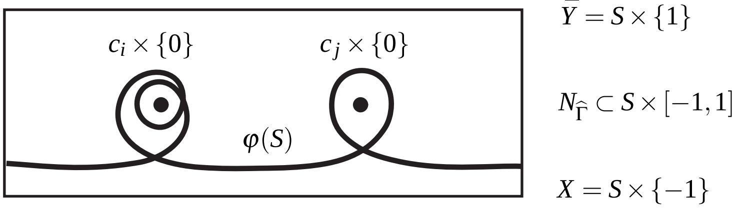

Figure 1: An immersion $\varphi : S \to N_{\widehat{\Gamma}}$.

Note that the bottom and top conformal structures of the ACM-sequence $\rho_n = B(\tau_c^n X, \tau_c^{2n} \overline{Y})$ converge to the same projective lamination $[c] \in \mathbb{P}\mathscr{ML}(S)$ in the Thurston compactifications of $T(S)$ and $T(\overline{S})$ but in different speeds. On the other hand, we remark that a sequence $\rho_n = B(\tau_c^n X, \tau_c^n \overline{Y})$ diverges and its top and bottom structures converge to $[c] \in \mathbb{P}\mathscr{ML}$ in the same speed. Moreover, Ohshika [Ohs98] showed that any sequence $\rho_n = B(X_n, \overline{Y}_n)$ diverge if the sequences X_n and $\overline{Y}_n$ converge to maximal and connected projective laminations $[\mu], [\nu] \in \mathbb{P}\mathscr{ML}(S)$ with the same support.

2.6. Pull-backs of limit sets

Suppose that a sequence $\Sigma_n \in Q(S)$ of quasi-fuchsian projective structures converges to $\Sigma_\infty \in \overline{Q(S)}$. Here we explain our fundamental idea on how to know what component of $Q(S)$ contains Σ_n. Note that the sequence $\rho_{\Sigma_n} \in \mathscr{QF}$ of their holonomy converges to $\rho_{\Sigma_\infty} \in \overline{\mathscr{QF}}$ and set $\Gamma_n = \rho_{\Sigma_n}(\pi_1(S))$ and $\Gamma_\infty = \rho_{\Sigma_\infty}(\pi_1(S))$. In addition, we assume that Γ_n converges geometrically to a Kleinian group $\widehat{\Gamma}$, which contains the algebraic limit Γ_∞. Since the sequence Λ_{Γ_n} converges to $\Lambda_{\widehat{\Gamma}}$ in the sense of Hausdorff ([KT90]), one see that the sequence $\Lambda_{\Sigma_n} \subset \Sigma_n$ of pull-backs also converges to $\widehat{\Lambda}_{\Sigma_\infty} \subset \Sigma_\infty$ in the sense of Hausdorff, where $\Lambda_{\Sigma_n} = f_{\Sigma_n}^{-1}(\Lambda_{\Gamma_n})/\pi_1(\Sigma_n)$ and $\widehat{\Lambda}_{\Sigma_\infty} = f_{\Sigma_\infty}^{-1}(\Lambda_{\widehat{\Gamma}})/\pi_1(\Sigma_\infty)$ (see Lemma 3.3 in [Ito00]). Here the sets $\Lambda_{\Sigma_n} \subset \Sigma_n$ and $\widehat{\Lambda}_{\Sigma_\infty} \subset \Sigma_\infty$ are compared via K_n-quasi-isometry maps $q_n : \Sigma_\infty \to \Sigma_n$ between hyperbolic surfaces Σ_∞ and Σ_n such that $K_n \to 1$ as $n \to \infty$. Now recall that Σ_n is in $\mathscr{Q}_{\lambda_n}$ if and only if Λ_{Σ_n} is a realization of $2\lambda_n$ on Σ_n. Therefore, the shape of $\widehat{\Lambda}_{\Sigma_\infty}$ in Σ_∞ give us information on the shape of $\Lambda_{\Sigma_n} \subset \Sigma_n$ and hence on the lamination $\lambda_n \in \mathscr{ML}_{\mathbb{N}}$ such that $\Sigma_n \in \mathscr{Q}_{\lambda_n}$.

3. Standardly convergent sequence in $\mathscr{QF}$

In this section, we survey our results in [Itob], one of which states that the grafting map $\Psi_\lambda : \mathscr{QF} \to P(S)$ takes every standardly convergent sequence to a convergent sequence.

3.1. Grafting for boundary groups

Let $\widehat{P}(S)$ denotes the one-point compactification $P(S) \cup \{\infty\}$ of $P(S)$. We will extend the grafting map $\Psi_\lambda : \mathscr{QF} \to \mathscr{Q}_\lambda$ to $\Psi_\lambda : \mathscr{QF} \sqcup \partial^\pm \mathscr{QF} \to \widehat{P}(S)$. To this end, we first recall another (but equivalent) definition of the grafting operation which was introduced by Bromberg [Bro02] so that it also makes sense for elements of $\partial^- \mathscr{QF}$.

Definition 3.1 (Grafting II)**.** We adopt Notation 2.1. Here we further assume that c separates S into two surfaces S_1 and S_2 with boundaries. (The non-separating case is described precisely in [Bro02].) Accordingly, Σ and Σ^- decompose into $\Sigma - c^+ = \Sigma_1 \sqcup \Sigma_2$ and $\Sigma^- - c^- = \Sigma_1^- \sqcup \Sigma_2^-$, respectively. Let i denotes either 1 or 2 and let $\Delta_i \subset \Omega_\Gamma^-$ be the connected component of the inverse image of $\Sigma_i^- \subset \Sigma^-$ whose closure $\overline{\Delta}_i$ contains $\tilde{c}^-$. Then the stabilizer subgroup $\Gamma_i = \mathrm{Stab}_\Gamma(\Delta_i)$ of $\Gamma \cong \pi_1(S)$ is identified with $\pi_1(S_i)$. Since Γ_i is a purely loxodromic free group with non-empty region of discontinuity, Maskit's result [Mas67] implies that Γ_i is a Schottky group. Note that the conformal boundary $\Omega_{\Gamma_i}/\Gamma_i$ of $M_{\Gamma_i} = \mathbb{H}^3/\Gamma_i$ with natural projective structure is containing both projective surfaces Σ_i and Σ_i^-. Then the *grafting* $\mathrm{Gr}_c(\Sigma)$ is obtained from projective surfaces $\Omega_{\Gamma_1}/\Gamma_1 - \Sigma_1^-$ and $\Omega_{\Gamma_2}/\Gamma_2 - \Sigma_2^-$ by gluing their boundaries without twisting (see Figure 2).

Observe that Definition 2.2 works well even for $\rho \in \partial^+ \mathscr{QF}$ whenever γ is loxodromic, because there still exists a $\langle\gamma\rangle$-invariant simple arc $\tilde{c}^+$ in non-degenerate

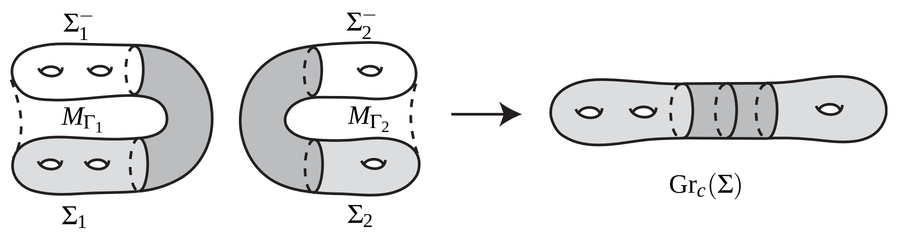

Figure 2: The grafting $\mathrm{Gr}_c(\Sigma)$ of Σ along c.

component Ω_Γ^+, for which $(\widehat{\mathbb{C}} - \tilde{c}^+)/\langle\gamma\rangle$ is still an annulus. On the other hand, Definition 3.1 works well for $\rho \in \partial^- \mathcal{QF}$ whenever every connected component of the parabolic locus $\mathrm{para}(\rho)$ of ρ intersects c essentially. In fact, in this case, Γ_1 and Γ_2 in Definition 3.1 are still Schottky groups, γ is still loxodromic, and there still exists a $\langle\gamma\rangle$-invariant simple arc $\tilde{c}^-$ in the non-degenerate component Ω_Γ^-. For general $\lambda \in \mathcal{ML}_\mathbb{N}$, we also obtain the grafting $\Psi_\lambda(\rho) \in P(S)$ of ρ along λ if the pair (λ, ρ) is *admissible*, or satisfies the following condition:

- $\rho \in \partial^+ \mathcal{QF}$ and $\mathrm{para}(\rho)$ and λ have no parallel component in common, or
- $\rho \in \partial^- \mathcal{QF}$ and every component of $\mathrm{para}(\rho)$ intersects λ essentially.

Otherwise, we set $\Psi_\lambda(\rho) = \infty \in \widehat{P}(S)$.

3.2. Continuity of grafting maps

One may expect that the extended grafting map $\Psi_\lambda : \mathcal{QF} \sqcup \partial^\pm \mathcal{QF} \to \widehat{P}(S)$ is also continuous at $\partial^\pm \mathcal{QF}$, but this is not the case for every $\lambda \in \mathcal{ML}_\mathbb{N}$; see Theorem 5.1. On the contrary, we have the following theorem, which implies that only exotically convergent sequences cause the non-continuity of the extended grafting maps.

Theorem 3.2 ([Itob]). *Let $\rho_n \in \mathcal{QF}$ be a sequence converging standardly to $\rho_\infty \in \partial^\pm \mathcal{QF}$. Then the sequence $\Psi_\lambda(\rho_n)$ also converges to $\Psi_\lambda(\rho_\infty)$ in $\widehat{P}(S)$ for every $\lambda \in \mathcal{ML}_\mathbb{N}$.*

Here and throughout, we let $\overline{\mathcal{Q}_\lambda}$ denote the closure of the component $\mathcal{Q}_\lambda$ of $Q(S)$ in $P(S)$, *not* in $\widehat{P}(S)$, and set $\partial\mathcal{Q}_\lambda = \overline{\mathcal{Q}_\lambda} - \mathcal{Q}_\lambda$. Then the above theorem tells us that $\Psi_\lambda(\rho)$ is surely contained in $\partial\mathcal{Q}_\lambda$ if the pair of $\lambda \in \mathcal{ML}_\mathbb{N}$ and $\rho \in \partial^\pm \mathcal{QF}$ is admissible. Recall that, as we observed in §2.4, a sequence $\Sigma_n \in \mathcal{Q}_0$ converges to $\Sigma_\infty \in \partial\mathcal{Q}_0$ if and only if $\rho_{\Sigma_n} \in \mathcal{QF}$ converges standardly to $\rho_{\Sigma_\infty} \in \partial^+ \mathcal{QF}$. Thus we obtain the following:

Corollary 3.3. *The grafting map* $\mathrm{Gr}_\lambda : \mathcal{Q}_0 \to \mathcal{Q}_\lambda$ *extends continuously to* $\mathrm{Gr}_\lambda : \overline{\mathcal{Q}_0} \to \widehat{P}(S)$ *for every* $\lambda \in \mathcal{ML}_\mathbb{N}$.

It is important to remark that we do not know whether $\mathscr{Q}_0$ is self-bumping at $\partial\mathscr{Q}_0$ or not, and thus we have to avoid this point. The following theorem plays an important roll in the proof of Theorem 3.2:

Theorem 3.4. *For a given $B(X,\overline{Y}) \in \mathscr{QF}$, set $\mathscr{B} = B_X \cup B_{\overline{Y}}$. Suppose that $\{\lambda_n\}$ is a sequence of distinct elements of $\mathscr{ML}_{\mathbb{N}}$. Then the sequence $\{\pi \circ \Psi_{\lambda_n}(\mathscr{B})\}$ eventually escapes any compact subset K of $T(S)$; that is, $\pi \circ \Psi_{\lambda_n}(\mathscr{B}) \cap K = \emptyset$ for all large enough n.*

We now outline the proof of Theorem 3.2. We only consider the case where the pair (λ, ρ_∞) is admissible and set $\Sigma_\infty = \Psi_\lambda(\rho_\infty) \in P(S)$. Let $\Phi : U \to P(S)$, $\rho_\infty \mapsto \Sigma_\infty$ be a univalent local branch of hol^{-1} defined on a neighborhood U of ρ_∞. Then the sequence $\Sigma_n = \Phi(\rho_n)$ converges to Σ_∞. If $\Sigma_n \in \mathscr{Q}_\lambda$ for all large enough n, then $\Sigma_n = \Psi_\lambda(\rho_n)$, and then we obtain the desired convergence $\Sigma_n = \Psi_\lambda(\rho_n) \to \Sigma_\infty = \Psi_\lambda(\rho_\infty)$. We show that $\Sigma_n \in \mathscr{Q}_\lambda$ by using the idea in §2.6; that is, we show that the pull backs $\Lambda_{\Sigma_n} \subset \Sigma_n$ of the limit sets Λ_{Γ_n} are realizations of 2λ. We assume that the sequence $\Gamma_n = \rho_n(\pi_1(S))$ converges geometrically to a Kleinian group $\widehat{\Gamma}$, which contains the algebraic limit $\Gamma_\infty = \rho_\infty(\pi_1(S))$. Then $\Lambda_{\Sigma_n} \subset \Sigma_n$ converges to the pull-back $\widehat{\Lambda}_{\Sigma_\infty} \subset \Sigma_\infty$ of $\Lambda_{\widehat{\Gamma}}$. Although it is difficult to understand the shape of $\widehat{\Lambda}_{\Sigma_\infty}$, we know a rough sketch of the subset $\Lambda_{\Sigma_\infty} \subset \widehat{\Lambda}_{\Sigma_\infty}$ in relation to λ from the definition of the grafting $\Sigma_\infty = \Psi_\lambda(\rho_\infty)$. Moreover, we can see that each connected component of $\widehat{\Lambda}_{\Sigma_\infty}$ contains that of Λ_{Σ_∞} by using [ACCS96, Lemma 2.4]. By combining the above observations, we see that Λ_{Σ_n} are realizations of 2λ in Σ_n for all large enough n. At this stage, we make use of Theorem 3.4 essentially, which asserts that the sequence Σ_n is contained in a finite union of components of $Q(S)$.

As a consequence of Theorems 3.2 and 3.4, Goldman's grafting theorem for quasi-fuchsian groups (Theorem 2.3) extends to all boundary b-groups, which is conjectured by Bromberg in [Bro02].

Theorem 3.5 ([Itob]). *For $\rho \in \partial^{\pm}\mathscr{QF}$, we have*

$$hol^{-1}(\rho) = \{\Psi_\lambda(\rho) \mid \lambda \in \mathscr{ML}_{\mathbb{N}}, \Psi_\lambda(\rho) \neq \infty\}.$$

4. Exotically convergent sequence in $\mathscr{QF}$

In this section, we shall show that ACM-sequences cause the bumping and self-bumping of components of $Q(S)$. Throughout this section, Figures 3 and 4 should be helpful for the reader to understand the arguments.

4.1. Exotic components bump to the standard one

We first show the following:

Theorem 4.1 ([Ito00]). *For any non-zero $\lambda \in \mathscr{ML}_{\mathbb{N}}$, we have $\overline{\mathscr{Q}_0} \cap \overline{\mathscr{Q}_\lambda} \neq \emptyset$.*

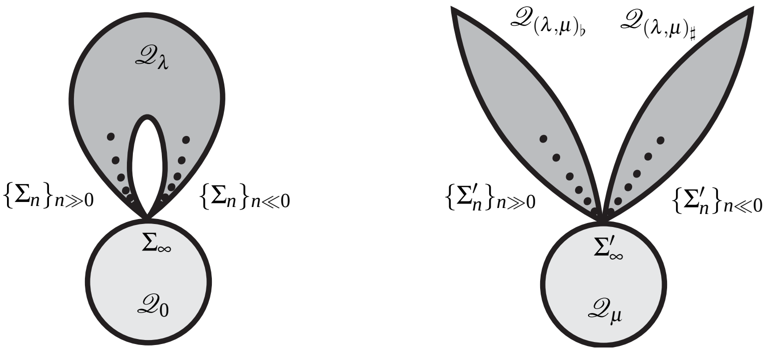

Figure 3: Sequences $\{\Sigma_n\}_{n\in\mathbb{Z}}$ and $\{\Sigma'_n\}_{n\in\mathbb{Z}}$.

Let $\{\rho_n\} \subset \mathscr{QF}$ be the ACM-sequence for λ as in (2.1), which converges exotically to some $\rho_\infty \in \partial^+\mathscr{QF} = hol(\partial\mathscr{Q}_0)$. Let Σ_∞ be the unique point in $\partial\mathscr{Q}_0$ with $hol(\Sigma_\infty) = \rho_\infty$ and let $\Phi : U \to P(S)$, $\rho_\infty \mapsto \Sigma_\infty$ be a univalent local branch of hol^{-1} defined on a neighborhood U of ρ_∞. Then the sequence $\Sigma_n = \Phi(\rho_n)$ converges to $\Sigma_\infty = \Phi(\rho_\infty)$. Since the convergence $\rho_n \to \rho_\infty$ is exotic, we see from Table 1 that Σ_n are exotic for all $|n| \gg 0$ (see also Theorem A.2 in [McM98]). Moreover, we see that $\Sigma_n \in \mathscr{Q}_\lambda$ for all $|n| \gg 0$ by using the idea in §2.6. In fact, one can observe that $\widehat{\Lambda}_{\Sigma_\infty} \subset \Sigma_\infty$ is a "decorated realization" of 2λ, that is, $\widehat{\Lambda}_{\Sigma_\infty}$ contains a realization of 2λ and is contained in a regular neighborhood of a realization of 2λ (see the left side of Figure 4). Since $\Lambda_{\Sigma_n} \subset \Sigma_n$ converge to $\widehat{\Lambda}_{\Sigma_\infty} \subset \Sigma_\infty$, the sets Λ_{Σ_n} turn out to be realizations of 2λ and thus $\Sigma_n \in \mathscr{Q}_\lambda$ for all $|n| \gg 0$. This implies that $\overline{\mathscr{Q}_0} \cap \overline{\mathscr{Q}_\lambda} \neq \emptyset$. We also remark that $\Sigma_n = \Psi_\lambda(\rho_n)$ hold for all $|n| \gg 0$.

4.2. Simultaneous bumping

We extend Theorem 4.1 to the following:

Theorem 4.2 ([Ito00]). *Let $\{\lambda_i\}_{i=1}^m$ be a finite subset of $\mathscr{ML}_\mathbb{N} - \{0\}$ such that $i(\lambda_i, \lambda_j) = 0$ for every $1 \le i < j \le m$. Then we have $\overline{\mathscr{Q}_0} \cap \overline{\mathscr{Q}_{\lambda_1}} \cap \cdots \cap \overline{\mathscr{Q}_{\lambda_m}} \neq \emptyset$.*

In fact, we can construct ACM-sequences

$$\rho_n^{(i)} = B(\tau_{\lambda_i}^n X_i, (\tau_{\underline{\lambda_i}} \circ \tau_{\lambda_i})^n \overline{Y_i})$$

for λ_i for each $1 \le i \le m$ so that all of these sequences $\rho_n^{(i)}$ $(i = 1, \dots, m)$ have the same algebraic limit $\rho_\infty \in \partial^+\mathscr{QF}$; see §5 in [Ito00]. Let $\Phi : U \to P(S)$ be a univalent local branch of hol^{-1} such that $\Phi(\rho_\infty) \in \partial\mathscr{Q}_0$ as in §4.1. Then for each i, we have a convergent sequence $\Phi(\rho_n^{(i)}) \to \Phi(\rho_\infty)$, which turns out to be $\Phi(\rho_n^{(i)}) \in \mathscr{Q}_{\lambda_i}$ for all $|n| \gg 0$ by the same argument as in §4.1. Thus we have $\overline{\mathscr{Q}_0} \cap \left(\bigcap_{i=1}^m \overline{\mathscr{Q}_{\lambda_i}}\right) \neq \emptyset$.

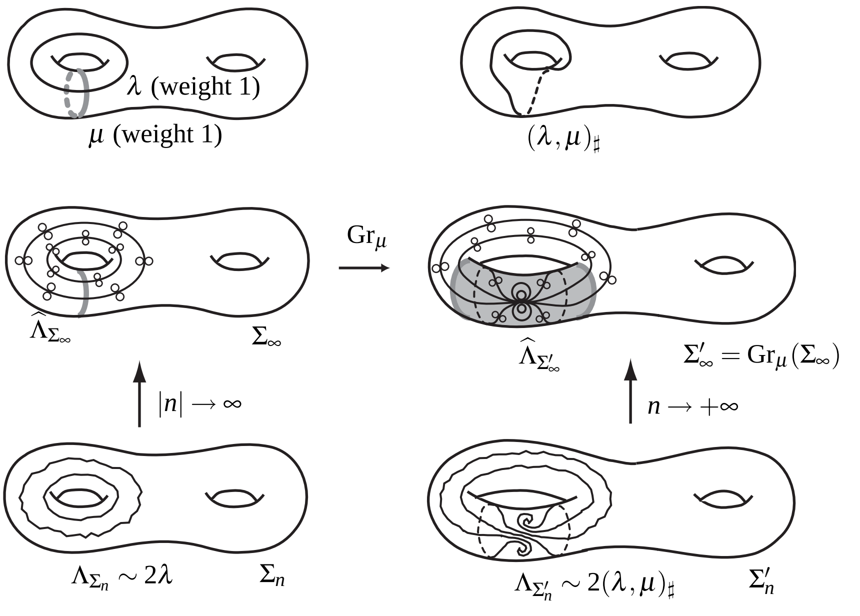

Figure 4: Schematic figure explaining the proof of Theorems 4.1 and 4.3.

4.3. Self-bumping of exotic components

Here we outline the proof of the following:

Theorem 4.3 ([Itoa])**.** *For any non-zero* $\lambda \in \mathscr{ML}_{\mathbb{N}}$*,* $\mathscr{Q}_\lambda$ *self-bumps.*

Let $\{\rho_n\}$ be the ACM-sequence for λ as in (2.1). Then we actually show that the subsequences $\{\rho_n\}_{n\gg 0}$, $\{\rho_n\}_{n\ll 0}$ of $\{\rho_n\}$ are contained in distinct components of $U \cap \mathscr{QF}$ for any sufficiently small neighborhood U of ρ_∞. This is a consequence of the following fact: in the same notation as in §2.6, both sequences $\{\Lambda_{\Gamma_n}\}_{n\gg 0}$, $\{\Lambda_{\Gamma_n}\}_{n\ll 0}$ converges to $\Lambda_{\widehat{\Gamma}}$ in the sense of Hausdorff but Λ_{Γ_n} $(n \gg 0)$ and Λ_{Γ_n} $(n \ll 0)$ are spiraling in opposite directions at each fixed point of rank-two parabolic subgroups of $\widehat{\Gamma}$.

Before outlining the proof, we recall the definition of operations $(\cdot,\cdot)_\sharp, (\cdot,\cdot)_\flat$: $\mathscr{ML}_{\mathbb{N}} \times \mathscr{ML}_{\mathbb{N}} \to \mathscr{ML}_{\mathbb{N}}$, which is closely observed in [Luo01]. For any two elements $\lambda, \mu \in \mathscr{ML}_{\mathbb{N}}$, new elements $(\lambda,\mu)_\sharp$ and $(\lambda,\mu)_\flat$ in $\mathscr{ML}_{\mathbb{N}}$ are obtained by taking realizations $\widehat{\lambda}, \widehat{\mu}$ of λ, μ so that the geometric intersection number of $\widehat{\lambda}$ and $\widehat{\mu}$ is minimal, and drawing "zigzag" paths on $\widehat{\lambda} \cup \widehat{\mu}$ under the rules in Figure 5 (see also Figure 6). Now let $\lambda, \mu \in \mathscr{ML}_{\mathbb{N}}$. We collect here some of basic properties of these operations:

(i) $(\lambda,\mu)_\sharp = (\mu,\lambda)_\flat$.

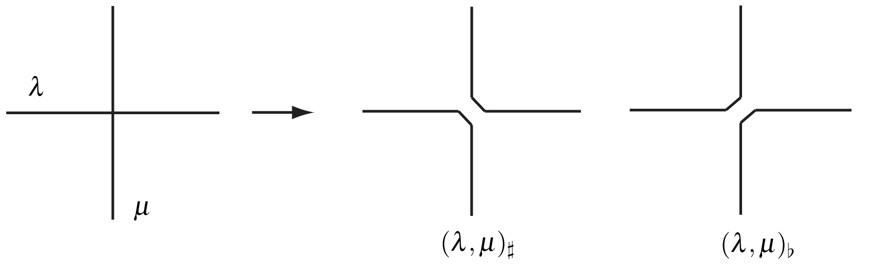

Figure 5: Rules to construct $(\lambda,\mu)_\sharp$ and $(\lambda,\mu)_\flat$.

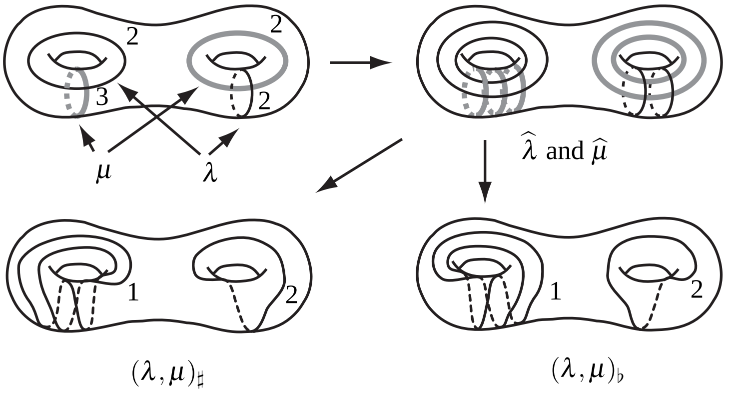

Figure 6: Examples of $(\lambda,\mu)_\sharp$ and $(\lambda,\mu)_\flat$.

(ii) $(\lambda,\mu)_\sharp \neq (\lambda,\mu)_\flat$ if and only if $i(\lambda,\mu) \neq 0$. If $i(\lambda,\mu) = 0$ then $(\lambda,\mu)_\sharp = (\lambda,\mu)_\flat = \lambda+\mu$.

(iii) Assume that every components of μ intersects λ. Then $((\lambda,\mu)_\sharp,\mu)_\flat = ((\lambda,\mu)_\flat,\mu)_\sharp = \lambda$, $((\lambda,\mu)_\sharp,\mu)_\sharp = (\lambda,2\mu)_\sharp$ and $((\lambda,\mu)_\flat,\mu)_\flat = (\lambda,2\mu)_\flat$.

We now go back to the proof of Theorem 4.3. Let ρ_n be the ACM-sequence for λ as in (2.1). We now take another non-zero element $\mu \in \mathscr{ML}_\mathbb{N}$ such that λ and μ have no parallel component in common. Then the pair (μ,ρ_∞) is admissible, and thus we have the grafting $\Psi_\mu(\rho_\infty) \in P(S)$ of ρ_∞ along μ. Let $\Phi' : U \to P(S)$, $\rho_\infty \mapsto \Psi_\mu(\rho_\infty)$ be a univalent local branch of hol^{-1} defined on a neighborhood U of ρ_∞. Then the sequence $\Sigma'_n = \Phi'(\rho_n)$ converges to $\Sigma'_\infty = \Psi_\mu(\rho_\infty)$ as $|n| \to \infty$. One of the crucial observations in [Itoa] is the following:

Proposition 4.4. *$\Sigma'_n \in \mathscr{Q}_{(\lambda,\mu)_\sharp}$ for all $n \gg 0$ and $\Sigma'_n \in \mathscr{Q}_{(\lambda,\mu)_\flat}$ for all $n \ll 0$.*

Skech of proof. If $i(\lambda,\mu) = 0$, it is easy to see that the set $\widehat{\Lambda}_{\Sigma'_\infty} \subset \Sigma'_\infty$ is a decorated realization of $\lambda+\mu = (\lambda,\mu)_\sharp = (\lambda,\mu)_\flat$ and that $\Sigma'_n \in \mathscr{Q}_{\lambda+\mu}$ for all $|n| \gg 0$. Suppose that $i(\lambda,\mu)\neq 0$. Then $\widehat{\Lambda}_{\Sigma'_\infty}\subset\Sigma'_\infty$ is a "decorated train-track" whose switches in $\widehat{\Lambda}_{\Sigma'_\infty}$ are pull-backs of rank-two parabolic fixed points in $\Lambda_{\widehat{\Gamma}}$ (see the right side of Figure 4). Since $\Lambda_{\Sigma'_n}\to\widehat{\Lambda}_{\Sigma'_\infty}$ as $|n|\to\infty$ and since $\{\Lambda_{\Sigma'_n}\}_{n\gg 0}$ and $\{\Lambda_{\Sigma'_n}\}_{n\ll 0}$ are spiraling in opposite directions at each switch of $\widehat{\Lambda}_{\Sigma'_\infty}\subset\Sigma'_\infty$, we see that the sets $\Lambda_{\Sigma'_n}\subset\Sigma'_n$ are realizations of $2(\lambda,\mu)_\sharp$ if $n\gg 0$ and of $2(\lambda,\mu)_\flat$ if $n\ll 0$. □

Therefore, if we further assume that $i(\lambda,\mu)\neq 0$, the sequences $\{\Sigma'_n\}_{n\gg 0}$, $\{\Sigma'_n\}_{n\ll 0}$ are contained in distinct components of $Q(S)$. This implies that $\{\rho_n\}_{n\gg 0}$, $\{\rho_n\}_{n\ll 0}$ are contained in distinct components of $U\cap\mathscr{QF}$, and hence that the sequences $\{\Sigma_n=\Phi(\rho_n)\}_{n\gg 0}$, $\{\Sigma_n=\Phi(\rho_n)\}_{n\ll 0}$, obtained in §4.1, are contained in distinct components of $\Phi(U)\cap\mathscr{Q}_\lambda$. Therefore $\mathscr{Q}_\lambda$ self-bumps at $\rho_\infty\in\overline{\mathscr{Q}_0}\cap\overline{\mathscr{Q}_\lambda}$. We also remark that we have $\Sigma'_n=\Psi_{(\lambda,\mu)_\sharp}(\rho_n)$ for all $n\gg 0$ and $\Sigma'_n=\Psi_{(\lambda,\mu)_\flat}(\rho_n)$ for all $n\ll 0$.

4.4. Bumping of any two components

As a consequence of the above arguments in this section, we obtain the following:

Theorem 4.5. *For any λ, $\mu\in\mathscr{ML}_\mathbb{N}$, we have $\overline{\mathscr{Q}_\lambda}\cap\overline{\mathscr{Q}_\mu}\neq\emptyset$.*

For a convenience of the reader, we give here the same proof in [Itoa].

Proof of Theorem 4.5. If $i(\lambda,\mu)=0$, we obtain the result from Theorem 4.2. Hence, we assume that $i(\lambda,\mu)\neq 0$. We decompose μ into $\mu=\mu'+\mu''$ so that $\mu',\mu''\in\mathscr{ML}_\mathbb{N}$ and that $i(\lambda,\mu)=i(\lambda,\mu')$. We first consider the case where $\mu''=0$. As observed in §4.1, there exists an element $\rho_\infty\in\overline{\mathscr{Q}_0}\cap\overline{\mathscr{Q}_{(\lambda,\mu)_\flat}}$ which is a limit of an ACM-sequence for $(\lambda,\mu)_\flat$. Then the same argument as in §4.3 reveals that $\Psi_\mu(\rho_\infty)\in\overline{\mathscr{Q}_\mu}\cap\overline{\mathscr{Q}_\lambda}$ since $\lambda=((\lambda,\mu)_\flat,\mu)_\sharp$. We next consider the case where $\mu''\neq 0$. Since $i(\lambda,\mu'')=0$ and $i(\mu',\mu'')=0$, we have $i((\lambda,\mu')_\flat,\mu'')=0$. Then as observed in §4.2, there exists a limit $\rho_\infty\in\overline{\mathscr{Q}_0}\cap\overline{\mathscr{Q}_{(\lambda,\mu')_\flat}}\cap\overline{\mathscr{Q}_{\mu''}}$ of ACM-sequences. Then we have $\Psi_{\mu'}(\rho_\infty)\in\overline{\mathscr{Q}_\lambda}\cap\overline{\mathscr{Q}_\mu}$ since $\lambda=((\lambda,\mu')_\flat,\mu')_\sharp$ and since $\mu=\mu'+\mu''$. □

5. Additional observations

Throughout this section, we suppose that $\{\rho_n\}$ is the ACM-sequence for λ as in (2.1). We have observed that $\lim_{n\to\pm\infty}\Psi_\lambda(\rho_n)=\Psi_0(\rho_\infty)$ in §4.1 and that

$$\lim_{n\to+\infty}\Psi_{(\lambda,\mu)_\sharp}(\rho_n)=\lim_{n\to-\infty}\Psi_{(\lambda,\mu)_\flat}(\rho_n)=\Psi_\mu(\rho_\infty) \tag{5.1}$$

in §4.3. Now let ν be an element of $\mathscr{ML}_\mathbb{N}$ such that $i(\nu,\lambda)\neq 0$ and that ν and λ have no parallel component in common. Then we have

$$\lim_{n\to+\infty}\Psi_\nu(\rho_n)=\Psi_{(\nu,\lambda)_\sharp}(\rho_\infty),\quad \lim_{n\to-\infty}\Psi_\nu(\rho_n)=\Psi_{(\nu,\lambda)_\flat}(\rho_\infty) \tag{5.2}$$

by substituting $\mu=(\nu,\lambda)_\sharp$ or $\mu=(\nu,\lambda)_\flat$ in (5.1). Since $i(\nu,\lambda)\neq 0$, we have $\Psi_{(\nu,\lambda)_\sharp}(\rho_\infty)\neq\Psi_{(\nu,\lambda)_\flat}(\rho_\infty)$. By choosing λ suitably for every ν, we obtain the following:

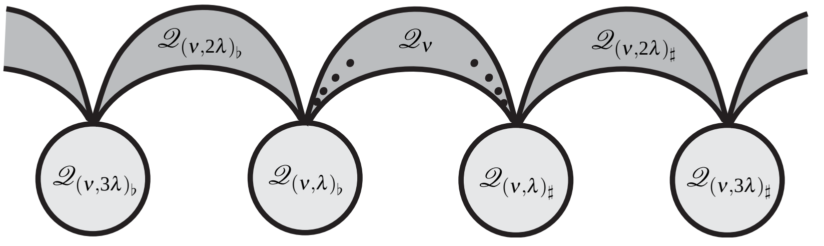

Figure 7: Analytic continuation of Ψ_ν along α.

Theorem 5.1. *The grafting map $\Psi_\nu : \mathscr{QF} \to P(S)$ does not extend continuously to $\partial\mathscr{QF}$ for every $\nu \in \mathscr{ML}_\mathbb{N}$.*

We now observe some properties of analytic continuations of local branches of hol^{-1}. Suppose that U is a sufficiently small neighborhood of $\rho_\infty \in \partial^+\mathscr{QF}$. Let $\alpha : S^1 = \mathbb{R} \cup \{\infty\} \to \mathscr{QF} \cup U$ be a continuous map such that $\alpha(n) = \rho_n$ for all $n \in \mathbb{Z}$ and that $\alpha(\infty) = \rho_\infty$. (We do not know whether we can choose α so that $\alpha(S^1) \subset \overline{\mathscr{QF}}$.) The closed curve $\alpha(S^1)$ in $R(S)$ is also denote by α. Since $\Psi_\lambda(\alpha(n)) \to \Psi_0(\rho_\infty)$ as $|n| \to \infty$, the branch $\Psi_\lambda : \mathscr{Q}_\lambda \to P(S)$ of hol^{-1} is continued analytically to a univalent local branch $\Phi : U \to P(S), \rho_\infty \mapsto \Psi_0(\rho_\infty)$ along both the paths $\alpha(\mathbb{R}_{\geq 0})$ and $\alpha(\mathbb{R}_{\leq 0})$, and hence there exists a lift $\tilde\alpha \subset \mathscr{Q}_\lambda \cup \Phi(U)$ of α for which $hol|_{\tilde\alpha} : \tilde\alpha \to \alpha$ is one-to-one. Note that the above argument does not imply that the map Ψ_λ extends to a univalent local branch $\mathscr{QF} \cup U \to P(S)$ of hol^{-1}. In fact, if $\eta_n \in \mathscr{QF} \cap U$ is a sequence converging standardly to ρ_∞, then $\Psi_\lambda(\eta_n)$ converges to ∞ in $\widehat{P}(S)$, not to $\Psi_0(\rho_\infty) \in P(S)$. Since $P(S)$ is contractible, $\tilde\alpha$ is contractible in $P(S)$, and hence α is contractible in $R(S)$. This implies that the bumping at $\rho_\infty \in \partial\mathscr{QF}$ of the two arms of $\mathscr{QF}$ containing $\{\rho_n\}_{n\gg 0}$ and $\{\rho_n\}_{n\ll 0}$ yields no non-trivial element of $\pi_1(R(S))$. On the other hand, let take $\nu \in \mathscr{ML}_\mathbb{N}$ as above and let us consider the analytic continuation of Ψ_ν. In this case, since $\lim_{n\to+\infty}\Psi_\nu(\alpha(n)) \neq \lim_{n\to-\infty}\Psi_\nu(\alpha(n))$ from (5.2), the analytic continuation of the local branch Ψ_ν along $\alpha \subset R(S)$ yields succeeding sequence of local branches

$$\ldots, \Psi_{(\nu,4\lambda)_\flat}, \Psi_{(\nu,2\lambda)_\flat}, \Psi_\nu, \Psi_{(\nu,2\lambda)_\sharp}, \Psi_{(\nu,4\lambda)_\sharp}, \cdots$$

(see Figure 7). Thus we obtain a lift $\tilde\alpha'$ of α in $P(S)$ for which $hol|_{\tilde\alpha'} : \tilde\alpha' \to \alpha$ is infinite-to-one. We sum up the arguments in this section:

Theorem 5.2. *There exists a contractible closed curve α in $R(S)$ whose pre-image $hol^{-1}(\alpha) \subset P(S)$ has connected components $\tilde\alpha, \tilde\alpha'$ such that the map $hol|_{\tilde\alpha} : \tilde\alpha \to \alpha$ is one-to-one and the map $hol|_{\tilde\alpha'} : \tilde\alpha' \to \alpha$ is infinite-to-one. Especially, the lift $\widetilde{hol} : P(S) \to \widetilde{R(S)}$ of the map $hol : P(S) \to R(S)$ to the universal cover is not an embedding.*

References

[AC96] J. W. Anderson and R. D. Canary, *Algebraic limits of Kleinian groups which rearrange the pages of a book*, Invent. Math. **126** (1996), 205–214.

[ACCS96] J. W. Anderson, R. D. Canary, M. Culler and P. B. Shalen, *Free Kleinian groups and volumes of hyperbolic 3-manifolds*, J. Diff. Goem. **44** (1996), 738–782.

[ACM00] J. W. Anderson, R. D. Canary and D. McCullough, *On the topology of deformation spaces of Kleinian groups*, Ann. of Math. **152** (2000), 693–741.

[Ber60] L. Bers, *Simultaneous uniformization*, Bull. Amer. Math. Soc. **66** (1960), 94–97.

[Bro97] J. F. Brock, *Iteration of mapping classes on a Bers slice: examples of algebraic and geometric limits of hyperbolic 3-manifolds*, in Lipa's Legacy (New York, 1995), Contemp. Math. **211**, Amer. Math. Soc., Providence, 1997, 81–106.

[Bro02] K. Bromberg, *Projective structures with degenerate holonomy and the Bers' density conjecture*, preprint 2002. `arXiv:mathGT/0211402`

[BH01] K. Bromberg and J. Holt, *Self-bumping of deformation spaces of hyperbolic 3-manifolds*, J. Diff. Geom. **57** (2001), 47–65.

[Can03] R. D. Canary, *Pushing the boundary*, in the Tradition of Ahlfors and Bers, III, Contemp. Math. **355** (2004), Amer. Math. Soc, 109-121.

[Ear81] C. J. Earle, *On variation of projective structures*, Riemann surfaces and related topics, Ann. Math. Studies **97** (1981), 87–99.

[Gol87] W. M. Goldman, *Projective structures with Fuchsian holonomy*, J. Diff. Geom. **25** (1987), 297–326.

[Hej75] D. A. Hejhal, *Monodromy groups and linearly polymorphic functions*, Acta Math. **135** (1975), 1–55.

[Hub81] J. H. Hubbard, *The monodromy of projective structures*, Riemann surfaces and related topics, Ann. Math. Studies **97** (1981), 257–275.

[Ito00] K. Ito, *Exotic projective structures and quasi-Fuchsian space*, Duke Math. J. **105** (2000), 185–209.

[Itoa] K. Ito, *Exotic projective structures and quasi-fuchsian spaces II*, preprint, `http://www.math.nagoya-u.ac.jp/~itoken/index.html`

[Itob] K. Ito, *On continuous extension of grafting maps*, preprint, `arXiv:mathGT/0411133`

[KT90] S. P. Kerckhoff and W. P. Thurston, *Non-continuity of the action of the modular group at Bers' boundary of Teichmuller space*, Invent. Math. **100** (1990), 25–47.

[Kra71] I. Kra, *A generalization of a theorem of Poincaré*, Proc. Amer. Math. Soc. **27** (1971), 299–302.

[Luo01] F. Luo, *Some applications of a multiplicative structure on simple loops in surfaces*, in Knots, braids, and mapping class groups—papers dedicated to Joan S. Birman (New York, 1998), AMS/IP Stud. Adv. Math., **24**, Amer. Math. Soc., Providence, 2001, 123–129.

[Mas67] B. Maskit, *A characterization of Schottky groups*, J. Analyse Math. **19** (1967), 227–230.

[MT98] K. Matsuzaki and M. Taniguchi, *Hyperbolic manifolds and Kleinian groups*, Oxford University Press, 1998.

[McM98] C. T. McMullen, *Complex earthquakes and Teichmuller theory*, J. Amer. Math. Soc. **11** (1998), 283–320.

[Ohs98] K. Ohshika, *Divergent sequences of Kleinian groups*, The Epstein birthday schrift, 419–450, Geom. Topol. Monogr., 1, Geom. Topol. Publ., Coventry, 1998.

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