Convergence and divergence of Kleinian punctured torus groups

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Abstract

In this paper we give a necessary and sufficient condition in which a sequence of Kleinian punctured torus groups converges. This result tells us that every *exotically* convergent sequence of Kleinian punctured torus groups is obtained by the method due to Anderson and Canary [AC]. Thus we obtain a complete description of the set of points at which the space of Kleinian punctured torus groups self-bumps. We also discuss geometric limits of sequences of Bers slices.

1 Introduction

One of the central issues in the theory of Kleinian groups is to understand the structures of deformation spaces of Kleinian groups. In this paper we consider Kleinian punctured torus groups, one of the simplest classes of Kleinian groups with a non-trivial deformation theory. Let S be a once-punctured torus. The deformation space $\mathcal{D}(S)$ of Kleinian punctured torus groups is the space of conjugacy classes of discrete faithful representations $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ which takes a loop surrounding the cusp to a parabolic element. Although the interior of $\mathcal{D}(S)$ (in the space of representations) is parameterized by a product of Teichmüller spaces of S, its boundary is quite complicated. For example, McMullen [Mc1] showed that $\mathcal{D}(S)$ self-bumps by using the method developed by Anderson and Canary [AC] (see also [BH]). Here we say that $\mathcal{D}(S)$ *self-bumps* if there is a point ρ on the boundary such that for any sufficiently small neighborhood of ρ, the intersection of the neighborhood with the interior of $\mathcal{D}(S)$ is disconnected. Furthermore, Bromberg [Brom] recently showed that $\mathcal{D}(S)$ is not even locally connected. We refer the reader to [Ca] for more information on the topology of deformation spaces of general Kleinian groups.

In this paper we characterize sequences in $\mathcal{D}(S)$ which give rise to the self-bumping of $\mathcal{D}(S)$. We now survey our results and their backgrounds. By fixing a pair of generators of $\pi_1(S)$, the Teichmüller space $\mathcal{T}(S)$ of S is naturally identified with the upper half plane $\mathbf{H}$. The Thurston's compactification of $\mathcal{T}(S)$ with the set $\mathcal{PL}(S)$ of projective measured laminations is identified with the closure $\overline{\mathbf{H}}$ of $\mathbf{H}$ in the Riemann sphere. Then the subset of $\mathcal{PL}(S)$ of simple closed curves on S corresponds to the set $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ of rational numbers. To each $\rho \in \mathcal{D}(S)$, one can associate an ordered pair of *end invariants* (ν^-, ν^+) which lies in $(\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$, where Δ is the diagonal of $\partial\mathbf{H} \times \partial\mathbf{H}$. It is known by Minsky [Mi] that all elements of $\mathcal{D}(S)$ are classified by their end invariants, and that the inverse

$$Q := \nu^{-1} : (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta \to \mathcal{D}(S)$$

of the map $\nu : \rho \mapsto (\nu^-, \nu^+)$ is a continuous bijection (although the map ν is not continuous as mentioned below). This result tells us that a convergent sequence $(x_n, y_n) \to (x_\infty, y_\infty)$ in $(\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$ induces a convergent sequence $Q(x_n, y_n) \to Q(x_\infty, y_\infty)$ in $\mathcal{D}(S)$. On the other hand, in this paper, we consider sequences $\{Q(x_n, y_n)\}_{n=1}^\infty$ in $\mathcal{D}(S)$ such that the sequences $(x_n, y_n) \in (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$ of their end invariants converge to some points $(x_\infty, x_\infty) \in \Delta$. Especially, we investigate conditions in which such a sequence $Q(x_n, y_n) \in \mathcal{D}(S)$ converges/diverges in $\mathcal{D}(S)$. If such a sequence converges, we say that it is an *exotically* convergent sequence. It is known by Ohshika [Oh] that if $x_\infty \in \mathbb{R} \setminus \mathbb{Q}$, then such a sequence $Q(x_n, y_n)$ diverges in $\mathcal{D}(S)$. Thus we concentrate our attention to the case of $x_\infty \in \hat{\mathbb{Q}}$. By changing bases of $\pi_1(S)$ if necessary, we may always assume that $x_\infty = \infty \in \hat{\mathbb{Q}}$. In this case, there are typical examples of exotically convergent sequences due to Anderson and Canary [AC]: Let τ denote the Dehn twist around the simple closed curve corresponding to $\infty \in \hat{\mathbb{Q}}$, which acts on $\mathcal{T}(S) = \mathbf{H}$ by $z \mapsto z + 1$. Then they showed that for given $x, y \in \mathbf{H}$ and $p \in \mathbb{Z}$, the sequence

$$\{Q(\tau^{pn}x, \tau^{(p+1)n}y)\}_{n=1}^\infty$$

converges in $\mathcal{D}(S)$, whereas the sequence $(\tau^{pn}x, \tau^{(p+1)n}y) \in (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$ converges to $(\infty, \infty) \in \Delta$ if $p \neq 0, -1$. Those sequences cause the non-continuity of the map ν and the self-bumping of $\mathcal{D}(S)$. We will show that all exotically convergent sequences are essentially obtained by the method of Anderson and Canary.

Our main result in this paper is Theorem 1.1 below (see Theorems 3.5 and 3.8). Roughly speaking, it states that if either $\{x_n\}$ or $\{y_n\}$ converge *horocyclically* to ∞ then $Q(x_n, y_n)$ diverges, and that if both $\{x_n\}$ and $\{y_n\}$ converge *tangentially* to ∞ then $Q(x_n, y_n)$ converges provided that the speeds of their convergence to ∞ are in the ratio of $p : p + 1$ for some integer p.

Theorem 1.1. *Suppose that a sequence $(x_n, y_n) \in (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$ converges to $(\infty, \infty) \in \Delta$ as $n \to \infty$. Then we have the following:*

(1) If $\operatorname{Im} x_n \to \infty$ or $\operatorname{Im} y_n \to \infty$, then the sequence $Q(x_n, y_n)$ diverges in $\mathcal{D}(S)$.

(2) Suppose that both $\operatorname{Im} x_n$ and $\operatorname{Im} y_n$ are bounded. (Then we have $|\operatorname{Re} x_n| \to \infty$ and $|\operatorname{Re} y_n| \to \infty$.) Furthermore, assume that there exist sequences $\{k_n\}$, $\{l_n\}$ of integers such that both $\{\tau^{k_n} x_n\}$, $\{\tau^{l_n} y_n\}$ converge in $\overline{\mathbf{H}} \setminus \{\infty\}$. Then the sequence $Q(x_n, y_n)$ converges in $\mathcal{D}(S)$ if and only if there exist integers p, q which satisfy

$$(p + 1)k_n - pl_n + q \equiv 0$$

for all n large enough.

This paper is organized as follows. In Section 2, we give the basic notion and definitions. In Section 3, we discuss conditions in which sequences in $\mathcal{D}(S)$ converge/diverge; our main tool is the "re-marking trick" on representations which is originally due to Kerckhoff and Thurston [KT] (see also [Brock1]). One of the key lemma in Section 3 is Theorem 3.9, whose proof is postponed until Section 6. To make use of the aspects of once-punctured torus groups in the succeeding sections, we recall Bers and Maskit slices, and a natural embedding of a Maskit slice into the complex plane in Sections 4. In section 5, we recall the Drilling Theorem due to Brock and Bromberg [BB], which plays an important role in the proof of Theorem 3.9. Using preparations in Sections 4 and 5, we prove Theorem 3.9 in Section 6. Applications of our results in Section 3 are discussed in Sections 7 and 8: In Section 7, we consider geometric limits of sequences of Bers slices, and obtain a condition in which the geometric limit of a sequence of Bers slices is strictly bigger than the limit of pointwise convergence (see Theorem 7.1). In Section 8, we obtain a complete description of the set of points at which $\mathcal{D}(S)$ self-bumps (see Theorem 8.3).

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2 Preliminaries

2.1 Teichmüller space

Let S be a once-punctured torus. Throughout this paper, we fix an ordered pair (α, β) of generators for $\pi_1(S)$, which determines a positively oriented ordered pair $([\alpha], [\beta])$ of generators for $H_1(S)$, where $[\alpha]$ and $[\beta]$ denote homology classes of α and β, respectively.

The Theichmüller space $\mathcal{T}(S)$ consists of pairs (f, X), where X is a hyperbolic Riemann surface of finite area and $f : S \to X$ is an orientation preserving homeomorphism. Two pairs (f_1, X_1) and (f_2, x_2) represent the same point in $\mathcal{T}(S)$ if there is a holomorphic isomorphism $g : X_1 \to X_2$ such that $g \circ f_1$ is isotopic to f_2.

The Teichmüller space $\mathcal{T}(S)$ is identified with the upper half plane $\mathbf{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0\}$ as follows: To a point x in $\mathbf{H}$, we associate the point $(f, X) \in \mathcal{T}(S)$ where X is the quotient $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}x)$ with one point removed, and $f : S \to X$ is an orientation preserving homeomorphism which take the curves $[\alpha]$ and $[\beta]$ to the images of the segments $[0, 1]$ and $[0, x]$ in X, respectively. In this identification, Thurston's compactification $\overline{\mathcal{T}}(S)$ of $\mathcal{T}(S)$ with the set $\mathcal{PL}(S)$ of projective measured laminations corresponds to the closure $\overline{\mathbf{H}} = \mathbf{H} \cup \hat{\mathbb{R}}$ of $\mathbf{H}$ in the Riemann sphere $\hat{\mathbb{C}}$, where $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. Then the simple closed curve that represent an unoriented homology class $\pm(s[\alpha] + t[\beta]) \in H_1(S)$ is identified with the rational number $-s/t \in \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. Especially, the homology classes $[\alpha]$, $[\beta]$ and $[\alpha^{-1}\beta]$ correspond to $1/0 = \infty$, 0 and 1 in $\hat{\mathbb{Q}}$, respectively.

We let $l_x(c)$ denote the hyperbolic length of a geodesic c on a Riemann surface $x \in \mathcal{T}(S)$.

2.2 Kleinian groups

A *Kleinian group* Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$, which acts on the hyperbolic 3-space $\mathbf{H}^3$ as isometries, and on the sphere at infinity $\partial \mathbf{H}^3 = \hat{\mathbb{C}}$ as conformal automorphisms. In this paper, we orient the ideal boundary $\partial \mathbf{H}^3 = \hat{\mathbb{C}}$ of the hyperbolic 3-space $\mathbf{H}^3$ such that the frame (f, n) has positive orientation if f is a positively oriented frame on $\hat{\mathbb{C}}$ and n is an inward-pointing vector. The *region of discontinuity* Ω_Γ for a Kleinian group Γ is the largest open subset of $\widehat{\mathbb{C}}$ on which Γ acts properly discontinuously, and the *limit set* Λ_Γ of Γ is its complement $\widehat{\mathbb{C}} - \Omega_\Gamma$.

Let $R(S)$ be the set of representations of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbb{C})$ which take the commutator of α and β to parabolic elements. A sequence $\rho_n \in R(S)$ is said to converge to $\rho \in R(S)$ if $\rho_n(\gamma) \to \rho(\gamma)$ in $\mathrm{PSL}_2(\mathbb{C})$ for every $\gamma \in \pi_1(S)$. Let $\mathcal{R}(S)$ denote the space of conjugacy classes $[\rho]$ of representations ρ in $R(S)$. A sequence $[\rho_n] \in \mathcal{R}(S)$ is said to converge to $[\rho] \in \mathcal{R}(S)$ if there exist representatives $\rho_n \in [\rho_n]$ and $\rho \in [\rho]$ such that $\rho_n \to \rho$. This topology on $\mathcal{R}(S)$ is known as the *algebraic topology*.

Let $\mathcal{D}(S) \subset \mathcal{R}(S)$ denote the set of discrete faithful representations, which is a closed subset of $\mathcal{R}(S)$ (see [Jø]) and is non-compact. It is known that there is an open neighborhood of $\mathcal{D}(S)$ in $\mathcal{R}(S)$ which is a 2-dimensional complex manifold (see Section 4.3 in [Ka] or Theorem 4.21 in [MT]).

2.3 End invariants

Here we review Minsky's work ([Mi]) on the classification of elements $[\rho] \in \mathcal{D}(S)$ by using pairs of their *end invariants* $(\nu^-, \nu^+) \in (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$, where Δ is the diagonal of $\partial \mathbf{H} \times \partial \mathbf{H}$.

Given $[\rho] \in \mathcal{D}(S)$ with $\Gamma = \rho(\pi_1(S))$, Bonahon's Theorem [Bo] guarantees that there exists an orientation preserving homeomorphism

$$\varphi : S \times (-1, 1) \to \mathbf{H}^3/\Gamma$$

which induces the representation ρ. The region of discontinuity Ω_Γ (possibly empty) decomposes into two parts Ω_Γ^-, Ω_Γ^+ such that Ω_Γ^-/Γ and Ω_Γ^+/Γ are the limits of $\varphi(S \times \{t\})$ in $(\mathbf{H}^3 \cup \Omega_\Gamma)/\Gamma$ as $t \to -1$ and $t \to +1$, respectively.

Suppose first that $[\rho]$ is in the interior $\mathrm{int}(\mathcal{D}(S))$ of $\mathcal{D}(S) \subset \mathcal{R}(S)$. Then its image $\Gamma = \rho(\pi_1(S))$ is a quasifuchsian group, i.e. Λ_Γ is a Jordan curve and Γ contains no element interchanging two components Ω_Γ^-, Ω_Γ^+ of Ω_Γ. In this case, the map φ extends to a homeomorphism

$$\varphi : S \times [-1, 1] \to (\mathbf{H}^3 \cup \Omega_\Gamma)/\Gamma.$$

Then the orientation preserving homeomorphism $\varphi|_{S \times \{-1\}} : S \to \Omega_\Gamma^-/\Gamma$ determines a point ν^- in $\mathcal{T}(S) = \mathbf{H}$. Similarly, the orientation preserving homeomorphism $\varphi|_{S \times \{1\}} : S \to \overline{\Omega_\Gamma^+/\Gamma}$ form S onto the complex conjugation $\overline{\Omega_\Gamma^+/\Gamma}$ of Ω_Γ^+/Γ determines a point ν^+ in $\mathcal{T}(S) = \mathbf{H}$. It is known by Bers [Be] that the map $\nu : [\rho] \mapsto (\nu^-, \nu^+)$ is a homeomorphism from $\mathrm{int}(\mathcal{D}(S))$ onto $\mathbf{H} \times \mathbf{H}$.

For general $[\rho] \in \mathcal{D}(S)$, the negative end invariant $\nu^- \in \overline{\mathbf{H}}$ is defined as follows: If Ω_Γ^-/Γ is homeomorphic to a once-punctured torus, then we define $\nu^- \in \mathbf{H}$ as above.

If Ω_Γ^-/Γ is a thrice-punctured sphere, let $c \in \hat{\mathbb{Q}}$ be a simple closed curve on S such that the map φ extends to a homeomorphism $(S \setminus c) \times \{-1\} \to \Omega_\Gamma^-/\Gamma$. Then we let $\nu^- = c \in \hat{\mathbb{Q}}$. Finally if Ω_Γ^-/Γ is empty, there is a sequence $c_n \in \hat{\mathbb{Q}}$ of simple closed curves on S such that their geodesic realizations c_n^* in $\mathbf{H}^3/\Gamma$ diverge to the negative end of $\mathbf{H}^3/\Gamma$ and that $c_n \in \hat{\mathbb{Q}}$ converges in $\hat{\mathbb{R}}$ to some irrational number c_∞. Then we let $\nu^- = c_\infty \in \hat{\mathbb{R}} \setminus \hat{\mathbb{Q}}$. Similarly $\nu^+ \in \overline{\mathbf{H}}$ is defined. Minsky's ending lamination theorem ([Mi]) tells us that all elements $[\rho] \in \mathcal{D}(S)$ are classified by their end invariants $(\nu^-, \nu^+) \in (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$. Furthermore he showed that the inverse $Q := \nu^{-1}$ of the map $\nu : [\rho] \mapsto (\nu^-, \nu^+)$ is continuous:

Theorem 2.1 (Minsky [Mi])**.** *The map*

$$Q = \nu^{-1} : (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta \to \mathcal{D}(S)$$

is a continuous bijection.

On the other hand, it is known by Anderson and Canary [AC] that ν is not continuous (see Theorem 3.3).

Given a representation $[\rho] = Q(x, y)$ in $\mathcal{D}(S)$, by abuse of notation, we also denote by $Q(x, y)$ the quotient manifold $\mathbf{H}^3/\rho(\pi_1(S))$ equipped with a homeomorphism $\varphi : S \times (-1, 1) \to \mathbf{H}^3/\rho(\pi_1(S))$ such that $\varphi_* = \rho$.

2.4 Action of $\mathrm{Mod}(S)$

The *mapping class group* $\mathrm{Mod}(S)$ is the group of isotopy classes of orientation-preserving homeomorphisms from S to itself. The action of $\sigma \in \mathrm{Mod}(S)$ on $\mathcal{T}(S)$ is defined by

$$\sigma(f, X) := (f \circ \sigma^{-1}, X).$$

Via our identification of $\mathcal{T}(S)$ with $\mathbf{H}$, $\mathrm{Mod}(S)$ is identified with $\mathrm{PSL}_2(\mathbb{Z})$. The action of $\mathrm{Mod}(S) = \mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{T}(S) = \mathbf{H}$ naturally extends to automorphisms of $\overline{\mathcal{T}}(S) = \overline{\mathbf{H}}$.

The action of $\sigma \in \mathrm{Mod}(S)$ on a representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ is defined by

$$\sigma \cdot \rho := \rho \circ \sigma_*^{-1},$$

where σ_* is the group automorphism of $\pi_1(S)$ induced by σ. Then the action of $\sigma \in \mathrm{Mod}(S)$ on $\mathcal{R}(S)$ is defined by

$$\sigma \cdot [\rho] := [\sigma \cdot \rho].$$

Note that these actions are compatible with the mapping $Q : (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta \to \mathcal{D}(S)$; i.e. we have

$$\sigma \cdot Q(x, y) = Q(\sigma x, \sigma y)$$

for every $\sigma \in \mathrm{Mod}(S)$ and $(x, y) \in (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$.

Throughout of this paper, we denote by $\tau \in \mathrm{Mod}(S)$ the Dehn twist around the curve $[\alpha]$, whose orientation is chosen so that the group isomorphism $\tau_* : \pi_1(S) \to \pi_1(S)$ satisfies

$$\tau_*(\alpha) = (\alpha) \quad \text{and} \quad \tau_*(\beta) = \alpha^{-1}\beta.$$

Then τ acts on $\mathbf{H} = \mathcal{T}(S)$ by

$$z \mapsto z + 1.$$

Given a representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, one can check that

$$(\tau \cdot \rho)(\alpha) = \rho(\alpha) \quad \text{and} \quad (\tau \cdot \rho)(\beta) = \rho(\alpha\beta),$$

and that $(\tau \cdot \rho)|_H \equiv \rho|_H$, where $H = \langle \alpha, \beta^{-1}\alpha\beta \rangle$ is the rank-2 free subgroup of $\pi_1(S)$ generated by α and $\beta^{-1}\alpha\beta$. (In general, if A is a subset of a group G, we denote by $\langle A \rangle$ the subgroup generated by A.)

2.5 Geometric convergence

We first recall the definition of the Hausdorff topology.

Definition 2.2 (Housdorff topology)**.** Let X be a locally compact metrizable space and let $\mathcal{C}(X)$ denote the set of closed subset of X. The *Hausdorff topology* on $\mathcal{C}(X)$ is defined by saying that a sequence $A_n \in \mathcal{C}(X)$ converges to $A \in \mathcal{C}(X)$ if

(1) For each $a \in A$, there exists a sequence $a_n \in A_n$ such that $a_n \to a$.

(2) If there exists a sequence $a_{n_j} \in A_{n_j}$ such that $a_{n_j} \to a$, then $a \in A$.

It is known that $\mathcal{C}(X)$ is compact. A sequence of Kleinian groups Γ_n is said to converge *geometrically* to a Kleinian group Γ_G if $\Gamma_n \to \Gamma_G$ in the Housdorff topology on $\mathcal{C}(\mathrm{PSL}_2(\mathbb{C}))$. A geometric convergence of Kleinian groups can be interpreted as a geometric convergence of framed manifolds in the sense of Gromov (see [CEG]): Fixing an orthonormal frame $\tilde{\omega}$ on a point in $\mathbf{H}^3$, there is a one-to-one correspondence between Kleinian groups Γ and framed hyperbolic 3-manifolds (N, ω) via the following relation

$$(N, \omega) = (\mathbf{H}^3, \tilde{\omega})/\Gamma.$$

Then a sequence Γ_n of Kleinian groups converges geometrically to a Kleinian group Γ_G if and only if the sequence $(N_n, \omega_n) = (\mathbf{H}^3, \tilde{\omega})/\Gamma_n$ of framed manifolds converges geometrically to $(N_G, \omega_G) = (\mathbf{H}^3, \tilde{\omega})/\Gamma_G$ in the sense of Gromov; that is, for any compact subset $\mathcal{K} \subset N_G$ containing ω_G and for all n large enough, there exist bilipschitz diffeomorphisms

$$f_n : \mathcal{K} \to N_n$$

onto their images such that $(f_n)_*(\omega_G) = \omega_n$ and that the bilipschitz constants $K(f_n)$ tend to 1 as $n \to \infty$. Here the bilipschitz constant $K(g)$ of a bilipschitz diffeomorphism $g : M \to N$ between Riemannian manifolds is defined to be the infimum over all K such that $\frac{1}{K} \leq \frac{|g_*(v)|}{|v|} \leq K$ for all $v \in TM$.

3 Conditions for convergence and divergence

Suppose that we are given a sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $(\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$ which converges to some $(x_\infty, y_\infty) \in \overline{\mathbf{H}} \times \overline{\mathbf{H}}$ as $n \to \infty$. In this section, we investigate conditions in which the sequence $\{Q(x_n, y_n)\}_{n=1}^{\infty}$ converges/diverges in $\mathcal{D}(S)$. Here we say that a sequence $Q(x_n, y_n)$ *diverges* in $\mathcal{D}(S)$ if it eventually exits any compact subset of $\mathcal{D}(S)$, or, in other words, it contains no convergent subsequence. It is immediately follows from Theorem 2.1 (which can be viewed as a refinement of Thurston's double limit theorem for punctured torus groups) that if (x_∞, y_∞) does not lie in Δ, then the sequence $Q(x_n, y_n)$ converges in $\mathcal{D}(S)$ and its limit is $Q(x_\infty, y_\infty)$. Thus we are interested in the case where $(x_\infty, y_\infty) \in \Delta$; i.e. $x_\infty = y_\infty \in \partial\mathbf{H}$. In other words, we are interested in *exotically* convergent sequences:

Definition 3.1. A convergent sequence $Q(x_n, y_n)$ in $\mathcal{D}(S)$ with $(x_n, y_n) \to (x_\infty, y_\infty) \in \overline{\mathbf{H}} \times \overline{\mathbf{H}}$ is said to be *standard* if $(x_\infty, y_\infty) \notin \Delta$, and *exotic* if $(x_\infty, y_\infty) \in \Delta$.

We can also eliminate the case where $x_\infty = y_\infty \in \hat{\mathbb{R}} \setminus \hat{\mathbb{Q}}$ by using the following result due to Ohshika [Oh]:

Theorem 3.2 (Ohshika)**.** *Let $x_\infty \in \hat{\mathbb{R}} \setminus \hat{\mathbb{Q}}$ and suppose that a sequence $(x_n, y_n) \in (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$ converges to $(x_\infty, x_\infty) \in \Delta$ as $n \to \infty$. Then the sequence $Q(x_n, y_n)$ diverges in $\mathcal{D}(S)$.*

Thus, in what follows, we concentrate our attention to the case where $x_\infty = y_\infty \in \hat{\mathbb{Q}}$. By changing bases of $\pi_1(S)$ if necessary, we may always assume that $x_\infty = y_\infty = \infty \in \hat{\mathbb{Q}}$. Recall that $[\alpha]$ is the simple closed curve on S corresponding to $\infty \in \hat{\mathbb{Q}}$, and that $\tau \in \mathrm{Mod}(S)$ denotes the Dehn twist around $[\alpha]$. It was first shown by Anderson and Canary [AC] (see also [Mc1]) that there exist exotically convergent sequences:

Theorem 3.3 (Anderson-Canary)**.** *Given $x, y \in \mathbf{H}$ and $p \in \mathbb{Z}$, the sequence*

$$\{Q(\tau^{pn}x, \tau^{(p+1)n}y)\}_{n=1}^{\infty} \tag{1}$$

in $\mathcal{D}(S)$ converges as $n \to \infty$, whereas the sequence $(\tau^{pn}x, \tau^{(p+1)n}y)$ in $(\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$ converges to $(\infty, \infty) \in \Delta$ when $p \neq 0, -1$.

We will see in Theorem 3.8 that every exotically convergent sequence in $\mathcal{D}(S)$ is essentially in the form of (1) in Theorem 3.3.

We now prepare our terminology.

Definition 3.4 (horocyclical/tangential convergence)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\overline{\mathbf{H}}$ converging to some $x_\infty \in \partial\mathbf{H}$ as $n \to \infty$. Then:

- $\{x_n\}$ is said to converge *horocyclically* to x_∞ if for any closed horoball $B \subset \overline{\mathbf{H}}$ touching $\partial\mathbf{H}$ at x_∞, $x_n \in B$ for all n large enough.
- $\{x_n\}$ is said to converge *tangentially* to x_∞ if for any closed horoball $B \subset \overline{\mathbf{H}}$ touching $\partial\mathbf{H}$ at x_∞, $x_n \notin B$ for all n large enough.

Note that, by definition, if $x_n \equiv x_\infty$ for all n large enough then $x_n \to x_\infty$ horocyclically. Note also that $x_n \to \infty \in \partial\mathbf{H}$ horocyclically if and only if $\operatorname{Im} x_n \to \infty$ (by letting $\operatorname{Im} \infty = \infty$), and that $x_n \to \infty \in \partial\mathbf{H}$ tangentially if and only if $\operatorname{Im} x_n$ are bounded and $|\operatorname{Re} x_n| \to \infty$. We also remark that for a given $c \in \hat{\mathbb{Q}}$, a horosphere touching $\partial\mathbf{H}$ at c is the set of Riemann surfaces $x \in \mathbf{H}$ with the same hyperbolic lengths $l_x(c)$ of the simple closed curve c, and that $x_n \to c$ horocyclically if and only if $l_{x_n}(c)$ tend to zero.

3.1 Horocyclic convergence

We first consider the case where either $\{x_n\}$ or $\{y_n\}$ converge horocyclically to ∞. (The case where both $\{x_n\}$ and $\{y_n\}$ converge tangentially to ∞ is discussed in Section 3.2.) Our main result in this case is the following:

Theorem 3.5. *Suppose that a sequence $(x_n, y_n) \in (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$ converges to $(\infty, \infty) \in \Delta$ as $n \to \infty$. Assume that either $\{x_n\}$ or $\{y_n\}$ converges horocyclically to ∞. Then the sequence $Q(x_n, y_n)$ diverges in $\mathcal{D}(S)$.*

The argument in the proof of Theorem 3.5 is similar to the argument of Minsky in [Mi, §12.3] proving the properness of the map $x \mapsto Q(x, \infty)$ on $\overline{\mathbf{H}} \setminus \{\infty\}$. We divide the proof of Theorem 3.5 into the following two cases:

case I Both $\{x_n\}$ and $\{y_n\}$ converge horocyclically to ∞, or

case II One of $\{x_n\}$ and $\{y_n\}$, say $\{x_n\}$, converges tangentially to ∞, and the other one, $\{y_n\}$, converges horocyclically to ∞.

Before proving case I of Theorem 3.5, we recall the foundations of the thin parts of hyperbolic 3-manifolds: Let M be a complete hyperbolic 3-manifold. Given $\epsilon > 0$, the ϵ-thin part $M^{<\epsilon}$ of M is the set of all points in M where the injective radius is less that ϵ. The Margulis Lemma implies that there is a universal constant $\epsilon_0 > 0$ such that each component T of the thin part $M^{<\epsilon_0}$ of M has a standard type: either T is an open solid-torus neighborhood of a short geodesic (called a Margulis tube), or T is the quotient of an open horoball $B \subset \mathbf{H}^3$ of a rank-1 or -2 parabolic group fixing B (called a rank-1 or -2 cusp).

Proof of case I. For simplicity, we only consider the case where neither x_n nor y_n equal to ∞ for every n large enough, but the proof of the general case is essentially the same. Since $\{x_n\}$, $\{y_n\}$ converge horocyclically to ∞, we have $x_n, y_n \in \mathbf{H}$ for every n large enough. Let ϵ_0 be the 3-dimensional Margulis constant and let $\epsilon < \epsilon_0$. Then $l_{x_n}([\alpha]) < \epsilon$ and $l_{y_n}([\alpha]) < \epsilon$ for all n large enough. We fix such n and denote the manifold $Q(x_n, y_n)$ by Q_n. Let c^* be the geodesic realization of $[\alpha]$ in Q_n, and let c^1, c^2 be its geodesic realizations on each connected components of the boundary $\partial\mathcal{C}(Q_n)$ of the convex core $\mathcal{C}(Q_n)$ of Q_n, where $\partial\mathcal{C}(Q_n)$ is regarded as a hyperbolic pleated surface. It is known by Sullivan (see [EM]) that there exists a constant $K > 0$, which does not depend on n, such that the hyperbolic lengths of c^1, c^2 are less than $K\epsilon$.

By modifying c^1, c^2 slightly in $\mathcal{C}(Q_n)$ in their homotopy classes, we may assume that both curves c^1, c^2 are unions of finitely many geodesic arcs. Let A be an immersed annulus in $\mathcal{C}(Q_n)$ with $\partial A = c^1 \sqcup c^2$ which is a finite union of geodesic triangles each of whose edges is a geodesic arc in ∂A or a geodesic arc joining a point in c^1 to a point in c^2. Then one can easily check that A is contained in the ϵ'-thin tube $\mathbf{T}_{\epsilon'}(c^*)$ where $\epsilon' = 2K\epsilon$.

Let d^* be the geodesic realization of $[\beta]$ in Q_n. Since $[\beta]$ intersects $[\alpha]$ in S, d^* must intersects A, and hence $\mathbf{T}_{\epsilon'}(c^*)$. We now let $n \to \infty$ and $\epsilon \to 0$. Then $\epsilon' = 2K\epsilon$ tends to zero and the distances in Q_n between $\partial\mathbf{T}_{\epsilon'}(c^*)$ and $\partial\mathbf{T}_{\epsilon_0}(c^*)$ diverge. Since $d^* \subset Q_n$ is not contained properly in $\mathbf{T}_{\epsilon_0}(c^*)$ for every n, the hyperbolic lengths of d^* in Q_n diverge as $n \to \infty$. Since the translation lengths of the image of $\beta \in \pi_1(S)$ diverge, the sequence $Q(x_n, y_n)$ of representation diverges in $\mathcal{D}(S)$. □

Before considering case II of Theorem 3.5, we prepare the following lemma:

Lemma 3.6. *Let $[\eta_n] \to [\eta_\infty]$ be a convergent sequence in $\mathcal{D}(S)$ and let $\eta_n \to \eta_\infty$ be a convergent sequence of their representatives. Given a sequence $\{k_n\}_{n=1}^\infty$ of integers, the sequence $\{\tau^{k_n} \cdot [\eta_n]\}$ converges (resp. diverges) in $\mathcal{D}(S)$ if and only if the sequence $\{\eta_n(\alpha)^{k_n}\}$ converges (resp. diverges) in* $\mathrm{PSL}_2(\mathbb{C})$.

Proof. We first show the statement for convergence. Let $\rho_n = \tau^{k_n} \cdot \eta_n$. Then we have

$$\rho_n(\alpha) = (\tau^{k_n} \cdot \eta_n)(\alpha) = \eta_n(\alpha),$$
$$\rho_n(\beta) = (\tau^{k_n} \cdot \eta_n)(\beta) = \eta_n(\alpha^{k_n}\beta) = \eta_n(\alpha)^{k_n}\eta_n(\beta).$$

Since $\eta_n \to \eta_\infty$ as $n \to \infty$, ρ_n converge in $\mathcal{D}(S)$ if and only if $\eta_n(\alpha)^{k_n}$ converge in $\mathrm{PSL}_2(\mathbb{C})$. It remains to show that the sequence $[\rho_n] = \tau^{k_n} \cdot [\eta_n]$ of conjugacy classes of ρ_n converges in $\mathcal{D}(S)$ if and only if ρ_n themselves converges. To this end, consider a rank-2 free subgroup $H = \langle \alpha, \beta^{-1}\alpha\beta \rangle$ of $\pi_1(S)$. Then we have $\rho_n|_H \equiv \eta_n|_H$ from Section 2.4, and thus $\rho_n|_H$ converge to $\eta_\infty|_H$ as $n \to \infty$. Now suppose that $[\rho_n]$ converge in $\mathcal{D}(S)$. Then there exist elements $\theta_n \in \mathrm{PSL}_2(\mathbb{C})$ such that $\theta_n \cdot \rho_n \cdot \theta_n^{-1}$ converge, and thus $\theta_n^{-1} \cdot (\rho_n|_H) \cdot \theta_n$ converge. Since $\rho_n|_H$ also converge and since H is a rank-2 free group, one can see that θ_n converge, and hence that ρ_n converge (see p.35 in [KT]). The converse is trivial.

The statement for divergence is deduced from the statement for convergence because a sequence diverges in a topological space if and only if it contains no convergent subsequence. This completes the proof. □

We will also need Minsky's pivot Theorem (Theorem 4.1 in [Mi]) in the arguments below. Recall that the complex translation length $\lambda(\gamma)$ of a loxodromic element $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ is defined to be

$$\lambda(\gamma) = \log(\gamma'(z)) = l + i\theta$$

with $\theta \in (-\pi, \pi]$, where $\gamma'(z)$ is the derivative at the repelling fixed point of γ.

Theorem 3.7 (Pivot Theorem [Mi])**.** *There exist positive constants ϵ, C which satisfy the following: Given an element $[\rho] = Q(x, y)$ in $\mathcal{D}(S)$, if the real part of $\lambda(\rho(\alpha))$ is less than ϵ then*

$$d_{\mathbf{H}}\left(\frac{2\pi i}{\lambda(\rho(\alpha))}, x - \overline{y} + i\right) < C,$$

where $d_{\mathbf{H}}$ *denotes the hyperbolic distance in* $\mathbf{H}$*.*

We now back to the proof of Theorem 3.5.

Proof of case II. Recall that $\tau \in \mathrm{Mod}(S)$ is the Dehn twist around $[\alpha] = \infty \in \hat{\mathbb{Q}}$ acting on $\overline{\mathbf{H}} = \overline{\mathcal{T}}(S)$ by $z \mapsto z+1$. To obtain the result, it suffices to show that any subsequence of $\{(x_n, y_n)\}_{n=1}^{\infty}$ contains a further subsequence such that $Q(x_n, y_n)$ diverges. Given a subsequence of $\{(x_n, y_n)\}_{n=1}^{\infty}$, pass to a further subsequence (which is also denoted by $\{(x_n, y_n)\}_{n=1}^{\infty}$) such that there exists a sequence $\{k_n\}$ of integers for which $\{\tau^{k_n} x_n\}$ converges to some $x'_\infty \in \overline{\mathbf{H}} \setminus \{\infty\}$ as $n \to \infty$. Note that $\{\tau^{k_n} y_n\}$ converges horocyclically to ∞ as well as $\{y_n\}$. Then we have a convergent sequence

$$\tau^{k_n} \cdot Q(x_n, y_n) = Q(\tau^{k_n} x_n, \tau^{k_n} y_n) \to Q(x'_\infty, \infty)$$

in $\mathcal{D}(S)$ and let $\eta_n \to \eta_\infty$ be a convergent sequence of their representatives. Observe from Lemma 3.6 that the sequence $Q(x_n, y_n) = \tau^{-k_n} \cdot [\eta_n]$ diverges in $\mathcal{D}(S)$ if and only if $\eta_n(\alpha)^{k_n}$ diverges in $\mathrm{PSL}_2(\mathbb{C})$. We will show that $\eta_n(\alpha)^{k_n} \to \infty$ below. Since $\eta_\infty(\alpha)$ is parabolic, the multiplier $\lambda(\eta_n(\alpha))$ of $\eta_n(\alpha)$ tends to zero as $n \to \infty$. It then follows form the Pivot Theorem (Theorem 3.7) that there exists a constant C such that

$$d_{\mathbf{H}}\left(\frac{2\pi i}{\lambda(\eta_n(\alpha))}, \tau^{k_n} x_n - \overline{\tau^{k_n} y_n} + i\right) < C$$

for all n large enough. Since $\{\tau^{k_n} x_n\}$ converges in $\overline{\mathbf{H}} \setminus \{\infty\}$, and since $\{\tau^{k_n} y_n\}$ converges horocyclically to ∞, one can see that $\lambda(\eta_n(\alpha))$ converges *horocyclically* to zero; i.e. for any $\epsilon > 0$, we have

$$\lambda(\eta_n(\alpha)) \in \{z \in \mathbb{C} \mid |z - \epsilon| \le \epsilon\}$$

for all n large enough. Then by Theorem 5.1 in [Mc2], the sequence $\langle \eta_n(\alpha) \rangle$ converges geometrically to $\langle \eta_\infty(\alpha) \rangle$. Thus $\eta_n(\alpha)^{k_n}$ diverge in $\mathrm{PSL}_2(\mathbb{C})$, and thus $Q(x_n, y_n)$ diverge in $\mathcal{D}(S)$ by Lemma 3.6. This completes the proof. □

3.2 Tangential convergence

We next consider the case where both $\{x_n\}$ and $\{y_n\}$ converge tangentially to ∞. Our main result in this case is the following:

Theorem 3.8. *Suppose that a sequence* $(x_n, y_n) \in (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$ *converges to* $(\infty, \infty) \in \Delta$ *as* $n \to \infty$*, and that both* $\{x_n\}$*,* $\{y_n\}$ *converge tangentially to* ∞*. Furthermore, assume that there exist sequences* $\{k_n\}$*,* $\{l_n\}$ *of integers such that both* $\{\tau^{k_n} x_n\}$*,* $\{\tau^{l_n} y_n\}$ *converge in* $\overline{\mathbf{H}} \setminus \{\infty\}$*. Then the sequence* $Q(x_n, y_n)$ *converges in* $\mathcal{D}(S)$ *if and only if there exist integers* p*,* q *which satisfy*

$$(p+1)k_n - p l_n + q \equiv 0$$

for all n *large enough.*

Remark. By passing to a subsequence if necessary, we may always assume that there exist sequences $\{k_n\}$, $\{l_n\}$ which satisfy the assumption in Theorem 3.8. We also remark that, since both $\{k_n\}$, $\{l_n\}$ are divergent sequences, the relations $(p+1)k_n - pl_n + q \equiv 0$ for all n large enough imply that $p \neq 0, -1$.

One of the key observations in the proof of Theorem 3.8 is the following theorem, whose proof is postponed until Section 6.

Theorem 3.9. *Let $u_n \to u_\infty$, $v_n \to v_\infty$ be convergent sequences in $\overline{\mathbf{H}} \setminus \{\infty\}$, and let $\{m_n\}_{n=1}^{\infty}$ be a divergent sequence of integers. Then we have a convergent sequence $Q(u_n, \tau^{m_n} v_n) \to Q(u_\infty, \infty)$ in $\mathcal{D}(S)$ and let $\eta_n \to \eta_\infty$ be a convergent sequence of their representatives. In this notation, we have the following:*

(1) $\lim_{n\to\infty} \eta_n(\alpha)^{m_n}$ *exists.*

(2) The group $\langle \delta, \hat{\delta} \rangle$ generated by $\delta = \eta_\infty(\alpha)$ and $\hat{\delta} = \lim_{n\to\infty} \eta_n(\alpha)^{m_n}$ is rank-2 parabolic.

(3) The sequence $\langle \eta_n(\alpha) \rangle$ of cyclic groups converges geometrically to $\langle \delta, \hat{\delta} \rangle$ as $n \to \infty$.

Remark. The essential point in Theorem 3.9 is that the sequence $\langle \eta_n(\alpha) \rangle$ converges geometrically without passing to a subsequence, and that $\lim_{n\to\infty} \eta_n(\alpha)^{m_n}$ is primitive in the geometric limit.

In the proof of Theorem 3.8, we also need the following lemma, which is originally due to Thurston (see [Th]). For the convenience of the reader we give a sketch of its proof following the argument in [Oh].

Lemma 3.10. *Suppose that a sequence $Q(x_n, y_n) \in \mathcal{D}(S)$ converges to $[\rho_\infty] \in \mathcal{D}(S)$ as $n \to \infty$, and that $\{x_n\}$ converges in $\overline{\mathbf{H}}$ to ∞. Then $\rho_\infty(\alpha)$ is parabolic.*

Sketch of Proof. By passing to a subsequence if necessary, we may assume that $x_n \in \mathbf{H}$ for all n or $x_n \in \partial\mathbf{H}$ for all n. Suppose first that $x_n \in \mathbf{H}$ for all n. Let c_n be the shortest simple closed geodesic on the Riemann surface x_n. Then there exists a sequence of positive real numbers $\{r_n\}$ tending to zero such that $r_n l_{x_n}(c_n) \to 0$ and that $\{r_n c_n\}$ converges in the set $\mathcal{ML}(S)$ of measured laminations on S to the simple closed curve $[\alpha]$. Now recall from [Brock2] that the length function

$$L : \mathcal{D}(S) \times \mathcal{ML}(S) \to \mathbb{R}$$

is defined by setting $L([\rho], \lambda)$ the length of λ on a pleated surface if λ is realizable, and zero otherwise. It was shown by Brock [Brock2] that L is continuous. It then follows from $L(Q(x_n, y_n), r_n c_n) \to 0$ that $L([\rho_\infty], [\alpha]) = 0$. This implies that $\rho_\infty(\alpha)$ is parabolic.

Suppose next that $x_n \in \partial\mathbf{H} = \mathcal{PL}(S)$ for all n. Then there exist representatives $\tilde{x}_n \in \mathcal{ML}(\mathcal{S})$ of $x_n \in \mathcal{PL}(S)$ such that $\tilde{x}_n \to [\alpha]$ in $\mathcal{ML}(S)$. Since $L(Q(x_n, y_n), \tilde{x}_n) = 0$, the continuity of L again implies that $L([\rho_\infty], [\alpha]) = 0$. Thus $\rho_\infty(\alpha)$ is parabolic also in this case. □

We now back to the proof of Theorem 3.8.

Proof of Theorem 3.8. We begin by fixing our notation. Let $u_n = \tau^{k_n} x_n$, $v_n = \tau^{l_n} y_n$. We denote by $u_\infty, v_\infty \in \overline{\mathbf{H}} \setminus \{\infty\}$ the limits of $\{u_n\}$, $\{v_n\}$, respectively. Let

$$[\rho_n] = Q(x_n, y_n),$$

and let $\eta_n = \tau^{k_n} \cdot \rho_n$. Then by setting $m_n = k_n - l_n$ we have

$$[\eta_n] = \tau^{k_n} \cdot Q(x_n, y_n) = Q(u_n, \tau^{m_n} v_n).$$

In this notation, suppose first that the sequence $[\rho_n] = Q(x_n, y_n)$ converges in $\mathcal{D}(S)$ to some $[\rho_\infty] \in \mathcal{D}(S)$, and assume that $\{\rho_n\}$ converges to ρ_∞.

We now claim that $m_n \to \infty$. Since $\rho_\infty(\alpha)$ is parabolic by Lemma 3.10, and since $\rho_n(\alpha) = \eta_n(\alpha)$ for all n, the multiplier $\lambda(\eta_n(\alpha))$ of $\eta_n(\alpha)$ tends to zero as $n \to \infty$. It then follows from the Pivot Theorem (Theorem 3.7) that there exists a constant C such that

$$d_{\mathbf{H}}\left(\frac{2\pi i}{\lambda(\eta_n(\alpha))},\, u_n - \overline{\tau^{m_n} v_n} + i\right) < C$$

for all n large enough. This implies that $\{u_n - \overline{\tau^{m_n} v_n} + i\}$ diverges in $\mathbf{H}$ as $n \to \infty$, and thus the claim $m_n \to \infty$ follows.

Therefore the sequence $[\eta_n] = Q(u_n, \tau^{m_n} v_n)$ converges to $Q(u_\infty, \infty)$ as $n \to \infty$. Since $\{\rho_n\}$ converges by assumption, and since $\eta_n|_H \equiv \rho_n|_H$ for all n, the same argument in Lemma 3.6 tells us that $\{\eta_n\}$ converges to a representative η_∞ of $Q(u_\infty, \infty)$ without taking conjugations. Applying Theorem 3.9 to the convergent sequence $Q(u_n, \tau^{m_n} v_n) \to Q(u_\infty, \infty)$ with converging representatives $\eta_n \to \eta_\infty$, we see that the sequence $\langle \eta_n(\alpha) \rangle$ of loxodromic cyclic groups converges geometrically to the rank-2 parabolic group $\langle \delta, \hat{\delta} \rangle$ generated by $\delta = \eta_\infty(\alpha)$ and $\hat{\delta} = \lim_{n\to\infty} \eta_n(\alpha)^{m_n}$. On the other hand, since the sequence $[\rho_n] = \tau^{-k_n} \cdot [\eta_n]$ converges in $\mathcal{D}(S)$ by assumption, it follows from Lemma 3.6 that $\{\eta_n(\alpha)^{-k_n}\}$ converges in $\mathrm{PSL}_2(\mathbb{C})$. Therefore $\{\eta_n(\alpha)^{-k_n}\}$ converges to an element $\hat{\delta}^p \delta^q \in \langle \delta, \hat{\delta} \rangle$ for some $p, q \in \mathbb{Z}$. Since $\{\eta_n(\alpha)^{pm_n + q}\}$ also converges to $\hat{\delta}^p \delta^q$, it follows from discreetness that $-k_n \equiv pm_n + q$ for all n large enough (see Lemma 3.6 in [JM]). Thus we obtain $(p+1)k_n - pl_n + q \equiv 0$ for all n large enough.

Conversely, suppose that $(p+1)k_n - pl_n + q \equiv 0$ hold for some $p, q \in \mathbb{Z}$ and for all n large enough. Since $-k_n \equiv pm_n + q$ and $k_n \to \infty$, we have $m_n \to \infty$. Therefore the sequence $[\eta_n] = Q(u_n, \tau^{m_n} v_n)$ converges to $Q(u_\infty, \infty)$ as $n \to \infty$. Assume that $\{\eta_n\}$ converges to a representative η_∞ of $Q(u_\infty, \infty)$. Then there exists $\lim_{n\to\infty} \eta_n(\alpha)^{m_n}$ by Theorem 3.9. Let $\delta = \eta_\infty(\alpha)$ and $\hat{\delta} = \lim_{n\to\infty} \eta_n(\alpha)^{m_n}$. Then $\eta_n(\alpha)^{-k_n} = \eta_n(\alpha)^{pm_n+q} \to \hat{\delta}^p \delta^q$ as $n \to \infty$. Thus the sequence $Q(x_n, y_n) = [\rho_n] = \tau^{-k_n} \cdot [\eta_n]$ converges in $\mathcal{D}(S)$ to some $[\rho_\infty]$ from Lemma 3.6. More precisely, the sequence $\rho_n = \tau^{-k_n} \cdot \eta_n$ converges to a representative ρ_∞ of the limit $[\rho_\infty]$ as follows:

$$\rho_n(\alpha) = (\tau^{-k_n} \cdot \eta_n)(\alpha) = \eta_n(\alpha) \to \eta_\infty(\alpha) = \rho_\infty(\alpha), \tag{2}$$

$$\rho_n(\beta) = (\tau^{-k_n} \cdot \eta_n)(\beta) = \eta_n(\alpha)^{-k_n} \eta_n(\beta) \to \hat{\delta}^p \delta^q \eta_\infty(\beta) = \rho_\infty(\beta). \tag{3}$$

This completes the proof. ∎

In Section 6, we will give an explicit description of the limit of an exotically convergent sequence in Theorem 3.8.

As a consequence of Theorem 3.8 we obtain a condition for divergence.

Corollary 3.11. *Suppose that a sequence* $(x_n, y_n) \in (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$ *converges to* $(\infty, \infty) \in \Delta$ *as* $n \to \infty$*, and that both* $\{x_n\}$*,* $\{y_n\}$ *converge tangentially to* ∞*. Let* $\{k_n\}$*,* $\{l_n\}$ *be sequences of integers such that both* $\{\tau^{k_n} x_n\}$*,* $\{\tau^{l_n} y_n\}$ *contain no subsequence converging to* ∞*. Then the sequence* $\{Q(x_n, y_n)\}_{n=1}^{\infty}$ *diverges in* $\mathcal{D}(S)$ *if and only if for every* $p \in \mathbb{Z}$ *the sequence* $\{(p+1)k_n - pl_n\}_{n=1}^{\infty}$ *diverges in* $\mathbb{Z}$*.*

Proof. Suppose that the sequence $\{Q(x_n, y_n)\}_{n=1}^{\infty}$ does not diverge; i.e. it contains a subsequence $\{Q(x_i, y_i)\}_{i=1}^{\infty}$ which converges in $\mathcal{D}(S)$. By passing to a further subsequence if necessary, we may assume that both $\{\tau^{k_i} x_i\}$, $\{\tau^{l_i} y_i\}$ converges in $\overline{\mathbf{H}} \setminus \{\infty\}$. Then by Theorem 3.8, there is an integer p such that $(p+1)k_i - pl_i$ converges in $\mathbb{Z}$ as $i \to \infty$. This implies that the sequence $\{(p+1)k_n - pl_n\}_{n=1}^{\infty}$ does not diverge in $\mathbb{Z}$. The converse is similarly obtained. □

4 Bers and Maskit slices

To proceed our argument by using the aspects of once-punctured torus groups, we now introduce Bers and Maskit slices of $\mathcal{D}(S)$, and a natural embedding of a Maskit slice into the complex plane $\mathbb{C}$.

Given $y \in \overline{\mathbf{H}}$, we define a slice $\mathcal{B}_y \subset \mathcal{D}(S)$ by

$$\mathcal{B}_y = \{Q(x, y) \,|\, x \in \overline{\mathbf{H}},\, (x, y) \notin \Delta\}.$$

A slice $\mathcal{B}_y$ is called a *Bers slice* if $y \in \mathbf{H}$, and a *Maskit slice* if $y \in \hat{\mathbb{Q}} \subset \partial\mathbf{H}$. For simplicity, we generally called a slice $\mathcal{B}_y$ for $y \in \overline{\mathbf{H}}$ as a Bers slice if there is no confusion. When $y \in \mathbf{H}$, the map

$$b_y : \overline{\mathbf{H}} \to \mathcal{B}_y, \quad x \mapsto Q(x, y)$$

is a homeomorphism onto its image $\mathcal{B}_y$ by Theorem 2.1. When $y \in \partial\mathbf{H}$, the map

$$b_y : \overline{\mathbf{H}} \setminus \{y\} \to \mathcal{B}_y, \quad x \mapsto Q(x, y)$$

is also a homeomorphism from Theorem 2.1 together with the arguments by Minsky [Mi] in the case of $y \in \hat{\mathbb{Q}}$, and by Ohshika [Oh] in the case of $y \in \hat{\mathbb{R}} \setminus \hat{\mathbb{Q}}$. Given $x \in \overline{\mathbf{H}}$, we similarly define a slice $\mathcal{B}_x^* \subset \mathcal{D}(S)$ by

$$\mathcal{B}_x^* = \{Q(x, y) \,|\, y \in \overline{\mathbf{H}},\, (x, y) \notin \Delta\}.$$

The Maskit slice $\mathcal{B}_\infty$ for $\infty \in \hat{\mathbb{Q}}$ can be naturally embedded into the complex plane $\mathbb{C}$ in the following way. Given $z, \mu \in \mathbb{C}$, we define elements of $\mathrm{PSL}_2(\mathbb{C})$ by

$$T_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad U_\mu = \begin{pmatrix} i\mu & i \\ i & 0 \end{pmatrix}.$$

(We will later use the fact that $U_\mu U_\nu^{-1} = T_{\mu-\nu}$ holds for any $\mu, \nu \in \mathbb{C}$.) Given $[\rho] \in \mathcal{R}(S)$ such that $\rho(\alpha)$ is parabolic, there is a unique $\mu \in \mathbb{C}$ such that ρ is conjugate to a representation $\boldsymbol{\rho}_\mu : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ which is defined by

$$\boldsymbol{\rho}_\mu(\alpha) = T_2, \quad \boldsymbol{\rho}_\mu(\beta) = U_\mu.$$

The representation $\boldsymbol{\rho}_\mu$ is normalized so that the fixed points of $\boldsymbol{\rho}_\mu(\alpha)$, $\boldsymbol{\rho}_\mu(\beta^{-1}\alpha\beta)$ and $\boldsymbol{\rho}_\mu(\alpha^{-1}\beta^{-1}\alpha\beta)$ coincide with ∞, 0 and -1, respectively. One can see that the map

$$\beta_\infty : \mathbb{C} \to \mathcal{R}(S), \quad \mu \mapsto [\boldsymbol{\rho}_\mu]$$

is a holomorphic embedding, and that from Theorem 2.1 the following holds:

$$\beta_\infty(\mathbb{C}) \cap \mathcal{D}(S) \quad = \quad \mathcal{B}_\infty \sqcup \mathcal{B}_\infty^*.$$

We let denote the preimages of $\mathcal{B}_\infty$ and $\mathcal{B}_\infty^*$ in $\mathbb{C}$ by

$$\mathcal{M} = \beta_\infty^{-1}(\mathcal{B}_\infty) \quad \text{and} \quad \mathcal{M}^* = \beta_\infty^{-1}(\mathcal{B}_\infty^*).$$

It is known that the subset $\mathcal{M} \subset \mathbb{C}$ is contained in the upper half plane $\mathbf{H}$, and that $\mathcal{M}^* = \{\overline{\mu} \mid \mu \in \mathcal{M}\}$ is the complex conjugation of $\mathcal{M}$. The set $\mathcal{M}$ is also called the Maskit slice. Now we define a bijective map

$$m : \overline{\mathbf{H}} \setminus \{\infty\} \to \mathcal{M}$$

by $m = \beta_\infty^{-1} \circ b_\infty$. Then we have $[\boldsymbol{\rho}_{m(x)}] = Q(x, \infty)$ and $[\boldsymbol{\rho}_{\overline{m(x)}}] = Q(\infty, x)$ for every $x \in \overline{\mathbf{H}} \setminus \{\infty\}$.

5 Drilling Theorem

In the proof of Theorem 3.9 in the next section, we will make use of the Drilling Theorem due to Brock and Bromberg [BB]. The following version appears in [Brom].

Theorem 5.1 (Drilling Theorem)**.** *Given $L > 1$ and $\epsilon < \epsilon_0$, there exists $l > 0$ such that the following holds. Let M be a complete hyperbolic structure on the interior of a compact 3-manifold with boundary, and γ a simple closed geodesic in M with length $\leq l$. Let $\hat{M}$ be a complete hyperbolic structure on $M \setminus \{\gamma\}$ such that the inclusion map $\phi : \hat{M} \to M$ extends to a conformal map between the conformal boundaries of M and $\hat{M}$. Then the inclusion map ϕ can be chosen so that a restriction of ϕ^{-1} is an L-bilipschitz diffeomorphism of pairs*

$$\phi^{-1} : (M \setminus \mathbf{T}_\epsilon, \partial \mathbf{T}_\epsilon) \to (\hat{M} \setminus \mathbf{P}_\epsilon, \partial \mathbf{P}_\epsilon)$$

where $\mathbf{T}_\epsilon \subset M^{<\epsilon}$ is the Margulis tube containing γ, and $\mathbf{P}_\epsilon \subset \hat{M}^{<\epsilon}$ is the rank-2 cusp which arises from the deletion of γ.

In the next section, we will apply the Drilling Theorem in the case where $M \cong S \times (-1,1)$ and $\hat{M} \cong S \times (-1,1) \setminus [\alpha] \times \{0\}$. Before starting the proof of Theorem 3.9, we prepare some notation. Given a representation $[\rho] = Q(x,y)$ in $\mathcal{D}(S)$, recall that the manifold $Q(x,y) = \mathbf{H}^3/\rho(\pi_1(S))$ is equipped with a marking homeomorphism

$$\varphi : N \to Q(x,y)$$

from $N = S \times (-1,1)$ onto $Q(x,y)$. Let $\hat{N} = S \times (-1,1) \setminus [\alpha] \times \{0\}$ be the manifold N with a simple closed curve $[\alpha] \times \{0\}$ removed. For every $x, y \in \overline{\mathbf{H}} \setminus \{\infty\}$, there is a complete hyperbolic maniford $\hat{Q}(x,y)$ with a homeomorphism

$$\hat{\varphi} : \hat{N} \to \hat{Q}(x,y)$$

such that the maps $\hat{\varphi}|_{S\times\{-1/2\}}$ and $\hat{\varphi}|_{S\times\{1/2\}}$ induce representations $Q(x,\infty)$ and $Q(\infty,y)$, respectively. A neighborhood of $[\alpha]\times\{0\}$ in $\hat{N}$ corresponds to the rank-2 cusp of $\hat{Q}(x,y)$. The manifold $\hat{Q}(x,y)$ is actually obtained as follows: First cut away the ends of the manifolds $Q(x,\infty)$, $Q(\infty,y)$ bounded by totally geodesic triply-punctured spheres, and next glue the resulting boundaries by orientation reversing isometry. A Kleinian group $\hat{\Gamma}$ uniformizing $\hat{Q}(x,y)$ is obtained by using parameters in the Maskit slice as follows: Let $\mu = m(x)$ and $\nu = m(y)$. Then $[\boldsymbol{\rho}_\mu] = Q(x,\infty)$ and $[\boldsymbol{\rho}_{\overline{\nu}}] = Q(\infty,y)$, and thus

$$Q(x,\infty) = \mathbf{H}^3/\langle T_2, U_\mu\rangle \quad \text{and} \quad Q(\infty,y) = \mathbf{H}^3/\langle T_2, U_{\overline{\nu}}\rangle.$$

Then the group

$$\hat{\Gamma} = \langle T_2,\, U_\mu,\, U_{\overline{\nu}}\rangle = \langle T_2,\, U_\mu,\, T_{\nu-\overline{\nu}}\rangle$$

generated by T_2, U_μ and $T_{\nu-\overline{\nu}} = U_\mu U_{\overline{\nu}}$ is a Kleinian group such that $\hat{Q}(x,y) = \mathbf{H}^3/\hat{\Gamma}$. Here the rank-2 parabolic subgroup $\langle T_2,\, T_{\mu-\overline{\nu}}\rangle$ of $\hat{\Gamma}$ corresponds to the fundamental group of the rank-2 cusp of $\hat{Q}(x,y)$.

6 Proof of Theorem 3.9

In this section, we prove Theorem 3.9, which is restated as in Theorem 6.1 below. We begin by fixing our notation: Let $u_n \to u_\infty$, $v_n \to v_\infty$ be convergent sequences in $\overline{\mathbf{H}} \setminus \{\infty\}$, and let $\mu = m(u_\infty)$, $\nu = m(v_\infty)$. Let $\{m_n\}$ be a divergent sequence of integers. Then we have a convergent sequence

$$Q(u_n, \tau^{m_n} v_n) \to Q(u_\infty, \infty) = [\boldsymbol{\rho}_\mu]$$

in $\mathcal{D}(S)$, and let $\eta_n \to \boldsymbol{\rho}_\mu$ be a convergent sequence of their representatives. Letting $\bar{\eta}_n = \tau^{-m_n} \cdot \eta_n$, we have another convergent sequence

$$[\bar{\eta}_n] = Q(\tau^{-m_n} u_n, v_n) \to Q(\infty, v_\infty) = [\boldsymbol{\rho}_{\overline{\nu}}]$$

in $\mathcal{D}(S)$. In this notation, we have the following:

Theorem 6.1. *(1) $\{\bar{\eta}_n\}$ converges to $\boldsymbol{\rho}_{\overline{\nu}}$ as $n \to \infty$.*

(2) $\lim_{n\to\infty} \eta_n(\alpha)^{m_n} = \boldsymbol{\rho}_\mu(\beta)\boldsymbol{\rho}_{\overline{\nu}}(\beta)^{-1} = T_{\mu-\overline{\nu}}$.

(3) The sequence $\langle \eta_n(\alpha)\rangle$ of loxodromic cyclic groups converges geometrically to the rank-2 parabolic group $\langle T_2, T_{\mu-\overline{\nu}}\rangle$ as $n \to \infty$.

Proof. (1) Consider a rank-2 free subgroup $H = \langle \alpha, \beta^{-1}\alpha\beta\rangle$ of $\pi_1(S)$. Then we have $\eta_n|_H = \bar{\eta}_n|_H$ for all n as observed in Section 2.4. On the other hand, one can easily check that $\boldsymbol{\rho}_\mu|_H = \boldsymbol{\rho}_{\overline{\nu}}|_H$ by definition. It then follows from $\eta_n \to \boldsymbol{\rho}_\mu$ that $\bar{\eta}_n|_H \to \boldsymbol{\rho}_{\overline{\nu}}|_H$ as $n \to \infty$. Now since $[\bar{\eta}_n] \to [\boldsymbol{\rho}_{\overline{\nu}}]$ as $n \to \infty$, we have $\bar{\eta}_n \to \boldsymbol{\rho}_{\overline{\nu}}$ by the same argument in Lemma 3.6.

(2) It follows from $\bar{\eta}_n = \tau^{-m_n} \cdot \eta_n$ that

$$\bar{\eta}_n(\beta) = (\tau^{-m_n} \cdot \eta_n)(\beta) = \eta_n(\alpha^{-m_n}\beta) = \eta_n(\alpha)^{-m_n}\eta_n(\beta),$$

and hence that $\eta_n(\alpha)^{m_n} = \eta_n(\beta)\bar{\eta}_n(\beta)^{-1}$. Thus we obtain

$$\lim_{n\to\infty} \eta_n(\alpha)^{m_n} = \boldsymbol{\rho}_\mu(\beta)\boldsymbol{\rho}_{\overline{\nu}}(\beta)^{-1} = U_\mu U_{\overline{\nu}}^{-1} = T_{\mu-\overline{\nu}}.$$

(3) We actually show that any subsequence of the sequence $\{\langle \eta_n(\alpha)\rangle\}_{n=1}^\infty$ contains a further subsequence which converges geometrically to $\langle T_2, T_{\mu-\overline{\nu}}\rangle$ as $n \to \infty$. Then the result follows because the set $\mathcal{C}(\mathrm{PSL}_2(\mathbb{C}))$ of closed subsets of $\mathrm{PSL}_2(\mathbb{C})$ is sequentially compact with respect to the geometric topology.

Let $\Gamma_n = \eta_n(\pi_1(S))$. Given a subsequence of the sequence $\{\Gamma_n\}_{n=1}^\infty$, there is a further subsequence (which is also denoted by $\{\Gamma_n\}_{n=1}^\infty$) converging geometrically to some Kleinian group Γ_G as $n \to \infty$. Let Θ_G denote the subgroup of Γ_G which is the geometric limit of the sequence $\langle \eta_n(\alpha)\rangle$. Note that Θ_G contains $\langle T_2, T_{\mu-\overline{\nu}}\rangle$. In addition, since $\langle \eta_n(\alpha)\rangle$ are abelian, one can easily check that Θ_G is also abelian. Thus the subgroup $\Theta_G \subset \Gamma_G$ corresponds to a rank-2 cusp of $\mathbf{H}^3/\Gamma_G$, and thus Θ_G is a rank-2 parabolic group with a finite index subgroup $\langle T_2, T_{\mu-\overline{\nu}}\rangle$. We will show that $\Theta_G = \langle T_2, T_{\mu-\overline{\nu}}\rangle$ below.

Note that, since η_n and $\bar{\eta}_n$ have the same image Γ_n, and since $\eta_n \to \boldsymbol{\rho}_\mu$, $\bar{\eta}_n \to \boldsymbol{\rho}_{\overline{\nu}}$ as $n \to \infty$, Γ_G contains images of $\boldsymbol{\rho}_\mu$ and $\boldsymbol{\rho}_{\overline{\nu}}$. Consider the subgroup

$$\hat{\Gamma}_\infty = \langle \boldsymbol{\rho}_\mu(\pi_1(S)), \boldsymbol{\rho}_{\overline{\nu}}(\pi_1(S))\rangle = \langle T_2, U_\mu, T_{\mu-\overline{\nu}}\rangle$$

of Γ_G generated by the images of $\boldsymbol{\rho}_\mu$ and $\boldsymbol{\rho}_{\overline{\nu}}$. Then $\hat{\Gamma}_\infty$ is a Kleinian group uniformizing the manifold $\hat{Q}(u_\infty, v_\infty)$ with a rank-2 cusp, and the subgroup $\langle T_2, T_{\mu-\overline{\nu}}\rangle$ of $\hat{\Gamma}_\infty$ corresponds to the fundamental group of the rank-2 cusp of $\hat{Q}(u_\infty, v_\infty)$. Especially $\langle T_2, T_{\mu-\overline{\nu}}\rangle$ is a maximal abelian subgroup of $\hat{\Gamma}_\infty$. Now let $\mu_n = m(u_n)$, $\nu_n = m(v_n)$, and consider Kleinian groups

$$\hat{\Gamma}_n = \langle \boldsymbol{\rho}_{\mu_n}(\pi_1(S)), \boldsymbol{\rho}_{\overline{\nu_n}}(\pi_1(S))\rangle = \langle T_2, U_{\mu_n}, T_{\mu_n-\overline{\nu_n}}\rangle$$

which uniformize the manifolds $\hat{Q}(u_n, v_n)$ with rank-2 cusps. Since $\boldsymbol{\rho}_{\mu_n} \to \boldsymbol{\rho}_\mu$ and $\boldsymbol{\rho}_{\overline{\nu_n}} \to \boldsymbol{\rho}_{\overline{\nu}}$ as $n \to \infty$, we obtain group isomorphisms

$$\hat{\chi}_n : \hat{\Gamma}_\infty \to \hat{\Gamma}_n$$

which converge to the identity map on $\hat{\Gamma}_\infty$ as $n \to \infty$. This isomorphism $\hat{\chi}_n$ is actually defined by

$$\hat{\chi}_n(T_2) = T_2, \quad \hat{\chi}_n(U_\mu) = U_{\mu_n} \quad \text{and} \quad \hat{\chi}_n(T_{\mu-\overline{\nu}}) = T_{\mu_n - \overline{\nu_n}}.$$

On the other hand, we will show in Proposition 6.2 below that the sequence $\{\hat{\Gamma}_n\}$ also converges to Γ_G as $n \to \infty$. Since the sequence $\{\hat{\chi}_n\}$ of *faithful* representations converges to the identity map, and since the sequence of their images $\{\hat{\Gamma}_n\}$ converges geometrically to Γ_G, a standard argument using Lemma 3.6 in [JM] reviles that any element $\gamma \in \Gamma_G$ with $\gamma^k \in \hat{\Gamma}_\infty$ for some $k \in \mathbb{Z}$ actually lies in $\hat{\Gamma}_\infty$ (see, for example, Lemma 7.26 in [MT]). Since $\langle T_2, T_{\mu-\overline{\nu}} \rangle \subset \hat{\Gamma}_\infty$ is a finite index subgroup of Θ_G we have $\Theta_G \subset \hat{\Gamma}_\infty$. In addition, since $\langle T_2, T_{\mu-\overline{\nu}} \rangle$ is a maximal abelian subgroup of $\hat{\Gamma}_\infty$ we have $\Theta_G = \langle T_2, T_{\mu-\overline{\nu}} \rangle$. This completes the proof. □

Proposition 6.2. *In the same notation as above, $\{\hat{\Gamma}_n\}$ converges geometrically to Γ_G.*

Proof. For simplicity, we denote the manifolds $Q(u_n, \tau^{m_n} v_n)$ and $\hat{Q}(u_n, v_n)$ by Q_n and $\hat{Q}_n$, respectively. Fix an orthonormal frame $\tilde{\omega}$ in $\mathbf{H}^3$ and set the orthonormal frames in the quotient manifolds as follows:

$$\begin{aligned} (Q_n, \omega_n) &= (\mathbf{H}^3, \tilde{\omega})/\Gamma_n, \\ (\hat{Q}_n, \hat{\omega}_n) &= (\mathbf{H}^3, \tilde{\omega})/\hat{\Gamma}_n \quad \text{and} \\ (N_G, \omega_G) &= (\mathbf{H}^3, \tilde{\omega})/\Gamma_G. \end{aligned}$$

Since $\{\Gamma_n\}$ converges geometrically to Γ_G, the sequence (Q_n, ω_n) of framed manifolds converge geometrically to (N_G, ω_G) in the sense of Gromov; i.e. for any compact subset $\mathcal{K}$ of N_G with $\omega_G \in \mathcal{K}$ and for all n large enough, there exist bilipschitz diffeomorphisms

$$f_n : \mathcal{K} \to Q_n$$

onto their images such that $(f_n)_*(\omega_G) = \omega_n$ and that $K(f_n) \to 1$ as $n \to \infty$. Our goal is to show that the sequence $(\hat{Q}_n, \hat{\omega}_n)$ also converges geometrically to (N_G, ω_G) in the sense of Gromov. Let ϵ_0 be the 3-dimensional Margulis constant and let $\epsilon < \epsilon_0$. Let $\mathbf{T}_\epsilon \subset Q_n^{<\epsilon}$ be the Margulis tube containing the geodesic realization of $[\alpha]$ in Q_n, and let $\mathbf{P}_\epsilon \subset \hat{Q}_n^{<\epsilon}$ be the rank-2 cusp corresponding to $[\alpha]$. Then by the Drilling Theorem (Theorem 5.1), there exist bilipschitz diffeomorphisms of pairs

$$\Phi_n : (Q_n \setminus \mathbf{T}_\epsilon, \partial \mathbf{T}_\epsilon) \to (\hat{Q}_n \setminus \mathbf{P}_\epsilon, \partial \mathbf{P}_\epsilon)$$

such that $K(\Phi_n) \to 1$ as $n \to \infty$, and that $\Phi_n \circ \varphi|_{S \times \{-1/2\}}$ induces the representation $[\boldsymbol{\rho}_{\mu_n}] = Q(u_n, \infty)$, where $\varphi : S \times (-1, 1) \to Q_n$ is a marking homeomorphism inducing the representation $[\eta_n] = Q(u_n, \tau^{m_n} v_n)$.

Now choose $\epsilon < \epsilon_0$ small enough so that $f_n(\mathcal{K})$ is disjoint from $\mathbf{T}_\epsilon$ in Q_n for every n large enough. Then we obtain bilipschitz diffeomorphisms

$$g_n := \Phi_n \circ f_n : \mathcal{K} \to \hat{Q}_n$$

onto its images such that $K(g_n) \to 1$ as $n \to \infty$. We will observe below that Φ_n can be chosen so that $(\Phi_n)_*(\omega_n) = \hat{\omega}_n$, and hence that $(g_n)_*(\omega_G) = \hat{\omega}_n$. Then $(\hat{Q}_n, \hat{\omega}_n)$ converge geometrically to (N_G, ω_G) in the sense of Gromov, and hence $\hat{\Gamma}_n$ converge geometrically to Γ_G. Now let

$$\widetilde{g}_n : \widetilde{\mathcal{K}} \to \mathbf{H}^3$$

be the lift of g_n which satisfies $\pi_{\hat{\Gamma}_n} \circ \widetilde{g}_n = g_n \circ \pi_{\Gamma_G}$, where $\pi_{\hat{\Gamma}_n} : \mathbf{H}^3 \to \mathbf{H}^3/\hat{\Gamma}_n$ and $\pi_{\Gamma_G} : \mathbf{H}^3 \to \mathbf{H}^3/\Gamma_G$ are the universal covering maps, and $\widetilde{\mathcal{K}} = \pi_{\Gamma_G}^{-1}(\mathcal{K})$. Then $\widetilde{g}_n$ conjugates $\boldsymbol{\rho}_\mu(\alpha)$, $\boldsymbol{\rho}_\mu(\beta) \in \Gamma_G$ to $\boldsymbol{\rho}_{\mu_n}(\alpha)$, $\boldsymbol{\rho}_{\mu_n}(\beta) \in \hat{\Gamma}_n$, respectively. Since $\boldsymbol{\rho}_{\mu_n} \to \boldsymbol{\rho}_\mu$ and $K(\widetilde{g}_n) \to 1$ as $n \to \infty$, $\widetilde{g}_n$ converges to the identity map uniformly on compact subsets of $\widetilde{\mathcal{K}}$. Thus we may assume that $(\widetilde{g}_n)_*(\tilde{\omega}) \to \tilde{\omega}$, and hence that $(g_n)_*(\omega_G) = \hat{\omega}_n$. This completes the proof. □

As a consequence of Theorem 6.1, we obtain a precise description of the limit of an exotically convergent sequence in Theorem 3.8.

Theorem 6.3. *Suppose that a sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $(\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$ satisfies the same assumption as in Theorem 3.8. Let $u_\infty, v_\infty \in \overline{\mathbf{H}} \setminus \{\infty\}$ be the limits of the sequences $u_n = \tau^{k_n} x_n$, $v_n = \tau^{l_n} y_n$, and let $\mu = m(u_\infty)$, $\nu = m(v_\infty)$. Assume that there exist integers p, q such that $(p+1)k_n - pl_n + q \equiv 0$ for all n large enough. Then we have*

$$\lim_{n \to \infty} Q(x_n, y_n) = [\boldsymbol{\rho}_\xi], \quad \xi = (p+1)\mu - p\overline{\nu} + 2q \in \mathcal{M} \sqcup \mathcal{M}^*.$$

Proof. We use the same notation as in the proof of Theorem 3.8; for example, $[\rho_n] = Q(x_n, y_n)$, $[\eta_n] = Q(u_n, \tau^{m_n} v_n)$ and $\eta_n = \tau^{k_n} \cdot \rho_n$. As observed in the proof of Theorem 3.8 we have $m_n \to \infty$. Thus the sequence $[\eta_n] = Q(u_n, \tau^{m_n} v_n)$ converges to $[\boldsymbol{\rho}_\mu] = Q(u_\infty, \infty)$ as $n \to \infty$. Assume that $\{\eta_n\}$ converges to $\boldsymbol{\rho}_\mu$. Then $\delta = \lim_{n\to\infty} \eta_n(\alpha)$ equals $\boldsymbol{\rho}_\mu(\alpha) = T_2$. In addition, it follows from Theorem 6.1 that $\hat{\delta} = \lim_{n\to\infty} \eta_n(\alpha)^{m_n}$ equals $\boldsymbol{\rho}_\mu(\beta)\boldsymbol{\rho}_{\overline{\nu}}(\beta)^{-1} = T_{\mu - \overline{\nu}}$. Therefore, we obtain from the equations (2), (3) in the proof of Theorem 3.8 that

$$\begin{aligned} &\rho_n(\alpha) \to \eta_\infty(\alpha) = T_2, \quad \text{and} \\ &\rho_n(\beta) \to \hat{\delta}^p \delta^q \eta_\infty(\beta) = (T_{\mu-\overline{\nu}})^p (T_2)^q U_\mu = U_\xi, \end{aligned}$$

where $\xi = (p+1)\mu - p\overline{\nu} + 2q$. Thus $\{\rho_n\}$ converges to $\boldsymbol{\rho}_\xi$, and the result follows. □

Note that the complex number $\xi = (p+1)\mu - p\overline{\nu} + 2q$ in Theorem 6.3 does not depend on the choices of $\{k_n\}$, $\{l_n\}$ in Theorem 3.8. One can also check this fact directly as follows. Let $k_n' \equiv k_n + u$ and $l_n' = l_n + v$ be another sequences for some $u, v \in \mathbb{Z}$. Then μ, ν change into $\mu' = \mu + 2u$, $\nu' = \nu + 2v$. On the other hand, let p', q' be integers which satisfy $(p'+1)k_n' - p'l_n' + q' \equiv 0$ for all n large enough. Then we see from $m_n = k_n - l_n \to \infty$ that $p' = p$ and $q' = q - (p+1)u + pv$. Thus we obtain the following:

$$(p'+1)\mu' - p'\overline{\nu'} + 2q' = (p+1)\mu - p\overline{\nu} + 2q.$$

7 Geometric limits of Bers slices

In this section, we consider limits of sequences of Bers slices. Let $y_n \to y_\infty$ be a convergent sequence in $\overline{\mathbf{H}}$. Then the maps $\{b_{y_n}\}_{n=1}^\infty$ converges to b_{y_∞} at each point in $\overline{\mathbf{H}} \setminus \{y_\infty\}$. On the other hand, after passing to a subsequence if necessary, the sequence $\{\mathcal{B}_{y_n}\}_{n=1}^\infty$ of images of b_{y_n} converges to a closed subset $\mathcal{B}_G \subset \mathcal{D}(S)$ in the Housdorff topology on $\mathcal{D}(S)$; that is,

(1) for any $[\rho] \in \mathcal{B}_G$ there exists a sequence $[\rho_n] \in \mathcal{B}_{y_n}$ such that $[\rho_n] \to [\rho]$, and

(2) if there exists a sequence $[\rho_{n_j}] \in \mathcal{B}_{y_{n_j}}$ such that $[\rho_{n_j}] \to [\rho]$ then $[\rho] \in \mathcal{B}_G$.

The set $\mathcal{B}_G$ is called the *geometric limit* of the sequence $\{\mathcal{B}_{y_n}\}$. One can easily check that $\mathcal{B}_{y_\infty} \subset \mathcal{B}_G$.

Theorem 7.1. *Let $y_n \to y_\infty$ be a convergent sequence in $\overline{\mathbf{H}}$, and suppose that the sequence $\{\mathcal{B}_{y_n}\}$ converges geometrically to $\mathcal{B}_G$. Then we have the following:*

(1) $\mathcal{B}_{y_\infty}$ is a proper subset of $\mathcal{B}_G$ if and only if $y_\infty \in \hat{\mathbb{Q}}$ and $\{y_n\}$ converges tangentially to y_∞.

(2) Suppose that $\{y_n\}$ converges tangentially to $\infty \in \hat{\mathbb{Q}}$, and that there exists a sequence $\{l_n\}$ of integers such that $\{\tau^{l_n} y_n\}$ converges to some $v_\infty \in \overline{\mathbf{H}} \setminus \{\infty\}$. Let $\nu = m(v_\infty)$. Then we have

$$\mathcal{B}_G = \{[\boldsymbol{\rho}_\xi] \in \mathcal{D}(S) \,|\, \xi \in \mathcal{M} \sqcup (\mathcal{M}^* + 2\overline{\nu})\}.$$

In particular $\mathcal{B}_\infty \subsetneq \mathcal{B}_G \subsetneq \mathcal{B}_\infty \sqcup \mathcal{B}_\infty^*$ holds.

Proof. (1) Assume first that $\mathcal{B}_{y_\infty} \subsetneq \mathcal{B}_G$. Then there exists a sequence $\{x_n\}$ in $\overline{\mathbf{H}}$ such that the sequence $Q(x_n, y_n) \in \mathcal{B}_{y_n}$ converges in $\mathcal{D}(S)$ to a point which does not lie in $\mathcal{B}_{y_\infty}$. Pass to a subsequence so that $\{x_n\}$ converges to some $x_\infty \in \overline{\mathbf{H}}$. If $(x_\infty, y_\infty) \notin \Delta$, then Theorem 2.1 implies that $Q(x_n, y_n) \to Q(x_\infty, y_\infty) \in \mathcal{B}_{y_\infty}$, which contradicts our assumption. Thus $(x_\infty, y_\infty) \in \Delta$, i.e. $x_\infty = y_\infty \in \partial\mathbf{H}$. Since the sequence $Q(x_n, y_n)$ converges in $\mathcal{D}(S)$, we have $y_\infty \in \hat{\mathbb{Q}}$ from Theorem 3.2, and $y_n \to y_\infty$ tangentially from Theorem 3.5. The converse is deduced from (2).

(2) We first show that for a given sequence $\{x_n\}$ in $\overline{\mathbf{H}}$, every accumulation point $[\rho_\infty]$ of the sequence $Q(x_n, y_n) \in \mathcal{B}_{y_n}$ lies in the set $\{[\boldsymbol{\rho}_\xi] \,|\, \xi \in \mathcal{M} \sqcup (\mathcal{M}^* + 2\overline{\nu})\}$. Pass to a subsequence so that the sequence $Q(x_n, y_n)$ converges to $[\rho_\infty]$, and that $\{x_n\}$ converges to some $x_\infty \in \overline{\mathbf{H}}$. If $x_\infty \neq \infty$ then $[\rho_\infty] = Q(x_\infty, \infty)$ by Theorem 2.1, and thus $[\rho_\infty] \in \{[\boldsymbol{\rho}_\xi] \,|\, \xi \in \mathcal{M}\}$. Now suppose that $x_\infty = \infty$. Since the sequence $Q(x_n, y_n)$ converges, $x_n \to \infty$ tangentially by Theorem 3.5. Recall that we are assuming that there is a sequence $l_n \in \mathbb{Z}$ such that $v_n = \tau^{l_n} y_n$ converges to $v_\infty = m^{-1}(\nu) \neq \infty$. By passing to a further subsequence if necessary, there is also a sequence $k_n \in \mathbb{Z}$ such that $u_n = \tau^{k_n} x_n$ converges to some $u_\infty \neq \infty$. Let $\mu = m(u_\infty)$. Then by Theorem 6.3 the sequence $Q(x_n, y_n)$ converges to $[\rho_\infty] = [\boldsymbol{\rho}_\xi]$ with $\xi = (p+1)\mu - p\overline{\nu} + 2q$ for

some $p, q \in \mathbb{Z}$. Since $Q(x_n, y_n)$ is an exotically convergent sequence, we have $p \neq 0, -1$. Therefore if $p \geq 1$ then $\xi \in \mathcal{M}$, and if $p \leq -2$ then $\xi \in \mathcal{M}^* + 2\overline{\nu}$.

Next we show that for every $\xi \in \mathcal{M} \sqcup (\mathcal{M}^* + 2\overline{\nu})$, $[\boldsymbol{\rho}_\xi]$ is the limit of some sequence $Q(x_n, y_n) \in \mathcal{B}_{y_n}$. If $\xi \in \mathcal{M}$ then by letting $x = m^{-1}(\xi)$ the sequence $Q(x, y_n) \in \mathcal{B}_{y_n}$ converges to $Q(x, \infty) = [\boldsymbol{\rho}_\xi]$. Next assume that $\xi \in \mathcal{M}^* + 2\overline{\nu}$. Then $\xi = -\mu + 2\overline{\nu}$ for some $\mu \in \mathcal{M}$. Let $u = m^{-1}(\mu)$ and $x_n = \tau^{-2l_n} u$. Then the sequence $(x_n, y_n) \in (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$ satisfies the condition of Theorem 6.3 in the case of $p = -2$ and $q = 0$, and thus the sequence $Q(x_n, y_n) \in \mathcal{B}_{y_n}$ converges to $[\boldsymbol{\rho}_\xi]$ with $\xi = -\mu + 2\overline{\nu}$. This completes the proof. □

8 Self-Bumping of $\mathcal{D}(S)$

In this section, we consider sequences which give rise to the self-bumping of $\mathcal{D}(S)$, and obtain a precise description of the set of points at which $\mathcal{D}(S)$ self-bumps.

Let p be a non-negative integer and define a subset $\mathcal{M}(p) \subset \mathcal{M}$ by

$$\mathcal{M}(p) \quad = \quad \{(p+1)\mu - p\overline{\nu} \in \mathbb{C} \mid \mu, \nu \in \mathcal{M}\}.$$

Note that $\mathcal{M}(0) = \mathcal{M}$, and that for every p the set $\mathcal{M}(p)$ is invariant under the translation $\xi \mapsto \xi + 2$. We let $\mathcal{M}^*(p) = \{\overline{\xi} \mid \xi \in \mathcal{M}(p)\}$ denote the complex conjugation of $\mathcal{M}(p)$.

Lemma 8.1. *We have $\mathcal{M}(p) \subset \mathcal{M}(1)$ for every $p \geq 2$.*

Proof. Let $p = k + 1$ with $k \geq 1$. Then for any $\mu, \nu \in \mathcal{M}$, we have

$$(p+1)\mu - p\overline{\nu} = 2\mu - ((k+1)\overline{\nu} - k\mu) \in \mathcal{M}(1)$$

since $(k+1)\overline{\nu} - k\mu$ lies in $\mathcal{M}^*(k) \subset \mathcal{M}^*$. □

For every $y \in \hat{\mathbb{Q}}$, choose an element σ_y of $\mathrm{Mod}(S)$ which takes the curve $[\alpha]$ to the simple closed curve associated to y. Such an element σ_y is unique up to pre-composition with a power of τ. Recall from Section 4 that the map $\beta_\infty : \mathcal{M} \sqcup \mathcal{M}^* \to \mathcal{B}_\infty \sqcup \mathcal{B}^*_\infty$ is defined by $\beta_\infty(\mu) = [\boldsymbol{\rho}_\mu]$. Using this map, we define a map

$$\beta_y : \mathcal{M} \sqcup \mathcal{M}^* \to \mathcal{B}_y \sqcup \mathcal{B}^*_y$$

by $\beta_y(\mu) := \sigma_y \cdot \beta_\infty(\mu) = [\boldsymbol{\rho}_\mu \circ (\sigma_y)_*^{-1}]$. Let $\mathcal{B}_y(p) = \beta_y(\mathcal{M}(p))$ and $\mathcal{B}^*_y(p) = \beta_y(\mathcal{M}^*(p))$.

Remark. If we regard $\sigma_y \in \mathrm{Mod}(S)$ as an element of $\mathrm{PSL}_2(\mathbb{Z})$, it is easy to check that the following diagram commute:

$$\begin{array}{ccc} \overline{\mathbf{H}} \setminus \{\infty\} & \xrightarrow{m} & \mathcal{M} \\ \big\downarrow \sigma_y & & \big\downarrow \beta_y \\ \overline{\mathbf{H}} \setminus \{y\} & \xrightarrow{b_y} & \mathcal{B}_y. \end{array}$$

Then we have $b_y^{-1}(\mathcal{B}_y(p)) = \sigma_y(m^{-1}(\mathcal{M}(p)))$.

Definition 8.2. The space $\mathcal{D}(S)$ is said to *self-bump* at $[\rho] \in \partial\mathcal{D}(S)$ if there exists a neighborhood U of $[\rho]$ such that for every neighborhood $V \subset U$ of $[\rho]$ the intersection $V \cap \mathrm{int}(\mathcal{D}(S))$ is disconnected.

We define the following two subsets of $\partial\mathcal{D}(S)$:

$$\begin{aligned}\partial^{\mathrm{bump}}\mathcal{D}(S) &= \{[\rho] \in \partial\mathcal{D}(S) \,|\, \mathcal{D}(S) \text{ self-bumps at } [\rho]\},\\ \partial^{\mathrm{exotic}}\mathcal{D}(S) &= \{[\rho] \in \partial\mathcal{D}(S) \,|\, \exists \text{ an exotically convergent sequence with limit } [\rho]\}.\end{aligned}$$

Theorem 8.3. *We have*

$$\partial^{\mathrm{bump}}\mathcal{D}(S) = \partial^{\mathrm{exotic}}\mathcal{D}(S) = \bigsqcup_{y \in \hat{\mathbb{Q}}} \left(\mathcal{B}_y(1) \sqcup \mathcal{B}_y^*(1)\right).$$

We divide the proof of Theorem 8.3 into the following two Lemmas 8.4 and 8.5; the former is a consequence of Theorems 3.8 and 6.3, and the latter is a consequence of Theorem 2.1.

Lemma 8.4. $\partial^{\mathrm{exotic}}\mathcal{D}(S) = \bigsqcup_{y \in \hat{\mathbb{Q}}} \left(\mathcal{B}_y(1) \sqcup \mathcal{B}_y^*(1)\right)$.

Proof. Let $[\rho] \in \partial^{\mathrm{exotic}}\mathcal{D}(S)$. Then by definition there is a convergent sequence $Q(x_n, y_n) \to [\rho]$ in $\mathcal{D}(S)$ such that (x_n, y_n) converges to some $(x_\infty, x_\infty) \in \Delta$. We have $x_\infty \in \hat{\mathbb{Q}}$ from Theorem 3.2. Our goal is to show that $[\rho] \in \mathcal{B}_{x_\infty}(1) \sqcup \mathcal{B}_{x_\infty}^*(1)$. By changing the generators of $\pi_1(S)$ if necessary, we may assume that $x_\infty = \infty$. One can see from Theorems 3.8 and 6.3 that the limit $[\rho]$ of the exotically convergent sequence $Q(x_n, y_n)$ equals $[\boldsymbol{\rho}_\xi]$ with $\xi = (p+1)\mu - p\overline{\nu} + 2q$ for some $\mu, \nu \in \mathcal{M}$ and $p, q \in \mathbb{Z}$, $p \neq 0, -1$. It then follows from Lemma 8.1 that $\xi \in \mathcal{M}(1) \sqcup \mathcal{M}^*(1)$, and hence that $[\rho] = [\boldsymbol{\rho}_\xi] \in \mathcal{B}_\infty(1) \sqcup \mathcal{B}_\infty^*(1)$.

To show the converse, it is enough to show that for a given $[\rho] \in \mathcal{B}_\infty(1) = \beta_\infty(\mathcal{M}(1))$, there is an exotically convergent sequence with limit $[\rho]$. Let $\xi = 2\mu - \overline{\nu} \in \mathcal{M}(1)$ such that $[\rho] = [\boldsymbol{\rho}_\xi]$, and let $u = m^{-1}(\mu)$, $v = m^{-1}(\nu)$. Then the sequence $Q(\tau^n u, \tau^{2n} v)$ converges to $[\boldsymbol{\rho}_\xi]$ as $n \to \infty$ by Theorem 6.3. This completes the proof. ☐

Lemma 8.5. $\partial^{\mathrm{bump}}\mathcal{D}(S) = \partial^{\mathrm{exotic}}\mathcal{D}(S)$.

Proof. Given $[\rho_0] = Q(x_0, y_0)$ in $\partial\mathcal{D}(S)$, choose an open neighborhood $\mathcal{N}(\Delta) \subset \overline{\mathbf{H}} \times \overline{\mathbf{H}}$ of Δ whose closure does not contain (x_0, y_0). Since $(\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \mathcal{N}(\Delta)$ is compact, it follows form Theorem 2.1 that the map

$$Q|_{(\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \mathcal{N}(\Delta)} : (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \mathcal{N}(\Delta) \to \mathcal{D}(S)$$

is a homeomorphism onto its image. Let U be an open neighborhood of $[\rho_0]$ such that the intersection $U_0 := U \cap Q((\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \mathcal{N}(\Delta))$ is connected.

Suppose first that $[\rho_0] = Q(x_0, y_0) \in \partial^{\mathrm{bump}}\mathcal{D}(S)$. We may assume that the neighborhood U of $[\rho_0]$ taken above also satisfies the condition in the Definition 8.2. Then there exists a sequence $Q(x_n, y_n) \in \mathrm{int}(\mathcal{D}(S)) \cap U$ such that $Q(x_n, y_n) \notin \mathrm{int}(U_0)$ and that $Q(x_n, y_n) \to [\rho_0]$. Pass to a subsequence so that (x_n, y_n) converges to some point (x_∞, y_∞) in the closure of $\mathcal{N}(\Delta)$. If $(x_\infty, y_\infty) \notin \Delta$ then $Q(x_n, y_n) \to Q(x_\infty, y_\infty) =$

$Q(x_0, y_0)$, which contradicts the injectivity of the map Q. Thus $(x_\infty, y_\infty) \in \Delta$, and thus $Q(x_n, y_n)$ is an exotically convergent sequence. Therefore $[\rho_0] \in \partial^{\mathrm{exotic}}\mathcal{D}(S)$.

Conversely, suppose that $[\rho_0] \in \partial^{\mathrm{exotic}}\mathcal{D}(S)$. Then by definition there is a convergent sequence $Q(x_n, y_n) \to [\rho_0]$ in $\mathcal{D}(S)$ such that (x_n, y_n) converges to some point in Δ. Then $(x_n, y_n) \in \mathcal{N}(\Delta)$ for all n large enough. Since $\mathrm{int}(\mathcal{D}(S))$ is dense in $\mathcal{D}(S)$ by Theorem 2.1, we may assume that $Q(x_n, y_n) \in \mathrm{int}(\mathcal{D}(S))$ without changing another situation. Then since the sequence $Q(x_n, y_n) \in \mathrm{int}(\mathcal{D}(S)) \setminus \mathrm{int}(U_0)$ converges to $[\rho]$, $\mathcal{D}(S)$ self-bumps at $[\rho_0]$. Therefore $[\rho_0] \in \partial^{\mathrm{bump}}\mathcal{D}(S)$. □

We now consider the set of points $(x, y) \in (\overline{\mathbf{H}} \times \overline{\mathbf{H}}) \setminus \Delta$ such that $Q(x, y) \in \partial^{\mathrm{bump}}\mathcal{D}(S)$. Note that, since the map $Q|_{\mathbf{H}\times\mathbf{H}} : \mathbf{H}\times\mathbf{H} \to \mathrm{int}(\mathcal{D}(S))$ is bijective, $Q(x, y)$ lies in $\partial\mathcal{D}(S)$ if and only if (x, y) lies in either $\mathbf{H} \times \partial\mathbf{H}$, $\partial\mathbf{H} \times \mathbf{H}$ or $\partial\mathbf{H} \times \partial\mathbf{H} \setminus \Delta$. As a consequence of Theorem 8.3 we then obtain the following:

Corollary 8.6. *(1) If $(x, y) \in (\partial\mathbf{H} \times \partial\mathbf{H}) \setminus \Delta$, then $\mathcal{D}(S)$ does not self-bump at $Q(x, y) \in \partial\mathcal{D}(S)$.*

(2) Let $(x, y) \in \mathbf{H} \times \partial\mathbf{H}$. Then $Q(x, y) \in \partial^{\mathrm{bump}}\mathcal{D}(S)$ if and only if $y \in \hat{\mathbb{Q}}$ and $x \in b_y^{-1}(\mathcal{B}_y(1))$.

Proof. (1) It follows immediately from Theorem 8.3.

(2) By Theorem 8.3, $Q(x, y) \in \partial^{\mathrm{bump}}\mathcal{D}(S)$ if and only if $y \in \hat{\mathbb{Q}}$ and $Q(x, y) \in \mathcal{B}_y(1)$. Since $Q(x, y) = b_y(x)$ by definition, we obtain the result. □

Remark. It is experimentally conjectured in [MSW] that $\min\{\mathrm{Im}\,\mu \mid \mu \in \mathcal{M}\} = 1.616.. > 1.5$. If it is true, we can show that $m^{-1}(\mathcal{M}(1)) \subset \{z \in \mathbb{C} \mid \mathrm{Im}\,z > 1\}$. Then by Shimizu-Leutbecher's lemma, we have $b_y^{-1}(\mathcal{B}_y(1)) \cap b_{y'}^{-1}(\mathcal{B}_{y'}(1)) = \emptyset$ for every y, $y' \in \hat{\mathbb{Q}}$, $y \neq y'$. It then follows from Corollary 8.6 (2) that for every $x \in \mathbf{H}$, the intersection $\partial\mathcal{B}_x \cap \partial^{\mathrm{bump}}\mathcal{D}(S)$ is empty or consists of exactly one point.

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