ON CONTINUOUS EXTENSIONS OF GRAFTING MAPS

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To the memory of Professor Nobuyuki Suita

Abstract. The definition of the grafting operation for quasifuchsian groups is extended by Bromberg [Br] to all b-groups. In this paper, we show that the extended grafting maps behave as continuous maps for every sequence which converges "standardly" to a boundary group, although the maps does not continuous in general. As a consequence of this result, we extend Goldman's grafting theorem for quasifuchsian groups to all boundary b-groups.

1. Introduction

In this paper we are interested in the behavior of the holonomy map from projective structures to representations. Especially, we study the continuity of the local inverse of the holonomy map at the boundary of the space of discrete faithful representations.

Let S be an oriented closed surface of genus $g > 1$. A projective structure on S is a (G, X)-structure where X is a Riemann sphere $\widehat{\mathbb{C}}$ and $G = \mathrm{PSL}_2(\mathbb{C})$ is the group of projective automorphism of $\widehat{\mathbb{C}}$. Let $P(S)$ denote the space of projective structures on S and $R(S)$ the space of conjugacy classes of representations $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$. Then the holonomy map $hol : P(S) \to R(S)$ takes a projective structure to its holonomy representation. It is known by Hejhal [He] that the map hol is a local homeomorphism but not a covering onto its image; see also §7.3.

Quasifuchsian space $\mathcal{QF} = \mathcal{QF}(S)$ is the subset of $R(S)$ of faithful representations with quasifuchsian images, which is known to be a connected, contractible open manifold. We now set $Q(S) = hol^{-1}(\mathcal{QF})$. Then Goldman [Go] showed that the set of components of $Q(S)$ is in one-to-one correspondence with the set $\mathcal{ML}_{\mathbb{N}} = \mathcal{ML}_{\mathbb{N}}(S)$ of integral measured laminations on S. Thus we obtain a decomposition $\bigsqcup_{\lambda \in \mathcal{ML}_{\mathbb{N}}} \mathcal{Q}_\lambda$ of $Q(S)$, where $\mathcal{Q}_\lambda$ is the connected component of $Q(S)$ associated to $\lambda \in \mathcal{ML}_{\mathbb{N}}$. We know that the map $hol|_{\mathcal{Q}_\lambda} : \mathcal{Q}_\lambda \to \mathcal{QF}$ is biholomorphic for each $\lambda \in \mathcal{ML}_{\mathbb{N}}$ and let $\Psi_\lambda : \mathcal{QF} \to \mathcal{Q}_\lambda$ denote the univalent local branch of hol^{-1}, which is called the *grafting map* for λ.

Although the map Ψ_λ does not extend continuously to the boundary $\partial\mathcal{QF} = \overline{\mathcal{QF}} - \mathcal{QF}$ in general, we obtain in this paper a sufficient condition under which two sequences ρ_n, ρ'_n in $\mathcal{QF}$ with the same limit in $\partial\mathcal{QF}$ are mapped by Ψ_λ to two sequences with the same limit. To state our results, we need to introduce the notion of standardly convergent sequence in $\mathcal{QF}$. Let $B : T(S) \times T(S) \to \mathcal{QF}$ be the parameterization of $\mathcal{QF}$ of Bers' simultaneous uniformization and let $\rho_n = B(X_n, Y_n) \in \mathcal{QF}$ be a sequence converging to $\rho \in \partial\mathcal{QF}$. Then we say that the

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sequence ρ_n converges *standardly* to ρ if there exists a compact subset K of $T(S)$ which contains all X_n or all Y_n; otherwise ρ_n converges *exotically* to ρ. We let $\partial^{\pm}\mathcal{QF} \subset \partial\mathcal{QF}$ denote the set of limits of standardly convergent sequences, or the set of all boundary b-groups; see §2.2. Then by using Bromberg's observation in [Br], the grafting map Ψ_λ is naturally extended to $\Psi_\lambda : \mathcal{QF} \sqcup \partial^{\pm}\mathcal{QF} \to \widehat{P}(S)$ for every $\lambda \in \mathcal{ML}_{\mathbb{N}}$, where $\widehat{P}(S) = P(S) \cup \{\infty\}$ denotes the one-point compactification of $P(S)$. Then one of our main result in this paper is the following:

Theorem 1.1 (Continuity)**.** *Let $\rho_n \in \mathcal{QF}$ be a sequence converging standardly to $\rho \in \partial^{\pm}\mathcal{QF}$. Then the sequence $\Psi_\lambda(\rho_n)$ converges to $\Psi_\lambda(\rho)$ in $\widehat{P}(S)$ for every $\lambda \in \mathcal{ML}_{\mathbb{N}}$.*

We mention here the non-continuity of the map Ψ_λ at $\partial^{\pm}\mathcal{QF}$. It is known by Anderson and Canary [AC] (see also [Mc]) that there exists an exotically convergent sequence in $\mathcal{QF}$ tending to a point in $\partial^{\pm}\mathcal{QF}$. In [Mc], McMullen made use of this sequence to show that the map $\Psi_0 : \mathcal{QF} \sqcup \partial^{\pm}\mathcal{QF} \to \mathcal{Q}_0 \cup \{\infty\}$ is not continuous, where $\mathcal{Q}_0$ is the component of projective structures with injective developing maps. Since the map *hol* is a local homeomorphism, this implies that the closure $\overline{\mathcal{QF}}$ is not a manifold with boundary; see [Mc]. Further, we have shown in [Ita] that the map $\Psi_\lambda : \mathcal{QF} \sqcup \partial^{\pm}\mathcal{QF} \to \widehat{P}(S)$ is not continuous at $\partial\mathcal{QF}$ for every $\lambda \in \mathcal{ML}_{\mathbb{N}}$. We refer the reader to an exposition [It3] for more information of exotically convergent sequences and their applications.

Along with Theorem 1.1, one of our essential observation in this paper is the following theorem. Now let $\pi : P(S) \to T(S)$ be the natural projection onto the Teichmüller space $T(S)$ of S. Further, for a given compact subset K of $T(S)$, we set $\mathcal{QF}_K = \{B(X,Y) \mid X \,\mathrm{or}\, Y \in K\}$ and call it the truncated quasifuchsian space for K.

Theorem 1.2 (Divergence)**.** *Let $\mathcal{QF}_K$ be the truncated quasifuchsian space for a compact subset K of $T(S)$ and let $\{\lambda_n\}$ be a sequence of distinct elements of $\mathcal{ML}_{\mathbb{N}}$. Then the sequence $\{\pi \circ \Psi_{\lambda_n}(\mathcal{QF}_K)\}$ eventually escapes any compact subset L of $T(S)$; that is, $\pi \circ \Psi_{\lambda_n}(\mathcal{QF}_K) \cap L = \emptyset$ for all large enough n.*

By combining Theorem 1.1 and Theorem 1.2, we obtain two theorems below: The first one is an extension of Goldman's grafting theorem and is conjectured by Bromberg in [Br]. The second one should be compared with the fact that any two components of $Q(S) = hol^{-1}(\mathcal{QF})$ have intersecting closures; see [It2]. Namely, Theorem 1.4 implies that only exotically convergent sequences cause the bumping of distinct components of $Q(S)$.

Theorem 1.3 (Grafting theorem for b-group)**.** *For every boundary b-groups $\rho \in \partial^{\pm}\mathcal{QF}$, all projective structures with holonomy ρ are obtained by grafting of ρ; that is, $hol^{-1}(\rho) = \{\Psi_\lambda(\rho) : \lambda \in \mathcal{ML}_{\mathbb{N}},\, \Psi_\lambda(\rho) \neq \infty\}$.*

Theorem 1.4 (Discreteness)**.** *Let $\mathcal{QF}_K$ be the truncated quasifuchsian space of a compact subset K of $T(S)$. Then the inverse image $hol^{-1}(\mathcal{QF}_K)$ of $\mathcal{QF}_K$ in $P(S)$ is discrete; that is, every connected component $\Psi_\lambda(\mathcal{QF}_K)$ of $hol^{-1}(\mathcal{QF}_K)$ has an open neighborhood in $P(S)$ which is disjoint from any other component.*

Remark. It is conjectured by Bers, Sullivan and Thurston that the closure of $\mathcal{QF}$ is equal to the space $AH(S) \subset R(S)$ of discrete, faithful representations. This conjecture is closely related to Thurston's ending lamination conjecture, which was recently solved affirmatively by Minsky et al. But we do not make use of these deep results in this paper.

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2. Preliminaries

2.1. **Quasifuchsian space.** A Kleinian group Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$, which acts on hyperbolic 3-space $\mathbb{H}^3$ as isometries and on the sphere at infinity $\widehat{\mathbb{C}}$ as conformal automorphisms. The union $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \widehat{\mathbb{C}}$ is naturally topologized as a closed 3-ball so that $\mathrm{PSL}_2(\mathbb{C})$ acts continuously on it. For a Kleinian group Γ, we let $\Omega_\Gamma \subset \widehat{\mathbb{C}}$ denote the region of discontinuity and $\Lambda_\Gamma = \widehat{\mathbb{C}} - \Omega_\Gamma$ the limit set. We associate to a Kleinian group Γ the following orbit spaces:

$$M_\Gamma = \mathbb{H}^3/\Gamma, \quad \overline{M_\Gamma} = (\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma, \quad \partial M_\Gamma = \Omega_\Gamma/\Gamma,$$

where ∂M_Γ is called the *conformal boundary* of M_Γ. In general if $\overline{M}$ is an oriented manifold with boundary ∂M, we orient ∂M by requiring that the frame (f, n) has positive orientation whenever f is a positively oriented frame on ∂M and n is an inward-pointing vector.

Let S be an oriented closed surface of genus $g > 1$. Let $R(S)$ be the space of conjugacy classes $[\rho]$ of representations $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ whose images $\rho(\pi_1(S))$ are non-abelian. The space $R(S)$ is equipped with the algebraic topology, the topology of convergence on generators up to conjugation. (By abuse of notation, we also denote $[\rho]$ by ρ if there is no confusion.) It is known that $R(S)$ is a $(6g-6)$-dimensional complex manifold (see Theorem 4.21 in [MT]).

Let $AH(S)$ be the subset of $R(S)$ of discrete, faithful representations, which is known to be a closed subset of $R(S)$ by Jørgensen [Jo]. Let $\rho_n \to \rho_\infty$ be an algebraically convergent sequence in $AH(S)$. Then it is known by Jørgensen and Marden [JM] that, by passing to a subsequence if necessary, the sequence $\Gamma_n = \rho_n(\pi_1(S))$ of Kleinian groups converges geometrically to some Kleinian group $\widehat{\Gamma}$, which contains the algebraic limit $\Gamma_\infty = \rho_\infty(\pi_1(S))$. Here Γ_n converges *geometrically* to $\widehat{\Gamma}$ if for any $\hat{\gamma} \in \widehat{\Gamma}$, there exist $\gamma_n \in \Gamma_n$ such that $\gamma_n \to \hat{\gamma}$ and if for every convergent sequence $\gamma_{n_j} \in \Gamma_{n_j}$ $(n_j \to \infty)$, the limit is contained in $\widehat{\Gamma}$. A sequence ρ_n converges *strongly* to ρ_∞ in $AH(S)$ if $\Gamma_n = \rho_n(\pi_1(S))$ converges geometrically to the algebraic limit $\Gamma_\infty = \rho_\infty(\pi_1(S))$.

Bonahon's theorem [Bo] guarantees that, for each $\rho \in AH(S)$ with $\Gamma = \rho(\pi_1(S))$, there exists an orientation preserving homeomorphism $\psi : S \times (-1, 1) \to M_\Gamma =: M_\rho$, where $(-1, 1)$ is an open interval. The conformal boundary ∂M_ρ (possibly empty) of M_ρ decomposes into two parts $\partial^+ M_\rho \sqcup \partial^- M_\rho$, where $\partial^+ M_\rho$ (resp. $\partial^- M_\rho$) is the limit of $\psi(S \times \{t\})$ in $\overline{M_\rho}$ as $t \to -1$ (resp. $t \to 1$). Associated to this decomposition, Ω_Γ decomposes into $\Omega_\Gamma^+ \sqcup \Omega_\Gamma^-$ for which $\Omega_\Gamma^+/\Gamma = \partial^+ M_\rho$ and $\Omega_\Gamma^-/\Gamma = \partial^- M_\rho$.

Let $\Gamma = \rho(\pi_1(S))$ be the image of some $\rho \in AH(S)$. Then Γ is called a *b-group* if exactly one of Ω_Γ^+ or Ω_Γ^- is non-empty, connected and simply connected. We also call $\rho \in AH(S)$ a b-group if its image is a b-group. For a b-group $\rho \in AH(S)$, we denote by $\mathrm{para}(\rho)$ its *parabolic locus*, a collection of homotopy classes of disjoint simple closed curves c on S such that $\rho(c) \in \Gamma$ is parabolic. Similarly $\Gamma = \rho(\pi_1(S))$ for $\rho \in AH(S)$ is called a *quasifuchsian group* if both Ω_Γ^+ and Ω_Γ^- are non-empty, connected and simply connected. Quasifuchsian space $\mathcal{QF} = \mathcal{QF}(S)$ is the subset of $R(S)$ of faithful representations with quasifuchsian images. It is known by Marden [Mar] and Sullivan [Su] that $\mathcal{QF}$ equals the interior of $AH(S)$. Hence $\mathcal{QF}$ is a $(6g-6)$-dimensional complex manifold in $R(S)$. On the other hand, since $AH(S)$ is closed, the closure of $\mathcal{QF}$ is contained in $AH(S)$. We denote by $\partial\mathcal{QF}$ the relative boundary of $\mathcal{QF}$ in $R(S)$, whose element is called a *boundary group*.

Now let $\rho \in \mathcal{QF}$. Then there exists an orientation preserving homeomorphism $\psi : S \times [-1,1] \to \overline{M_\rho}$ which induces the representation $\psi_* = \rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$. Moreover, it provides orientation preserving homeomorphisms $\psi|_{S\times\{-1\}} : S \to \partial^+ M_\rho$ and $\psi|_{S\times\{1\}} : \bar{S} \to \partial^- M_\rho$, where $\bar{S}$ denotes S with its orientation reversed. Hence $\rho \in \mathcal{QF}$ determines a pair of marked Reimann surfaces $(\partial^+ M_\rho, \partial^- M_\rho) \in T(S) \times T(\bar{S})$ in the product of the Teichmüller spaces. On the other hand, Bers [Be1] showed that each pair $(X, \bar{Y}) \in T(S) \times T(\bar{S})$ has the unique simultaneous uniformization $\rho = B(X, \bar{Y}) \in \mathcal{QF}$. Therefore the map

$$B : T(S) \times T(\bar{S}) \to \mathcal{QF}$$

gives us a global parameterization of $\mathcal{QF}$. We define *vertical* and *horizontal Bers slices* in $\mathcal{QF}$ by $B_X = \{B(X,\bar{Y}) : \bar{Y} \in T(\bar{S})\}$ and $B_{\bar{Y}} = \{B(X,\bar{Y}) : X \in T(S)\}$, respectively. It is known by Bers that both B_X and $B_{\bar{Y}}$ are precompact in $R(S)$, whose frontiers are denoted by ∂B_X and $\partial B_{\bar{Y}}$. A representation $\rho \in AH(S)$ is called a *Bers boundary group* if it is contained in the frontier of some Bers slice.

2.2. **Sequences in quasifuchsian space.** We introduce the notion of "standard" and "exotic" convergence for a sequence $\rho_n \in \mathcal{QF}$ tending to a limit $\rho \in \partial\mathcal{QF}$. For a given subset K of $T(S)$, we set $\bar{K} = \{\bar{X} \in T(\bar{S}) : X \in K\} \subset T(\bar{S})$, where $\bar{X} \in T(\bar{S})$ denotes the complex conjugation of $X \in T(S)$.

Definition 2.1 (Standard and exotic convergence)**.** Suppose that a sequence $\rho_n = B(X_n, \bar{Y}_n)$ in $\mathcal{QF}$ converges to $\rho \in \partial\mathcal{QF}$. Then the sequence ρ_n is said to converge *standardly* to ρ if there exists a compact subset $K \subset T(S)$ such that (i) $X_n \in K$ for all n, or (ii) $\bar{Y}_n \in \bar{K}$ for all n. Otherwise, we say that ρ_n converges *exotically* to ρ.

We let $\partial^+\mathcal{QF}$ and $\partial^-\mathcal{QF}$ denote the subsets of $\partial\mathcal{QF}$ of all limits of standardly convergent sequences of type (i) and (ii), respectively. We set $\partial^\pm\mathcal{QF} = \partial^+\mathcal{QF} \sqcup \partial^-\mathcal{QF}$. A b-group $\rho \in AH(S)$ is said to be a *boundary b-group* if it is contained in $\partial\mathcal{QF}$.

Lemma 2.2. *The set $\partial^\pm\mathcal{QF}$ equals the set of all boundary b-groups. Moreover we have*

$$\partial^+\mathcal{QF} = \bigsqcup_{X \in T(S)} \partial B_X, \quad \partial^-\mathcal{QF} = \bigsqcup_{\bar{Y} \in T(\bar{S})} \partial B_{\bar{Y}}.$$

Proof. It is trivial that every Bers boundary group is a boundary b-group. Next we show that every element of $\partial^\pm\mathcal{QF}$ is a Bers boundary group. Suppose that $\rho_n = B(X_n, \bar{Y}_n)$ converges standardly to $\rho \in \partial^+\mathcal{QF}$ (the proof for $\rho \in \partial^-\mathcal{QF}$ is parallel). Then there is a compact $K \subset T(S)$ such that $X_n \in K$ for all n. By passing to a subsequence if necessary, X_n converges to some $X \in K$. Now let us consider a new sequence $\rho'_n = B(X, \bar{Y}_n)$ in B_X. Then this sequence ρ'_n also converges to ρ because maximal dilatations of quasiconformal automorphism of $\widehat{\mathbb{C}}$ conjugating ρ_n to ρ'_n tend to 1 as $n \to \infty$. This implies that $\rho \in \partial B_X$ and hence that ρ is a Bers boundary group. Finally, we can see that every boundary b-group is an element of $\partial^{\pm}\mathcal{QF}$ from the next theorem, which is essentially due to Brock, Bromberg, Evans and Souto [BBES]. □

Theorem 2.3 (Brock-Bromberg-Evans-Souto)**.** *Every boundary b-group is an element of $\partial^{\pm}\mathcal{QF}$.*

We outline the proof of Theorem 2.3 by following the arguments in [BBES]. In the argument, we also obtain Corollary 2.4 below, which is required in the proof of Theorem 4.1.

Corollary 2.4. *Let $\rho \in \partial^{\pm}\mathcal{QF}$ with parabolic locus* $\mathrm{para}(\rho)$*. Then there exists a sequence $\rho_n = B(X_n, \bar{Y}_n)$ in $\mathcal{QF}$ which converges standardly to ρ and which satisfies $l_{\bar{Y}_n}(\mathrm{para}(\rho)) \to 0$ if $\rho \in \partial^{+}\mathcal{QF}$ or $l_{X_n}(\mathrm{para}(\rho)) \to 0$ if $\rho \in \partial^{-}\mathcal{QF}$. Here $l_{\bar{Y}_n}(\mathrm{para}(\rho))$ denotes the total sum of hyperbolic lengths of components of* $\mathrm{para}(\rho)$ *on $\bar{Y}_n$, and so on.*

Outline of the proof of Theorem 2.3. Let $\rho \in \partial\mathcal{QF}$ be a boundary b-group with parabolic locus $\mathrm{para}(\rho)$, which is possibly empty. We may assume that the positive part $\partial_c^{+} M_\rho$ of the conformal boundary of $M_\rho = \mathbb{H}^3/\rho(\pi_1(S))$ is homeomorphic to S, and thus determine a point $X \in T(S)$. By Theorem 3.1 in [BBES], of which the drilling theorem plays an important role in the proof, there exists a strongly convergent sequence $\rho_n \to \rho$ in $\overline{\mathcal{QF}}$ such that each ρ_n is a geometrically finite representation with $\mathrm{para}(\rho_n) = \mathrm{para}(\rho)$. Since the sequence ρ_n converges strongly to ρ, note that $X_n = \partial_c^{+} M_{\rho_n}$ converges to X in $T(S)$. Now fix n for a while. Then by the same argument as in §5 in [BBES], one can take a sequence $\rho_{n,k} = B(X_n, \bar{Y}_{n,k})$ in B_{X_n} which converges to $\rho_n \in \partial B_{X_n}$ as $k \to \infty$ and which satisfies $\lim_{k\to\infty} l_{\bar{Y}_{n,k}}(\mathrm{para}(\rho)) = 0$. Then we can choose a sequence $\rho'_n = B(X_n, \bar{Y}_n)$ from $\{\rho_{n,k}\}_{n,k\in\mathbb{N}}$ such that $\rho'_n \to \rho$ and that $\lim_{n\to\infty} l_{\bar{Y}_n}(\mathrm{para}(\rho)) = 0$ by a diagonal argument. Especially, $\rho \in \partial^{\pm}\mathcal{QF}$. □

We remark that the set $\partial\mathcal{QF} - \partial^{\pm}\mathcal{QF}$ is not empty; for instance, it contains limits of sequences which appear in Thurston's double limit theorem. On the other hand, Anderson and Canary [AC] showed that there exists a sequence in $\mathcal{QF}$ which converges exotically to some point in $\partial^{\pm}\mathcal{QF}$.

2.3. **Space of projective structures.** We only give a brief summary of projective structures and refer to [It1] and elsewhere for more details.

A projective structure on S is a (G, X)-structure where X is the Riemann sphere $\widehat{\mathbb{C}}$ and $G = \mathrm{PSL}_2(\mathbb{C})$ is the group of projective automorphism of $\widehat{\mathbb{C}}$. Let $P(S)$ be the space of marked projective structures on S. A projective structure $\Sigma \in P(S)$ has an underlying conformal structure $\pi(\Sigma) \in T(S)$. It is known that $P(S)$ is the holomorphic affine bundle over $T(S)$ with the projection $\pi : P(S) \to T(S)$, and that $P(S)$ is a $(6g-6)$-dimensional complex manifold.

A projective structure Σ on S can be lifted to that $\widetilde{\Sigma}$ on $\widetilde{S}$, where $\widetilde{S} \to S$ is the universal cover on which $\pi_1(S)$ acts as a covering group. Since $\widetilde{\Sigma}$ is simply connected, we obtain a developing map $f_\Sigma : \widetilde{S} \to \widehat{\mathbb{C}}$ by continuing charts of $\widetilde{\Sigma}$ analytically. In addition, the developing map induces a holonomy representation $\rho_\Sigma : \pi_1(S) \cong \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ which satisfies $f_\Sigma \circ \gamma = \rho_\Sigma(\gamma) \circ f_\Sigma$ for every $\gamma \in \pi_1(S)$. We remark that the pair (f_Σ, ρ_Σ) is determined uniquely up to $\mathrm{PSL}_2(\mathbb{C})$.

We now define the *holonomy map*

$$hol : P(S) \to R(S)$$

by $\Sigma \mapsto [\rho_\Sigma]$. Then Hejhal [He] showed that the map *hol* is a local homeomorphism and Earle [Ea] and Hubbard [Hu] independently showed that the map is holomorphic:

Theorem 2.5 (Hejhal, Earle and Hubbard). *The holonomy map* $hol : P(S) \to R(S)$ *is a holomorphic local homeomorphism.*

We denote by $Q(S) = hol^{-1}(\mathcal{QF})$ the set of projective structures with quasifuchsian holonomy. An element of $Q(S)$ is said to be *standard* if its developing map is injective; otherwise it is *exotic*. We denote by $\mathcal{Q}_0 \subset Q(S)$ the set of all standard projective structures. Now let $\rho = B(X, \bar{Y}) \in \mathcal{QF}$ with image $\Gamma = \rho(\pi_1(S))$. Then the quotient surface $\Sigma = \Omega_\Gamma^+/\Gamma$ can be regarded as a standard projective structure on S with bijective developing map $f_\Sigma : \widetilde{S} \to \Omega_\Gamma^+$, with holonomy representation $\rho_\Sigma = \rho$, and with underlying conformal structure $X \in T(S)$. Let

$$\Psi_0 : \mathcal{QF} \to \mathcal{Q}_0$$

be the map defined by the correspondence $\rho \mapsto \Omega_\Gamma^+/\Gamma$ as described above. Then the map Ψ_0 turns out to be a univalent local branch of hol^{-1} onto the connected component $\mathcal{Q}_0$ of $Q(S)$, which is called the *standard component*. It is known by Bers [Be2] that every Bers slice $B_X \subset \mathcal{QF}$ is embedded by the map Ψ_0 into a bounded domain $\Psi_0(B_X)$ of a fiber $\pi^{-1}(X) \subset P(S)$. Then one can see that the map $hol|_{\mathcal{Q}_0} : \overline{\mathcal{Q}_0} \to \mathcal{QF} \sqcup \partial^+\mathcal{QF}$ is bijective, where $\overline{\mathcal{Q}_0}$ is the closure of $\mathcal{Q}_0$ in $P(S)$.

3. Grafting

3.1. **Grafting maps on quasifuchsian space.** We let $\mathcal{ML}_{\mathbb{N}} = \mathcal{ML}_{\mathbb{N}}(S)$ denote the set of integral points of measured laminations on S. In other words, each element of $\lambda \in \mathcal{ML}_{\mathbb{N}}$ is a homotopy class of disjoint union $\sqcup_{i=1}^{l} k_i c_i$ of homotopically distinct simple closed curves c_i on S with positive integer k_i weights. We do not distinguish the homotopy class $\lambda \in \mathcal{ML}_{\mathbb{N}}$ and its representative if there is no confusion. The "zero-lamination" 0 is also contained in $\mathcal{ML}_{\mathbb{N}}$. In what follows, the parabolic locus $\mathrm{para}(\rho)$ of a *b*-group ρ is also regarded as an element of $\mathcal{ML}_{\mathbb{N}}$.

For each non-zero $\lambda \in \mathcal{ML}_{\mathbb{N}}$, we will explain how to obtain the grafting map $\mathrm{Gr}_\lambda : \mathcal{Q}_0 \to P(S)$, which satisfies $hol \circ \mathrm{Gr}_\lambda \equiv hol$ on $\mathcal{Q}_0$. We will give two equivalent definitions of grafting operation in Definition 3.1; the first one is as usual and the second one is introduced by Bromberg in [Br] so that it also makes sense for elements of $\partial^-\mathcal{QF}$.

We first fix our notation and situation in Definition 3.1 as follows: we assume that λ is a simple closed curve c of weight one for simplicity. Let $\Sigma \in \mathcal{Q}_0$ be a standard projective structure with holonomy representation $\rho_\Sigma \in \mathcal{QF}$. We let $\Omega_\Gamma = \Omega_\Gamma^+ \sqcup \Omega_\Gamma^-$ denote the region of discontinuity of the quasifuchsian group $\Gamma = \rho_\Sigma(\pi(S))$. Then we have projective structures $\Sigma = \Omega_\Gamma^+/\Gamma = \Psi_0(\rho)$ on S and $\Sigma^- = \Omega_\Gamma^-/\Gamma$ on $\bar{S}$. Let $\gamma \in \Gamma$ be a representative of the homotopy class of c and let $\widetilde{c}^+ \subset \Omega_\Gamma^+$ and $\widetilde{c}^- \subset \Omega_\Gamma^-$ be $\langle\gamma\rangle$-invariant simple arcs. Then the quotient curve $c^+ = \widetilde{c}^+/\langle\gamma\rangle$ is contained in Σ and $c^- = \widetilde{c}^-/\langle\gamma\rangle$ in Σ^-. We let $T = \Omega_{\langle\gamma\rangle}/\langle\gamma\rangle$ be the quotient torus with the projective structure which is induced from that of $\widehat{\mathbb{C}}$. Then T also contains the curves c^+ and c^-.

Definition 3.1 (Grafting)**.** In the situation as described above, the *grafting* $\mathrm{Gr}_c(\Sigma)$ of $\Sigma \in \mathcal{Q}_0$ along c is a projective structure obtained by the following (equivalent) procedures (see Figure 1):

I: We obtain $\mathrm{Gr}_c(\Sigma)$ by cutting both Σ and T along c^+ and glueing their boundaries without twisting.

II: Here we further assume that c separates S into two surfaces S_1 and S_2 with boundaries. (The non-separating case is described precisely in [Br].) Accordingly, Σ and Σ^- decompose into $\Sigma - c^+ = \Sigma_1 \sqcup \Sigma_2$ and $\Sigma^- - c^- = \Sigma_1^- \sqcup \Sigma_2^-$, respectively. Let $\Delta_i \subset \Omega_\Gamma^-$ $(i = 1, 2)$ be the connected component of the inverse image of $\Sigma_i^- \subset \Sigma^- = \Omega_\Gamma^-/\Gamma$ whose closure $\overline{\Delta}_i$ contains $\widetilde{c}^-$ in its boundary. Then the stabilizer subgroup $\Gamma_i = \mathrm{Stab}_\Gamma(\Delta_i)$ of $\Gamma \cong \pi_1(S)$ is identified with $\pi_1(S_i)$. Since Γ_i is a purely loxodromic free group with non-empty region of discontinuity, Maskit's result [Mas] implies that Γ_i is a Schottky group. Note that the conformal boundary $\partial M_{\Gamma_i} = \Omega_{\Gamma_i}/\Gamma_i$ of $M_{\Gamma_i} = \mathbb{H}^3/\Gamma_i$ with natural projective structure contains both projective surfaces Σ_i and Σ_i^-. Then $\mathrm{Gr}_c(\Sigma)$ is obtained from projective surfaces $\Omega_{\Gamma_1}/\Gamma_1 - \Sigma_1^-$ and $\Omega_{\Gamma_2}/\Gamma_2 - \Sigma_2^-$ by gluing their boundaries without twisting.

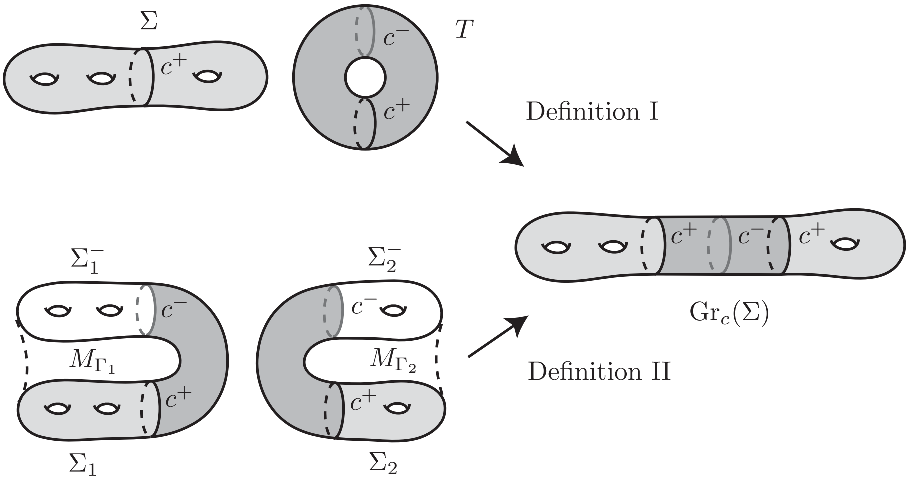


FIGURE 1. The grafting $\mathrm{Gr}_c(\Sigma)$ of Σ along c.

Observe that the Definitions I and II are equivalent. The grafting $\mathrm{Gr}_\lambda(\Sigma)$ of Σ along general $\lambda = \sqcup k_i c_i \in \mathcal{ML}_\mathbb{N}$ is also defined by linearity. An important fact is that the grafting operation does not change the holonomy representation; that is, $hol(\mathrm{Gr}_\lambda(\Sigma)) = hol(\Sigma)$ is always satisfied.

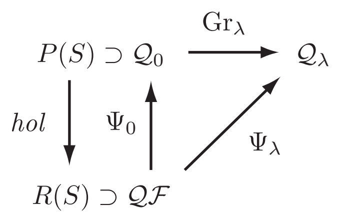
We define the *grafting map*

$$\mathrm{Gr}_\lambda : \mathcal{Q}_0 \to P(S)$$

for $\lambda \in \mathcal{ML}_\mathbb{N}$ by $\Sigma \mapsto \mathrm{Gr}_\lambda(\Sigma)$. Since the map Gr_λ satisfies $hol \circ \mathrm{Gr}_\lambda \equiv hol$ on $\mathcal{Q}_0$ and since $hol|_{\mathcal{Q}_0} : \mathcal{Q}_0 \to \mathcal{QF}$ is a biholomorphic map with its inverse $\Psi_0 : \mathcal{QF} \to \mathcal{Q}_0$, the map Gr_λ takes $\mathcal{Q}_0$ biholomorphically onto the image $\mathrm{Gr}_\lambda(\mathcal{Q}_0)$, which is denoted by $\mathcal{Q}_\lambda$. Hence we obtain a univalent local branch

$$\Psi_\lambda : \mathcal{QF} \to \mathcal{Q}_\lambda$$

of hol^{-1} which is defined by $\Psi_\lambda = \mathrm{Gr}_\lambda \circ \Psi_0$ (see the commutative diagram below). The map Ψ_λ is also called the *grafting map* for λ. By Goldman's grafting theorem



[Go] below, we obtain the decomposition $\bigsqcup_{\lambda \in \mathcal{ML}_\mathbb{N}} \mathcal{Q}_\lambda$ of $Q(S)$ into its connected components.

Theorem 3.2 (Goldmann [Go])**.** *For every $\rho \in \mathcal{QF}$, all projective structures with holonomy ρ are obtained by grafting of ρ; that is, we have $hol^{-1}(\rho) = \{\Psi_\lambda(\rho) : \lambda \in \mathcal{ML}_\mathbb{N}\}$.*

3.2. **Extension of grafting maps.** Let $\widehat{P}(S)$ denotes the one-point compactification $P(S) \cup \{\infty\}$ of $P(S)$. We now extend the grafting map $\Psi_\lambda : \mathcal{QF} \to \mathcal{Q}_\lambda$ to $\Psi_\lambda : \mathcal{QF} \sqcup \partial^\pm \mathcal{QF} \to \widehat{P}(S)$.

We first suppose that $\lambda \in \mathcal{ML}_\mathbb{N}$ is a simple closed curve c of weight 1. Then Definition 3.1.I also works well for $\rho \in \partial^+ \mathcal{QF}$ whenever c is not a component of para(ρ). In fact, in this case, γ is still loxodromic and there still exists a $\langle \gamma \rangle$-invariant simple arc $\tilde{c}^+$ in the non-degenerate component Ω_Γ^+. Therefore we can obtain the projective structure $\Psi_\lambda(\rho)$ by glueing $\Omega_\Gamma^+/\Gamma - c^+$ and $\Omega_{\langle\gamma\rangle}/\langle\gamma\rangle - c^+$ in the same way as in Definition 3.1.I.

On the other hand, Definition 3.1.II also works well for $\rho \in \partial^- \mathcal{QF}$ whenever every connected component of para(ρ) intersects c essentially. In fact, in this case, γ is still loxodromic, there still exists a $\langle \gamma \rangle$-invariant simple arc $\tilde{c}^-$ in the non-degenerate component Ω_Γ^-, and the groups Γ_1 and Γ_2 in Definition 3.1.II are still Schottky groups. Therefore we can obtain the projective structure $\Psi_\lambda(\rho)$ in the same way as in Definition 3.1.II.

By the same argument as above, we can also define the grafting $\Psi_\lambda(\rho)$ of $\rho \in \partial^\pm \mathcal{QF}$ for general $\lambda \in \mathcal{ML}_\mathbb{N}$ if the pair (λ, ρ) satisfies the following condition:

Definition 3.3 (Admissible)**.** The pair (λ, ρ) of $\lambda \in \mathcal{ML}_\mathbb{N}$ and $\rho \in \partial^\pm \mathcal{QF}$ is said to be *admissible*

- if $\rho \in \partial^+ \mathcal{QF}$ and if para(ρ) and λ have no parallel component in common; or
- if $\rho \in \partial^- \mathcal{QF}$ and if every component of para(ρ) intersects λ essentially.

If the pair (λ, ρ) is not admissible, we set $\Psi_\lambda(\rho) = \infty \in \widehat{P}(S)$. Now we obtain the extended grafting map

$$\Psi_\lambda : \mathcal{QF} \sqcup \partial^\pm \mathcal{QF} \to \widehat{P}(S),$$

which is also denoted by the same symbol Ψ_λ. Theorem 4.1 in the next section asserts that the map Ψ_λ takes every standardly convergent sequence $\mathcal{QF} \ni \rho_n \to \rho \in \partial^\pm \mathcal{QF}$ to a convergent sequence $\Psi_\lambda(\rho_n) \to \Psi_\lambda(\rho)$ in $\hat{P}(S)$. On the contrary, the map Ψ_λ does not take an exotically convergent sequence $\mathcal{QF} \ni \rho_n \to \rho \in \partial^\pm \mathcal{QF}$

to a convergent sequence in general (cf. [AC], [Mc]). By studying this phenomena more closely, we obtained the following:

Theorem 3.4 ([Ita])**.** *For any $\lambda \in \mathcal{ML}_{\mathbb{N}}$, the grafting map $\Psi_\lambda : \mathcal{QF} \sqcup \partial^{\pm}\mathcal{QF} \to \widehat{P}(S)$ is not continuous.*

4. Continuity

One of our main results in this paper is the following:

Theorem 4.1 (Continuity)**.** *Let $\rho_n \in \mathcal{QF}$ be a sequence which converges standardly to $\rho \in \partial^{\pm}\mathcal{QF}$. Then the sequence $\Psi_\lambda(\rho_n)$ converges to $\Psi_\lambda(\rho)$ in $\widehat{P}(S)$ for every $\lambda \in \mathcal{ML}_{\mathbb{N}}$.*

We divide the proof of Theorem 4.1 into two parts:

- the pair (λ, ρ) is admissible, i.e. $\Psi_\lambda(\rho) \in P(S)$; and
- the pair (λ, ρ) is not admissible, i.e. $\Psi_\lambda(\rho) = \infty$.

In this section, we give the proof of Theorem 4.1 for admissible pairs, and the proof for non-admissible pairs is given in §6 after preparing some divergence property in §5. Here we recall the definition of the topology on $P(S)$ which is suitable for our context; see [CEG, 1.5.4] or [KT, 2.1].

Definition 4.2 (Topology on $P(S)$)**.** A sequence Σ_n of projective structures converges to Σ in $P(S)$ if there exist orientation preserving C^1-diffeomorphisms $\varphi_n : \Sigma \to \Sigma_n$ consistent with their markings such that $f_{\Sigma_n} \circ \widetilde{\varphi}_n$ converge to f_Σ in compact-C^1 topology on $C^1(\widetilde{\Sigma}, \widehat{\mathbb{C}})$ by choosing lifts $\widetilde{\varphi}_n : \widetilde{\Sigma} \to \widetilde{\Sigma}_n$ of φ_n and developing maps $f_{\Sigma_n} : \widetilde{\Sigma}_n \to \widehat{\mathbb{C}}$, $f_\Sigma : \widetilde{\Sigma} \to \widehat{\mathbb{C}}$ suitably.

Now suppose that the pair (λ, ρ) of $\lambda \in \mathcal{ML}_{\mathbb{N}}$ and $\rho \in \partial^s\mathcal{QF}$ is admissible, where s denotes $+$ or $-$. Then the essential point in the proof of Theorem 4.1 for admissible pairs is that, since the sequence ρ_n converges standardly to ρ (but not necessarily strongly), there exist quasiconformal maps $\widetilde{g}_n : \Omega^s_\Gamma \to \Omega^s_{\Gamma_n}$ inducing group isomorphisms $\rho_n \circ \rho^{-1} : \rho(\pi_1(S)) = \Gamma \to \Gamma_n = \rho_n(\pi_1(S))$ which converge locally uniformly to the identity map in Ω^s_Γ (cf. [Be2]).

Proof of Theorem 4.1 for admissible pairs. Suppose that the pair (λ, ρ) is admissible, i.e. $\Psi_\lambda(\rho) \in P(S)$. We assume that λ is a simple closed curve c of weight 1 for simplicity and use the same notation as in Definition 3.1. We first consider the case of $\rho \in \partial^+\mathcal{QF}$. Set $\Gamma_n = \rho_n(\pi_1(S))$ and $\Gamma = \rho(\pi_1(S))$. Since the sequence ρ_n converges standardly to $\rho \in \partial^+\mathcal{QF}$, there exist quasiconformal maps $\widetilde{g}_n : \Omega^+_\Gamma \to \Omega^+_{\Gamma_n}$ inducing group isomorphisms $\rho_n \circ \rho^{-1} : \Gamma \to \Gamma_n$ which converge locally uniformly to the identity map in Ω^+_Γ. We may assume that the maps $\widetilde{g}_n : \Omega^+_\Gamma \to \Omega^+_{\Gamma_n}$ are C^1-diffeomorphisms and that $\widetilde{g}_n \to id$ in the compact-C^1 topology on $C^1(\Omega^+_\Gamma, \widehat{\mathbb{C}})$. The maps $\widetilde{g}_n$ descend to C^1-diffeomorphisms $g_n : \Sigma \to \Sigma_n$, where $\Sigma = \Omega^+_\Gamma/\Gamma$ and $\Sigma_n = \Omega^+_{\Gamma_n}/\Gamma_n$. The the maps $g_n : \Sigma \to \Sigma_n$ satisfy the conditions in Definition 4.2 and hence Σ_n converge to Σ in $P(S)$.

On the other hand, since $\rho_n(\gamma) \to \rho(\gamma)$ in $\mathrm{PSL}_2(\mathbb{C})$, there exist C^1-diffeomorphisms $\widetilde{h}_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ conjugating $\rho(\gamma)$ to $\rho_n(\gamma)$ such that $\widetilde{h}_n \to id$ in the compact-C^1 topology on $C^1(\widehat{\mathbb{C}}, \widehat{\mathbb{C}})$. The maps $\widetilde{h}_n$ descend to C^1-diffeomorphisms $h_n : T \to T_n$, where $T = \Omega_{\langle\rho(\gamma)\rangle}/\langle\rho(\gamma)\rangle$ and $T_n = \Omega_{\langle\rho_n(\gamma)\rangle}/\langle\rho_n(\gamma)\rangle$. Then the maps $h_n : T \to T_n$ satisfy the conditions in Definition 4.2 and hence T_n converge to T as marked projective structures.

Now recall that the grafting $\Psi_c(\rho)$ for $\rho \in \partial^+\mathcal{QF}$ is obtained from projective surfaces $\Sigma - c^+$ and $T - c^+$ by glueing their boundaries:

$$\Psi_c(\rho) = \left(\Sigma - c^+\right) \sqcup \left(T - c^+\right).$$

Similarly, we have

$$\Psi_c(\rho_n) = \left(\Sigma_n - g_n(c^+)\right) \sqcup \left(T_n - g_n(c^+)\right)$$

by regarding the curve $g_n(c^+) = \widetilde{g}_n(\tilde{c}_n)/\langle\rho_n(\gamma)\rangle \subset \Sigma_n$ also as a curve in T_n. To show that $\Psi_c(\rho_n) \to \Psi_c(\rho)$ in $P(S)$, we construct C^1-diffeomorphisms

$$\varphi_n : \Psi_c(\rho) \to \Psi_c(\rho_n)$$

satisfying the condition in Definition 4.2 by glueing the maps $g_n|_{\Sigma - c^+}$ and $h_n|_{T-c^+}$ piece by piece. More precisely, for $j = 1, 2$, let R_j be a regular neighborhood of c^+ in Σ such that $R_1 \subset R_2$ and let $\widetilde{R}_j \subset \Omega_\Gamma^+$ be the $\langle\rho(\gamma)\rangle$-invariant preimage of R_j. Then $R_j = \widetilde{R}_j/\langle\rho(\gamma)\rangle$ is also regarded as a regular neighborhood of $c^+ \subset T = \Omega_{\langle\rho(\gamma)\rangle}/\langle\rho(\gamma)\rangle$. Since both $\widetilde{g}_n$, $\widetilde{h}_n$ converge to id on $\widetilde{R}_2$, we can modify the maps $h_n : T \to T_n$ so that $h_n \equiv g_n$ on R_1 and that they still satisfy the condition in Definition 4.2. Then we obtain C^1-diffeomorphisms $\varphi_n : \Psi_c(\rho) \to \Psi_c(\rho_n)$ by setting $\varphi_n \equiv g_n$ on $\Sigma - c^+$ and $\varphi_n \equiv h_n$ on $T - c^+$. From the construction of φ_n, we can see that the maps $f_{\Psi_c(\rho_n)} \circ \widetilde{\varphi}_n : \widetilde{\Psi_c(\rho)} \to \widehat{\mathbb{C}}$ converge to $f_{\Psi_c(\rho)}$ in the compact-C^1 topology on $C^1\left(\widetilde{\Psi_c(\rho)}, \widehat{\mathbb{C}}\right)$ and hence that $\Psi_c(\rho_n) \to \Psi_c(\rho)$ in $P(S)$. Thus we complete the proof of Theorem 4.1 in this case.

Next we consider the case of $\rho \in \partial^-\mathcal{QF}$. The argument is similar to the case of $\rho \in \partial^+\mathcal{QF}$. Since the sequence ρ_n converges standardly to $\rho \in \partial^-\mathcal{QF}$, there exist quasiconformal maps $\widetilde{g}_n : \Omega_\Gamma^- \to \Omega_{\Gamma_n}^-$ inducing group isomorphisms $\rho_n \circ \rho^{-1} : \Gamma \to \Gamma_n$ such that $\widetilde{g}_n \to id$ locally uniformly in Ω_Γ^-. We may assume that $\widetilde{g}_n : \Omega_\Gamma^- \to \Omega_{\Gamma_n}^-$ are C^1-maps and that $\widetilde{g}_n \to id$ in the compact-C^1 topology on $C^1(\Omega_\Gamma^-, \widehat{\mathbb{C}})$. Then the map $\widetilde{g}_n$ descends to C^1-diffeomorphisms $g_n : \Sigma^- \to \Sigma_n^-$, where $\Sigma^- = \Omega_\Gamma^-/\Gamma$ and $\Sigma_n^- = \Omega_{\Gamma_n}^-/\Gamma_n$.

Recall that we are assuming that $S - c = S_1 \sqcup S_2$. We let i denote 1 or 2. Then $\Gamma_{i,n} = \rho_n(\pi_1(S_i))$ and $\Gamma_i = \rho(\pi_1(S_i))$ are Schottky groups. Since $\rho_n|_{\pi_1(S_i)} \to \rho|_{\pi_1(S_i)}$ and since Γ_i are quasiconformally stable [Mar], there exist quasiconformal maps $\widetilde{h}_{i,n} : \Omega_{\Gamma_i} \to \Omega_{\Gamma_{i,n}}$ inducing group isomorphisms $\rho_n \circ \rho^{-1}|_{\Gamma_i} : \Gamma_i \to \Gamma_{i,n}$ such that $\widetilde{h}_{i,n} \to id$ locally uniformly in Ω_{Γ_i}. We may assume that $\widetilde{h}_{i,n} : \Omega_{\Gamma_i} \to \Omega_{\Gamma_{i,n}}$ are C^1-maps and that $\widetilde{h}_{i,n} \to id$ in the compact-C^1 topology on $C^1(\Omega_{\Gamma_i}, \widehat{\mathbb{C}})$. Then the map $\widetilde{h}_{i,n}$ descends to a C^1-diffeomorphism $h_{i,n} : \Theta_i \to \Theta_{i,n}$, where we set $\Theta_i = \Omega_{\Gamma_i}/\Gamma_i$ and $\Theta_{i,n} = \Omega_{\Gamma_{i,n}}/\Gamma_{i,n}$.

Now recall that the grafting $\Psi_\lambda(\rho)$ for $\rho \in \partial^-\mathcal{QF}$ is obtained as follows:

$$\Psi_c(\rho) = \left(\Theta_1 - \Sigma_1^-\right) \sqcup \left(\Theta_2 - \Sigma_2^-\right),$$

where $\Sigma^- - c^- = \Sigma_1^- \sqcup \Sigma_2^-$. Similarly we have

$$\Psi_c(\rho_n) = \left(\Theta_{1,n} - \Sigma_{1,n}^-\right) \sqcup \left(\Theta_{2,n} - \Sigma_{2,n}^-\right)$$

by setting $\Sigma_{i,n}^- = g_n(\Sigma_i^-) \subset \Sigma_n^-$ for $i = 1, 2$. Since both $\widetilde{g}_n$, $\widetilde{h}_{i,n}$ converge to id, we can modify the maps $h_{i,n} : \Theta_i \to \Theta_{i,n}$ so that $h_{i,n} \equiv g_n$ on Σ_i^- and that they still satisfy the condition in Definition 4.2. We now define the map

$$\varphi_n : \Psi_c(\rho) \to \Psi_c(\rho_n)$$

by $\varphi_n \equiv h_{i,n}$ on $\Theta_{i,n}$ for $i = 1, 2$. Then one can see that $\Psi_c(\rho_n) \to \Psi_c(\rho)$ in $P(S)$. Thus we have completed the proof of Theorem 4.1 for admissible pairs. □

Corollary 4.3. *Suppose that two pairs* (λ, ρ), (μ, ρ) *are both admissible, where* $\lambda, \mu \in \mathcal{ML}_{\mathbb{N}}, \lambda \neq \mu$ *and* $\rho \in \partial^{\pm}\mathcal{QF}$. *Then* $\Psi_\lambda(\rho) \neq \Psi_\mu(\rho)$.

Proof. We first remark that $\Psi_\lambda(\eta) \neq \Psi_\mu(\eta)$ for all $\eta \in \mathcal{QF}$ (cf. [Go]). Now let $\mathcal{QF} \ni \rho_n \to \rho \in \partial^{\pm}\mathcal{QF}$ be a standardly convergent sequence. Then Theorem 4.1 for admissible pairs implies that $\Psi_\lambda(\rho_n) \to \Psi_\lambda(\rho)$ and $\Psi_\mu(\rho_n) \to \Psi_\mu(\rho)$. Suppose that $\Psi_\lambda(\rho) = \Psi_\mu(\rho)$. Then, since the map *hol* is a local homeomorphism, we have $\Psi_\lambda(\rho_n) = \Psi_\mu(\rho_n)$ for all large n, which is a contradiction. □

5. Length estimates and divergence

The aim of this section is to prove Theorem 5.5 (Divergence). A corollary of this theorem (Corollary 5.6) guarantees some uniqueness: let $\Sigma_n \to \Sigma$ be a convergent sequence in $P(S)$ such that the sequence $\rho_{\Sigma_n} \in \mathcal{QF}$ converges standardly to $\rho_\Sigma \in \partial^{\pm}\mathcal{QF}$. Then the sequence Σ_n is eventually contained in one component of $Q(S)$. This property plays an important role in the succeeding sections.

5.1. **Length-Modulus inequality.** Recall that the *modulus* $\mathrm{Mod}(A)$ of a conformal annulus A is defined uniquely as the ratio of the height and the circumference of an Euclidean annulus which is conformally equivalent to A. The following inequality is a direct consequence of the geometric and analytic definitions of the extremal length.

Lemma 5.1. *Let* X *be a complete hyperbolic surface of finite area* $\mathrm{Area}(X) < \infty$ *and let* A *be an essential annular domain in* X. *Then we have*

$$\mathrm{Mod}(A)\, l_X(c)^2 \le \mathrm{Area}(X),$$

where $l_X(c)$ *is the hyperbolic length of the homotopy class* c *of a core curve of* A.

Proof. Let $E_X(c)$ denotes the extremal length of the homotopy class c in X. From the analytical definition of $E_X(c)$, we have

$$E_X(c) := \sup_\rho \frac{\left(\inf_{c'} \int_{c'} \rho(z)\,|dz|\right)^2}{\iint \rho(z)^2\,|dz|^2} \ge \frac{(l_X(c))^2}{\mathrm{Area}(X)},$$

where the supremum is taken over all metrics ρ consistent with the conformal structure of X and the infimum is taken over all closed curves c' in the homotopy class c. On the other hand, from the geometrical definition of $E_X(c)$, we have

$$E_X(c) := \frac{1}{\sup_{A' \subset X} \mathrm{Mod}(A')} \le \frac{1}{\mathrm{Mod}(A)},$$

where the supremum is taken over all annuli $A' \subset X$ whose core curve is in the homotopy class c. From the above two inequality, the desired inequality follows. □

5.2. **Quasiconformal deformations.** We introduce the notion of a quasiconformal deformation of a projective structure with quasifuchsian holonomy, which was developed by Shiga and Tanigawa in [ST]. Let $\Sigma \in Q(S)$ and let $\rho_\Sigma \in \mathcal{QF}$ be its holonomy. Suppose that $\rho' \in \mathcal{QF}$ is a quasiconformal deformation of ρ_Σ induced by a quasiconformal automorphism $q : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, whose Beltrami differential is denoted by μ. Then we obtain a new projective structures Σ' with holonomy ρ' which is characterized as follows:

(1) There is a quasiconformal map $\tilde{\varphi} : \widetilde{\Sigma} \to \widetilde{\Sigma}'$ whose Beltrami differential is equal to the pull-back $f_\Sigma^*(\mu)$ of μ via the developing map $f_\Sigma : \widetilde{\Sigma} \to \widehat{\mathbb{C}}$. Moreover, the map $\tilde{\varphi} : \widetilde{\Sigma} \to \widetilde{\Sigma}'$ descends to a quasiconformal map $\varphi : \Sigma \to \Sigma'$, which is consistent with their markings.
(2) The developing map of Σ' is defined by $f_{\Sigma'} = q \circ f_\Sigma \circ \tilde{\varphi}^{-1} : \widetilde{\Sigma}' \to \widehat{\mathbb{C}}$.

Here we say that a map between projective structures is *quasiconformal* if it is a quasiconformal map between their underlying conformal structures. We call Σ' the *quasiconformal deformation* of Σ. We remark that every grafting map $\Psi_\lambda : \mathcal{QF} \to \mathcal{Q}_\lambda$ is obtained by quasiconformal deformations of some fixed $\Sigma \in \mathcal{Q}_\lambda$ and its holonomy $\rho_\Sigma \in \mathcal{QF}$.

5.3. **Length estimates.**

Theorem 5.2. *Fix $X \in T(S)$ arbitrarily. Suppose that $\lambda \in \mathcal{ML}_\mathbb{N}$ contains a weighted simple closed curve kc of weight $k \in \mathbb{N}$. Let $\rho \in B_X \cup B_{\bar{X}}$ and let $X' \in T(S)$ denotes the underlying conformal structure of $\Psi_\lambda(\rho)$. Then we have*

$$l_{X'}(c)^2 \leq \frac{4(g-1)\, l_X(c)}{k},$$

where g denotes the genus of S.

Remark. We make use of the inequality above mostly in the following form:

$$\frac{l_{X'}(c)}{l_X(c)} \leq \sqrt{\frac{4(g-1)}{k \cdot l_X(c)}},$$

which ensures that if $k \cdot l_X(c)$ is large enough then the Teichmüller distance $d_{T(S)}(X, X')$ between X and X' is also large.

Proof of Theorem 5.2. We first treat the case where λ is a simple closed curve c of weight one. Let $\rho^0 = B(X, \bar{X}) \in \mathcal{QF}$ be the Fuchsian representation with image $\Gamma^0 = \rho^0(\pi_1(S))$. We normalize ρ^0 so that $\Omega^+_{\Gamma^0}$ equals the upper half plane $H = \{x+iy \in \mathbb{C} : y > 0\}$, $\Omega^-_{\Gamma^0}$ equals the lower half plane $H^* = \{x+iy \in \mathbb{C} : y < 0\}$, and that the hyperbolic element $\gamma = \rho^0(c) \in \Gamma^0$ fixes the positive imaginary axis $i\mathbb{R}_+$. We let Σ^0 denote the standard projective structure $\Psi_0(\rho^0) = \Omega^+_{\Gamma^0}/\Gamma^0$. Recall that the grafting $\Sigma^0_c := \Psi_c(\rho^0) = \mathrm{Gr}_c(\Sigma^0)$ of Σ^0 along c is obtained from Σ^0 by cutting along c and inserting the annulus $A = (\mathbb{C} - i\mathbb{R}_+)/\langle\gamma\rangle$. We set $H - i\mathbb{R}_+ = H' \sqcup H''$ and let $A_1 = H'/\langle\gamma\rangle$, $A_2 = H^*/\langle\gamma\rangle$ and $A_3 = H''/\langle\gamma\rangle$. Then $A = A_1 \sqcup A_2 \sqcup A_3 \subset \Sigma^0_c$. Note that $\mathrm{Mod}(A_1) = \mathrm{Mod}(A_3) = \frac{\pi}{2l_X(c)}$ and $\mathrm{Mod}(A_2) = \frac{\pi}{l_X(c)}$.

Now let $\rho \in B_X \cup B_{\bar{X}}$ be a quasiconformal deformation of $\rho^0 = B(X, \bar{X})$ which is induced by a quasiconformal map $q : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Associated to the deformation ρ of ρ^0, we obtain a quasiconformal deformation $\Sigma_c = \Psi_c(\rho)$ of $\Sigma^0_c = \Psi_c(\rho^0)$ and a quasiconformal map $\varphi : \Sigma^0_c \to \Sigma_c$. Here we claim that $\mathrm{Mod}(\varphi(A)) \geq \frac{\pi}{l_X(c)}$. In fact, if $\rho \in B_X$, we may assume that q is conformal on the upper half plane

$H = \Omega^+_{\Gamma^0}$. The the map φ takes $A_1 \sqcup A_3 \subset \Sigma^0_c$ conformally into $\varphi(A) \subset \Sigma_c$ and thus we obtain $\mathrm{Mod}(\varphi(A)) \geq \mathrm{Mod}(\varphi(A_1)) + \mathrm{Mod}(\varphi(A_3)) = \frac{\pi}{l_X(c)}$. Similarly, if $\rho \in B_{\bar{X}}$, we may assume that q is conformal on the lower half plane $H^* = \Omega^-_{\Gamma^0}$. Then the map φ takes $A_2 \subset \Sigma^0_c$ conformally into $\varphi(A) \subset \Sigma_c$ and thus we obtain $\mathrm{Mod}(\varphi(A)) \geq \mathrm{Mod}(\varphi(A_2)) = \frac{\pi}{l_X(c)}$.

Recall that X' denote the underlying conformal structure of $\Sigma_c = \Psi_c(\rho)$. Lemma 5.1 then implies that $\mathrm{Mod}(\varphi(A))\, l_{X'}(c)^2 \leq \mathrm{Area}(X') = 4\pi(g-1)$. By combining this inequality with the inequality $\mathrm{Mod}(\varphi(A)) \geq \frac{\pi}{l_X(c)}$, the result follows in the case where $\lambda = c$.

Next suppose that $\lambda \in \mathcal{ML}_{\mathbb{N}}$ contains a weighted simple closed curve kc with $k \in \mathbb{N}$. Then the argument above reveals that $\Sigma_\lambda = \Psi_\lambda(\rho)$ contains an annulus $\tilde{A}$ which is a union of succeeding k parallel annuli each of whose modulus $\geq \frac{\pi}{l_X(c)}$. Then $\mathrm{Mod}(\tilde{A}) \geq \frac{k\pi}{l_X(c)}$ and hence we have obtained the desired inequality in general. □

Corollary 5.3. *For every non-zero $\lambda \in \mathcal{ML}_{\mathbb{N}}$, the map $\pi \circ \Psi_\lambda : \mathcal{QF} \to T(S)$ takes every horizontal or vertical Bers slice to a non-precompact proper subset of $T(S)$.*

Proof. Suppose that $kc \subset \lambda$. Then Theorem 5.2 tells us that the set $\pi \circ \Psi_\lambda(B_X \cup B_{\bar{X}})$ is contained in the proper subset

$$\left\{ X' \in T(S) : l_{X'}(c) \leq \sqrt{\frac{4(g-1)\, l_X(c)}{k}} \right\}$$

of $T(S)$. Now let $\rho \in B_X \cup B_{\bar{X}}$. From the construction of $\Psi_\lambda(\rho)$, one can see that the length $l_{X'}(c)$ of c in $X' = \pi \circ \Psi_\lambda(\rho)$ can be made arbitrarily small by varying $\rho = B(X, \bar{Y})$ in B_X so that $l_{\bar{Y}}(c) \to 0$, or $\rho = B(Y, \bar{X})$ in $B_{\bar{X}}$ so that $l_Y(c) \to 0$. This implies that both $\pi \circ \Psi_\lambda(B_X)$ and $\pi \circ \Psi_\lambda(B_{\bar{X}})$ are not precompact in $T(S)$. □

Compare the above result with a result which was independently obtained by Gallo [Ga] and Tanigawa [Ta]:

Theorem 5.4 (Gallo, Tanigawa)**.** *For every $\lambda \in \mathcal{ML}_{\mathbb{N}}$, the map $\pi \circ \Psi_\lambda$ takes the Fuchsian space $\mathcal{F} = \{B(X, \bar{X}) \in \mathcal{QF} : X \in T(S)\}$ bijectively onto $T(S)$.*

5.4. **Divergence.** We define a *truncated quasifuchsian space* for a compact subset K of $T(S)$ by

$$\mathcal{QF}_K = \{B(X, \bar{Y}) \in \mathcal{QF} : X \,\mathrm{or}\, Y \in K\},$$

which is precompact in $R(S)$. Note that $\mathcal{QF}_K = \bigcup_{X \in K} B_X \cup B_{\bar{X}}$ is satisfied.

Theorem 5.5 (Divergence)**.** *Suppose that $\mathcal{QF}_K$ is the truncated quasifuchsian space for some compact subset K of $T(S)$ and that $\{\lambda_n\}$ is a sequence of distinct elements of $\mathcal{ML}_{\mathbb{N}}$. Then the sequence $\{\pi \circ \Psi_{\lambda_n}(\mathcal{QF}_K)\}$ eventually escapes any compact subset L of $T(S)$; that is, $\pi \circ \Psi_{\lambda_n}(\mathcal{QF}_K) \cap L = \emptyset$ for all large enough n.*

Proof. We first show that for a given $X \in K$ there exists a sequence of weighted simple closed curves $k_n c_n \subset \lambda_n$ which satisfies $k_n \cdot l_X(c_n) \to \infty$ as $n \to \infty$. In fact, suppose for contradiction that both k and $l_X(c)$ are bounded for all weighted simple closed curves $kc \subset \lambda_n$ and for all n. Then the number of such simple closed curves c is also bounded. Therefore, the set $\{kc \subset \lambda_n \mid n \in \mathbb{N}\}$ is finite and hence that the set $\{\lambda_n \in \mathcal{ML}_{\mathbb{N}} \mid n \in \mathbb{N}\}$ is also finite, which is a contradiction. Now recall from Lemma 3.1 in [Wo] that for any $X, X' \in T(S)$ and any simple closed curves c on S, the ratio $l_{X'}(c)/l_X(c)$ is compared with the Teichmüller distance $d = d_{T(S)}(X, X')$ between

X and X' as $e^{-d} \leq l_{X'}(c)/l_X(c) \leq e^d$. Thus the sequence $\{k_n c_n\}$ taken above also satisfies $m_n := \min\{k_n \cdot l_X(c_n) \,|\, X \in K\} \to \infty$ as $n \to \infty$, since $K \subset T(S)$ is compact.

Now we may assume that $K \subset L$ without loss of generality. Let $\rho \in \mathcal{QF}_K$. Then $\rho \in B_X \cup B_{\bar{X}}$ for some $X \in K$. We let $X_n \in T(S)$ denote the underlying conformal structure of $\Psi_{\lambda_n}(\rho)$. Then we have

$$\frac{l_{X_n}(c_n)}{l_X(c_n)} \leq \sqrt{\frac{4(g-1)}{k_n \cdot l_X(c_n)}} \leq \sqrt{\frac{4(g-1)}{m_n}}$$

from Theorem 5.2 and the observation above. Thus there exists $N > 0$, which does not depend on $X \in K$, such that for all $n > N$, $d_{T(S)}(X, X_n)$ is greater that the diameter of L. Since $X \in K \subset L$, this implies that $X_n \notin L$ for all $n > N$. Thus we obtain $\pi \circ \Psi_{\lambda_n}(\mathcal{QF}_K) \cap L = \emptyset$ for all $n > N$. □

As an immediate consequence of Theorem 5.5, we obtain the following:

Corollary 5.6. *Let $\Sigma_n \to \Sigma$ be a convergent sequence in $P(S)$ such that $\rho_n = hol(\Sigma_n) \in \mathcal{QF}$, $\rho = hol(\Sigma) \in \partial^{\pm}\mathcal{QF}$, and that the sequence ρ_n converges standardly to ρ. Then the sequence Σ_n is eventually contained in one connected component of $\mathcal{Q}(S)$.*

Proof. From Theorem 5.5, there exists an infinite subsequence $\{\Sigma_{n_j}\}_{j=1}^{\infty}$ of $\{\Sigma_n\}_{n=1}^{\infty}$ which is contained in a component $\mathcal{Q}_\mu$ of $Q(S)$ for some $\mu \in \mathcal{ML}_{\mathbb{N}}$. Then we have $\Sigma = \Psi_\mu(\rho)$ because $\Sigma_{n_j} = \Psi_\mu(\rho_{n_j})$ converges to Σ by assumption and to $\Psi_\mu(\rho)$ by Theorem 4.1 for admissible pairs. If there exists another infinite subsequence of $\{\Sigma_n\}_{n=1}^{\infty}$ contained in $\mathcal{Q}_\nu$ for $\nu \neq \mu$, we also have $\Sigma = \Psi_\nu(\rho)$, which contradicts Corollary 4.3. Thus we have obtained the result. □

6. Continuity, revisited

We now go back to the proof of Theorem 4.1 for non-admissible pairs. To this end, we first prepare the following:

Lemma 6.1. *Suppose that the pair (λ, ρ) of $\lambda \in \mathcal{ML}_{\mathbb{N}}$ and $\rho \in \partial^{\pm}\mathcal{QF}$ is not admissible. Let $\rho_n = B(X_n, \bar{Y}_n)$ be a sequence in $\mathcal{QF}$ which converges standardly to ρ and which satisfies $l_{\bar{Y}_n}(\mathrm{para}(\rho)) \to 0$ if $\rho \in \partial^{+}\mathcal{QF}$ or $l_{X_n}(\mathrm{para}(\rho)) \to 0$ if $\rho \in \partial^{-}\mathcal{QF}$. Then $\Psi_\lambda(\rho_n) \to \infty$ in $\widehat{P}(S)$.*

Remark. For every $\rho \in \partial^{\pm}\mathcal{QF}$, Corollary 2.4 tells us that there exists a convergent sequence $\rho_n \to \rho$ which satisfies the conditions in the lemma above.

Proof of Lemma 6.1. We first suppose that $\rho \in \partial^{+}\mathcal{QF}$. Since the pair (λ, ρ) is not admissible, λ and $\mathrm{para}(\rho)$ have parallel components in common and let c be one of such components. Then since $l_{\bar{Y}_n}(c) \to 0$, there are annular neighborhoods A_n of c in $\bar{Y}_n$ such that $\mathrm{Mod}(A_n) \to \infty$ as $n \to \infty$. One can see that the annulus A_n is conformally embedded in the grafted part of $\Psi_\lambda(\rho_n)$ for every n. Then the underlying conformal structures of $\Psi_\lambda(\rho_n)$ diverge in $T(S)$, and hence $\Psi_\lambda(\rho_n) \to \infty$ in $\widehat{P}(S)$.

Next suppose that $\rho \in \partial^{-}\mathcal{QF}$. Since the pair (λ, ρ) is not admissible, there is a component c of $\mathrm{para}(\rho)$ which does not intersect λ essentially. Then since $l_{X_n}(c) \to 0$, there are annular neighborhoods A_n of c in X_n such that $\mathrm{Mod}(A_n) \to \infty$ as $n \to \infty$. One can see that the annulus A_n is conformally embedded in $\Psi_\lambda(\rho_n)$

for every n. Then by the same argument as above, we see that $\Psi_\lambda(\rho_n) \to \infty$ in $\widehat{P}(S)$. □

Proof of Theorem 4.1 for non-admissible pairs. Suppose that the pair (λ, ρ) is not admissible, i.e. $\Psi_\lambda(\rho) = \infty$. We will show that $\Psi_\lambda(\rho_n) \to \Psi_\lambda(\rho) = \infty$ for every standardly convergent sequence $\mathcal{QF} \ni \rho_n \to \rho \in \partial^{\pm}\mathcal{QF}$. To obtain a contradiction, we suppose that the sequence $\Psi_\lambda(\rho_n)$ has a subsequence (which is denoted by the same symbols) converging to some $\Sigma \in P(S)$. In addition, let $\mathcal{QF} \ni \rho'_n \to \rho$ be a standardly convergent sequence which satisfies the condition in Lemma 6.1. Since the map *hol* is a local homeomorphism, there is a convergent sequence $\Sigma'_n \to \Sigma$ in $P(S)$ for which $hol(\Sigma'_n) = \rho'_n$. From Corollary 5.6, the sequence Σ'_n is eventually contained in a component $\mathcal{Q}_\mu$ of $Q(S)$ for some $\mu \in \mathcal{ML}_{\mathbb{N}}$. Since $\Sigma'_n = \Psi_\mu(\rho'_n)$ converge to $\Sigma \neq \infty$, we see that the pair (μ, ρ) is admissible by Lemma 6.1. Further, Theorem 4.1 for admissible pairs implies that $\Sigma = \Psi_\mu(\rho)$. Since the sequence ρ_n also converges standardly to ρ, $\Psi_\mu(\rho_n) \to \Psi_\mu(\rho) = \Sigma$. Now we obtain two sequences $\Psi_\lambda(\rho_n)$, $\Psi_\mu(\rho_n)$ both of which converge to Σ. But since $\lambda \neq \mu$, this contradicts the fact that the map *hol* is a local homeomorphism. Thus we have shown Theorem 4.1. □

Corollary 6.2. *For every $\lambda \in \mathcal{ML}_{\mathbb{N}}$ and any compact subset $K \subset T(S)$, the map*

$$\Psi_\lambda|_{\overline{\mathcal{QF}_K}} : \overline{\mathcal{QF}_K} \to \widehat{P}(S)$$

is continuous, where $\overline{\mathcal{QF}_K}$ is the closure of $\mathcal{QF}_K$ in $R(S)$.

Proof. We only need to show that $\Psi_\lambda(\rho_n) \to \Psi_\lambda(\rho)$ in $\widehat{P}(S)$ for a convergent sequence $\rho_n \to \rho$ in $\partial\mathcal{QF} \cap \overline{\mathcal{QF}_K}$. By passing to a subsequence if necessary, we may assume that $\Psi_\lambda(\rho_n)$ converge to some Σ in $\widehat{P}(S)$. Let $\mathcal{QF} \ni \rho_{n,k} \to \rho_n$ $(k \to \infty)$ be a standardly convergent sequence for each n. Then $\Psi_\lambda(\rho_{n,k}) \to \Psi_\lambda(\rho_n)$ $(k \to \infty)$ by Theorem 4.1. By a diagonal argument, we can choose a convergent sequence $\mathcal{QF} \ni \rho'_n \to \rho$ from $\{\rho_{n,k}\}_{n,k\in\mathbb{N}}$ such that $\Psi_\lambda(\rho'_n) \to \Sigma$. On the other hand, since the sequence ρ'_n converges standardly to ρ, we have $\Psi_\lambda(\rho'_n) \to \Psi_\lambda(\rho)$ by Theorem 4.1. Therefore $\Sigma = \Psi_\lambda(\rho)$ holds and the result follows. □

Let $\overline{\mathcal{Q}_0}$ denote the closure of $\mathcal{Q}_0$ in $P(S)$, *not* in $\widehat{P}(S)$. Note that a sequence $\Sigma_n \in \mathcal{Q}_0$ converges to $\Sigma \in \partial\mathcal{Q}_0 = \overline{\mathcal{Q}_0} - \mathcal{Q}_0$ in $P(S)$ if and only if the sequence $\rho_{\Sigma_n} \in \mathcal{QF}$ converges standardly to $\rho_\Sigma \in \partial^+\mathcal{QF}$. Hence we obtain the following corollary as a consequence of Corollary 6.2:

Corollary 6.3. *For every $\lambda \in \mathcal{ML}_{\mathbb{N}}$, the grafting map $\mathrm{Gr}_\lambda : \mathcal{Q}_0 \to \mathcal{Q}_\lambda$ extends continuously to $\mathrm{Gr}_\lambda : \overline{\mathcal{Q}_0} \to \widehat{P}(S)$, where $\overline{\mathcal{Q}_0}$ is the closure of $\mathcal{Q}_0$ in $P(S)$.*

7. Grafting theorem for b-groups and other results

In this section, we give some applications of Theorem 4.1 (Continuity) and Theorem 5.5 (Divergence).

7.1. **Grafting theorem for boundary b-groups.** The following theorem is an extension of Goldman's grafting theorem [Go] (Theorem 3.2) and is conjectured by Bromberg in [Br].

Theorem 7.1. *For every boundary b-groups $\rho \in \partial^{\pm}\mathcal{QF}$, all projective structure with holonomy ρ are obtained by grafting of ρ; that is, we have*

$$hol^{-1}(\rho) = \{\Psi_\lambda(\rho) : \lambda \in \mathcal{ML}_{\mathbb{N}}, \Psi_\lambda(\rho) \in P(S)\}.$$

Proof. Let $\rho \in \partial^{\pm}\mathcal{QF}$ and $\Sigma \in hol^{-1}(\rho)$. We only need to show that $\Sigma = \Psi_\lambda(\rho)$ for some $\lambda \in \mathcal{ML}_{\mathbb{N}}$. Now let $\rho_n \in \mathcal{QF}$ be a sequence converging standardly to ρ. Then the sequence ρ_n is contained in $\mathcal{QF}_K$ for some compact $K \subset T(S)$. Since the map hol is a local homeomorphism, there exists a convergent sequence $\Sigma_n \to \Sigma$ of projective structures with $hol(\Sigma_n) = \rho_n$ for all n. It follows from Corollary 5.6 that there exist an element $\lambda \in \mathcal{ML}_{\mathbb{N}}$ such that $\Sigma_n \in \mathcal{Q}_\lambda$ and hence $\Sigma_n = \Psi_\lambda(\rho_n)$ for all large enough n. Then $\Psi_\lambda(\rho_n) \to \Sigma$ as $n \to \infty$. On the other hand, since the sequence ρ_n converges standardly to ρ, Theorem 4.1 implies that $\Psi_\lambda(\rho_n) \to \Psi_\lambda(\rho)$ in $\widehat{P}(S)$. Therefore $\Sigma = \Psi_\lambda(\rho)$ and the result follows. □

7.2. **Discreteness.** In [Ita], we have shown the following:

Theorem 7.2. *Any pair of connected components of $hol^{-1}(\mathcal{QF})$ have intersecting closures in $P(S)$.*

On the contrary, the next theorem implies that only exotically convergent sequences cause the bumping of distinct components of $Q(S)$.

Theorem 7.3. *Let $\mathcal{QF}_K$ be the truncated quasifuchsian space for some compact subset K of $T(S)$. Then every connected component of $hol^{-1}(\mathcal{QF}_K)$ has an open neighborhood in $P(S)$ which is disjoint from any other component.*

Proof. Suppose for contradiction that there exist $\rho \in \overline{\mathcal{QF}_K} \cap \partial\mathcal{QF}$ and $\lambda \in \mathcal{ML}_{\mathbb{N}}$ such that $\Psi_\lambda(\rho)$ is a limit of a sequence Σ_n in $hol^{-1}(\mathcal{QF}_K) - \Psi_\lambda(\mathcal{QF}_K)$. Note that the sequence $\rho_n = hol(\Sigma_n) \in \mathcal{QF}_K$ converges standardly to ρ. From Corollary 5.6, the sequence Σ_n is eventually contained in a component $\mathcal{Q}_\mu$ of $Q(S)$ with $\lambda \neq \mu$, and hence $\Sigma_n = \Psi_\mu(\rho_n)$. Since $\Sigma_n \to \Psi_\lambda(\rho)$ by assumption and since $\Psi_\mu(\rho_n) \to \Psi_\mu(\rho)$ by Theorem 4.1, we have $\Psi_\lambda(\rho) = \Psi_\mu(\rho)$, which contradicts Corollary 4.3. □

By the same argument as in the above theorem, we also obtain the following:

Theorem 7.4. *Let $\lambda \in \mathcal{ML}_{\mathbb{N}}$ and $\Sigma \in \partial\mathcal{Q}_\lambda$. Suppose that every sequence in $\mathcal{QF}$ which converges to $hol(\Sigma)$ is standard. Then every sequence $\Sigma_n \in Q(S)$ which converges to Σ is eventually contained in $\mathcal{Q}_\lambda$; i.e. there is no bumping of distinct components at Σ.*

7.3. **Obstructions for *hol* to be a covering map.** Here we observe how the map hol fail to be a covering map. We first fix our terminology:

Definition 7.5. Let $\mathcal{X}, \mathcal{Y}$ be arcwise-connected topological spaces. A surjective local homeomorphism $f : \mathcal{Y} \to \mathcal{X}$ is said to be a *covering map* or a *weak-covering map* if for any $x \in \mathcal{X}$ there is a neighborhood U of x such that $f|_V : V \to U$ is a homeomorphism

- for any connected component V of $f^{-1}(U)$, or
- for any connected component V of $f^{-1}(U)$ which contains a preimage of x,

respectively.

Remark. Our definition of covering map is the usual one. One can observe that a weak-covering map $f : \mathcal{Y} \to \mathcal{X}$ is a covering map if and only if, for any arc α in $\mathcal{X}$

with a starting point $x \in \mathcal{X}$ and for any $y \in f^{-1}(x)$, there is a lift $\tilde{\alpha} \subset \mathcal{Y}$ of α with the stating point y. We also remark that a surjective local homeomorphism $f : \mathcal{Y} \to \mathcal{X}$ is not necessarily a weak-covering map; in fact, for $x \in \mathcal{X}$ and $y \in f^{-1}(x)$, the choice of a neighborhood U of x which induces a homeomorphism $f|_V : V \to U$ from a neighborhood V of y depends on the choice of y in general.

Although the holonomy map $hol : P(S) \to R(S)$ is a local homeomorphism and the map $hol|_{Q(S)} : Q(S) \to \mathcal{QF}$ is a covering map, Hejhal showed the following:

Theorem 7.6 ([He, Theorem 8]). *The holonomy map $hol : P(S) \to R(S)$ is not a covering map onto its image.*

We explain this fact in our context. Let $\rho \in \partial^{\pm}\mathcal{QF}$ with $\mathrm{para}(\rho) \neq \emptyset$. Then there are $\lambda, \mu \in \mathcal{ML}_{\mathbb{N}}$ such that $\Psi_\lambda(\rho) \in P(S)$ and $\Psi_\mu(\rho) = \infty$. One can find a path $\alpha : [0,1] \to R(S)$ with $\alpha(0) \in \mathcal{QF}$ and $\alpha(1) = \rho$ for which there is a lift of the path α with the starting point $\Psi_\lambda(\alpha(0)) \in \mathcal{Q}_\lambda$ but there is no lift with the starting point $\Psi_\mu(\alpha(0)) \in \mathcal{Q}_\mu$. This implies that the map hol is not a covering map. We remark that this argument is the same to that of Hejhal [He], but he made use of a path $\alpha : [0,1] \to R(S)$ of Schottky representations.

Moreover we claim in Corollary 7.8 below that the covering map $hol|_{Q(S)} : Q(S) \to \mathcal{QF}$ does not even extend to a weak-covering. The essential observation is the following:

Theorem 7.7. *Let $\rho \in \partial^{\pm}\mathcal{QF}$. For every $\lambda \in \mathcal{ML}_{\mathbb{N}}$ with $\Psi_\lambda(\rho) \in P(S)$, we let $\Phi_\lambda : U_\lambda \to P(S)$, $\rho \mapsto \Psi_\lambda(\rho)$ be a univalent local branch of hol^{-1} defined on a neighborhood U_λ of ρ. Then ρ can not be an interior point of $\bigcap_\lambda U_\lambda$, where the intersection is taken over all $\lambda \in \mathcal{ML}_{\mathbb{N}}$ with $\Psi_\lambda(\rho) \in P(S)$.*

Proof. We may assume that $\rho \in \partial B_X$ for some $X \in T(S)$. Then there exists a sequence ρ_n of geometrically finite b-groups converging to ρ in ∂B_X for which the parabolic locus $\lambda_n = \mathrm{para}(\rho_n) \in \mathcal{ML}_{\mathbb{N}}$ has no parallel component in common with $\mathrm{para}(\rho)$ (possibly empty) for every n; see [It2, Theorem 5.5]. Then since $\Psi_{\lambda_n}(\rho) \neq \infty$ there exists a univalent local branch $\Phi_{\lambda_n} : U_{\lambda_n} \to P(S)$, $\rho \mapsto \Psi_{\lambda_n}(\rho)$ defined on a neighborhood U_{λ_n} of ρ for every n. On the other hand, since $\Psi_{\lambda_n}(\rho_n) = \infty$, we have $\rho_n \notin U_{\lambda_n}$ for every n. Therefore we have obtained the result. □

Corollary 7.8. *Let $O \subset R(S)$ be an open subset which properly contains $\mathcal{QF}$. Then the map $hol|_{hol^{-1}(O)} : hol^{-1}(O) \to O$ is not a weak-covering.*

Proof. Let $\rho \in \partial\mathcal{QF} \cap O$ and let U be a neighborhood of ρ contained in O. Then one can find a boundary b-group $\rho' \in \partial^{\pm}\mathcal{QF}$ in U. Then the result follows from the above theorem. □

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