On some analytic properties of deformation spaces of Kleinian groups

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Motivation (or a trigger)

Extension problem of holomorphic motions

Let $\phi : \Delta_K \times E \to \hat{\mathbb{C}}$ be a holomorphic motion of E over Δ_K, where $K \Subset \Delta$ is an AB-removable compact subset of the unit disk Δ, and $E \subset \hat{\mathbb{C}}$ is a closed set. Then, we show the following:

(Beck-Jiang-Mitra-S.)
Let $\{\Omega_1, \Omega_2, \cdots\}$ be the set of connected components of $\hat{\mathbb{C}} - E$.

(1) Every Ω_i is simply connected, then $\forall \phi$ can be extended to a holomorphic motion

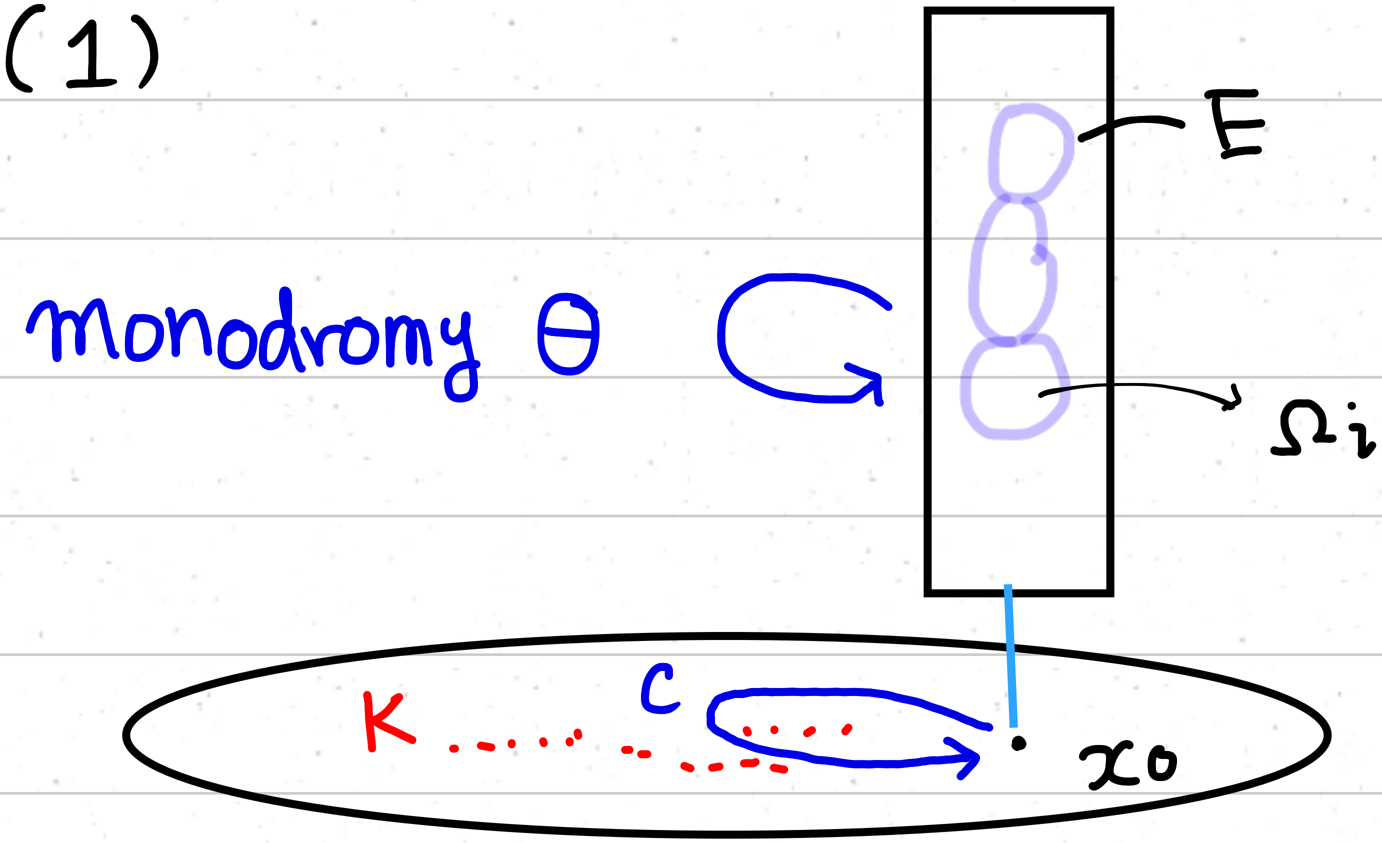
$$\hat{\phi} : \Delta_K \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

of $\hat{\mathbb{C}}$ over Δ_K.

(2) If $^{\exists}\Omega_i$ is not simply connected, then $^{\exists}\phi : \Delta^* \times E \to \hat{\mathbb{C}}$ holomorphic motion of E over $\Delta^* = \{0 < |z| < 1\}$ such that ϕ cannot be extended to a holo. motion $\hat{\phi}$ of $\hat{\mathbb{C}}$ over Δ^*.

Sketch of proof

(1)



$^\exists \theta : \pi_1(\Delta_K, x_0) \to QC(\hat{\mathbb{C}})/\underset{E}{\sim}$

s.t. $\theta([c])\big|_E = \mathrm{id}.$

$\Rightarrow \theta$ is trivial

($^\forall \Omega_i$ is simply connected)

$\Rightarrow$ For any finite subset $E_n \subset E$,

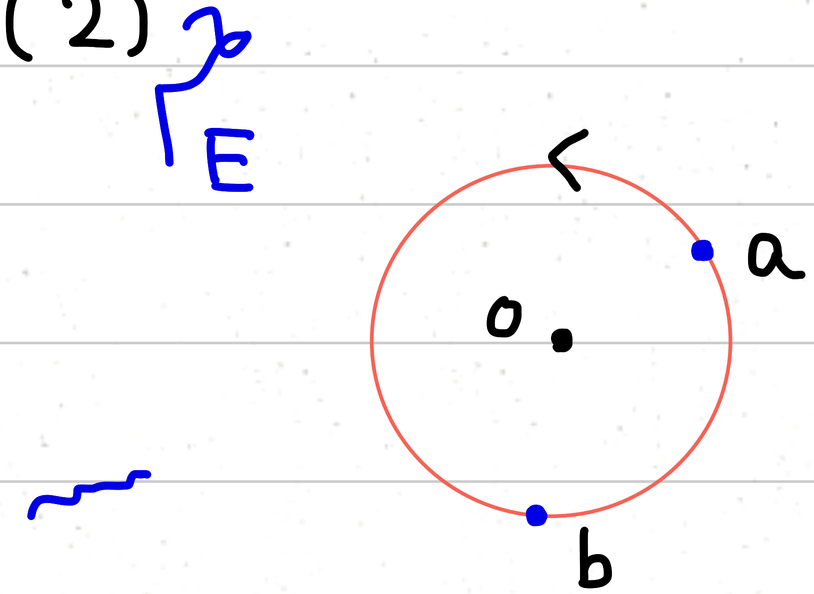
we see $\theta|_{E_n} : \pi_1(\Delta_K, x_0) \to \mathrm{Mod}(\hat{\mathbb{C}} \setminus E_n)$

is also trivial $\Rightarrow {}^\exists \Phi : \Delta_K \to T(E_n)$ holo.

$\Rightarrow \exists\, \Phi_n : \Delta_K \to T(\hat{\mathbb{C}} \setminus \dot{E}_n)$ holo. map induced by the holo. motion.

$\Rightarrow \Phi_n$ is extended to $\exists\, \tilde{\Phi}_n : \Delta \to T(\hat{\mathbb{C}} \setminus E_n)$

(2)



$a, b, 0 \in E$

$$\Delta^* \ni \lambda \quad \begin{array}{l} a \mapsto \lambda a \\ b \mapsto \lambda b \\ z \mapsto z \quad \forall z \in E - \{a, b\} \end{array}$$

$\Rightarrow$ the monodromy is not trivial.

We consider the group equivariant version.

$G_0 < PSL(2,\mathbb{C})$: a Kleinian group (non-elementary)

$Hom(G_0, PSL(2,\mathbb{C})) = \{\rho : G_0 \to PSL(2,\mathbb{C}) \text{ homo.}\}/_{\text{conj}}$

V : a connected complex mfd $\ni x_0$ base point

Consider $\{\theta_x\}_{x\in V} \subset Hom(G_0, PSL(2,\mathbb{C}))$

satisfying

(1) $\theta_{x_0} = id.$

(2) For each $x \in V$, $\exists U_x$: a neighborhood of x

s.t. $U_x \ni z \mapsto \tilde{\theta}_z(g) \in PSL(2,\mathbb{C})$ is holomorphic.

for $\forall g \in G_0$. representative

We call $\mathcal{G} := (G_0, V, \{\theta_x\}_{x\in V})$ a holomorphic family of G_0 over V.

Thm (Bers, Earle-Kra-Krushkal)

$\mathcal{G} = (G_0, \Delta, \{\theta_x\}_{x\in\Delta})$: a holo. family of G_0 over $\Delta = \{|z|<1\}$.

Suppose that θ_x is an isomorphism for $^\forall x \in \Delta$, discrete and type preserving. $\Rightarrow$ $\mathcal{G}$ is a quasi-conformal deformations of G_0.

Thm 1

$K \Subset \Delta$: AB-removable compact subset of Δ

G_0 : a Kleinian group

Suppose that $^\forall$ component of $\Omega(G_0)$ is simply connected.

Then $^\forall \mathcal{G} = (G_0, \Delta_K, \{\theta_x\}_{x\in\Delta_K})$ $(\Delta_K := \Delta \setminus K)$

which is type-preserving,

is extended to $^\exists \tilde{\mathcal{G}} = (G_0, \Delta, \{\tilde{\theta}_x\}_{x\in\Delta})$ over Δ.

Cor. 1

A generalization of McMullen's disk convexity thm for $QF(S)$.

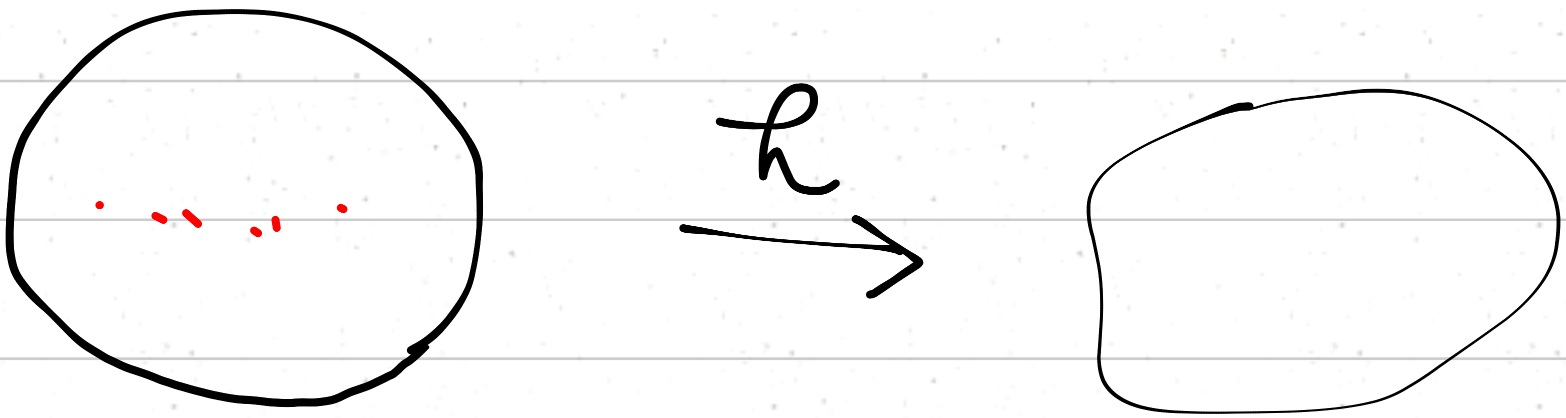
the space of Q-Fuchsian groups for S

Disk convexity:

$h : \Delta_k \cup \partial\Delta \to \mathrm{Hom}(\Gamma_0, PSL(2,\mathbb{C}))$ holo.

Suppose that $h(\partial\Delta) \subset QF(S)$ &

type preserving $\Rightarrow h(\Delta_k) \subset QF(S)$.

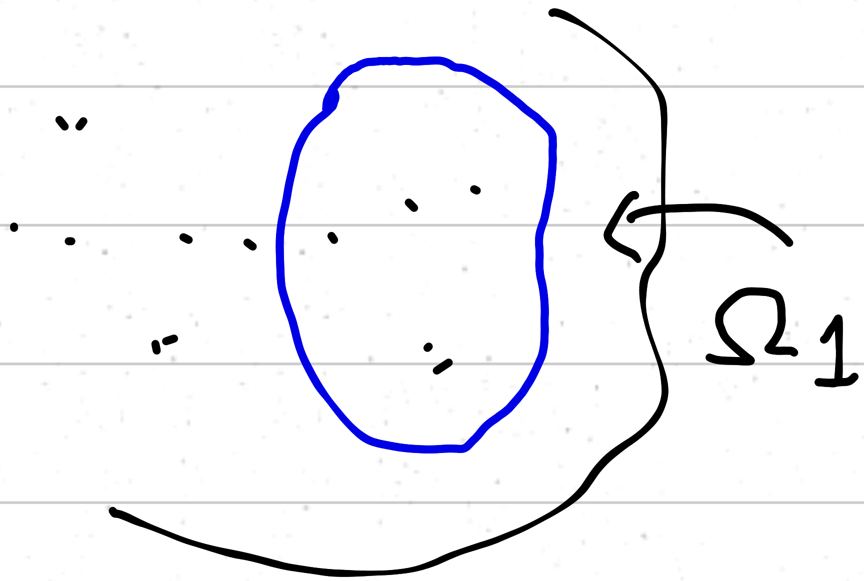


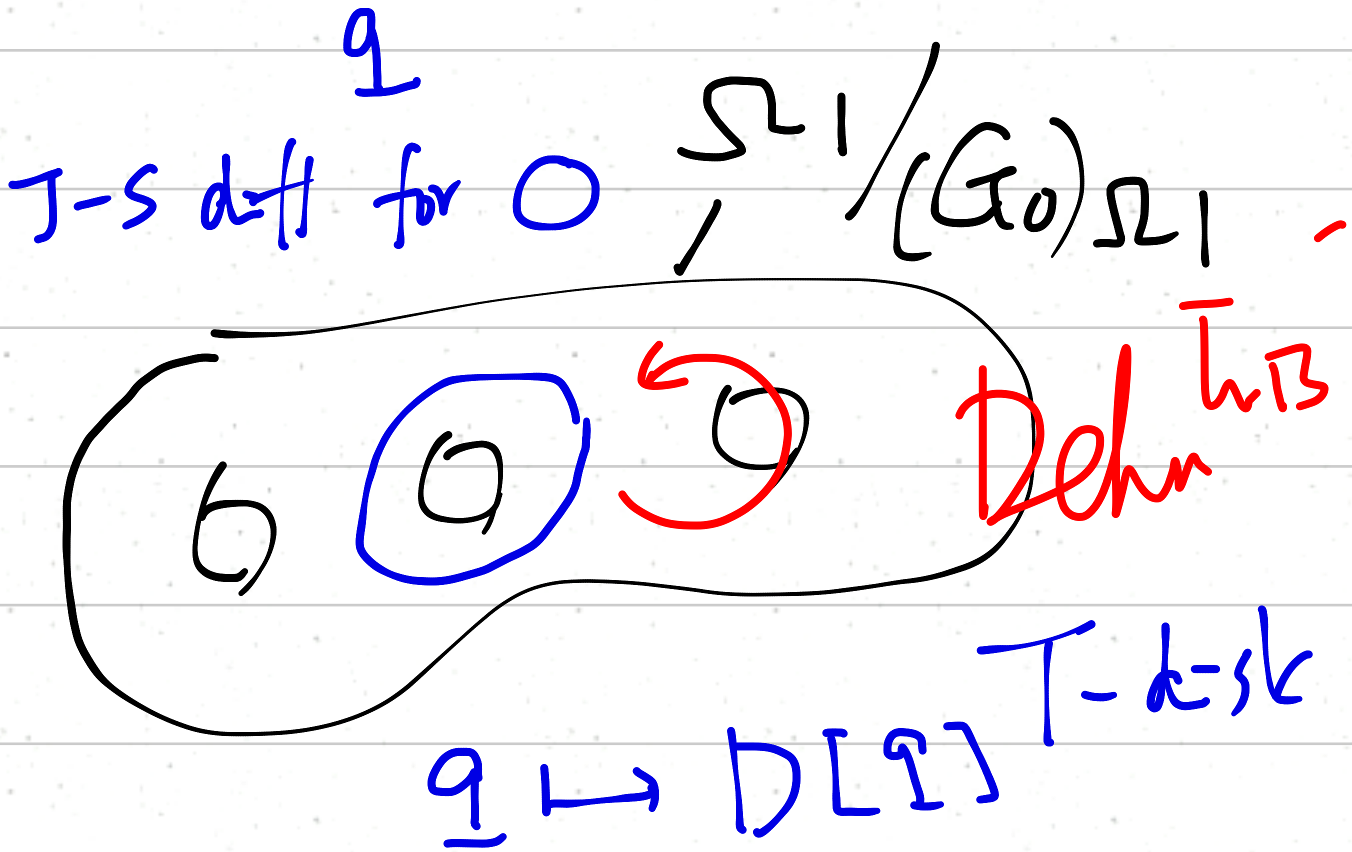
Thm 2.

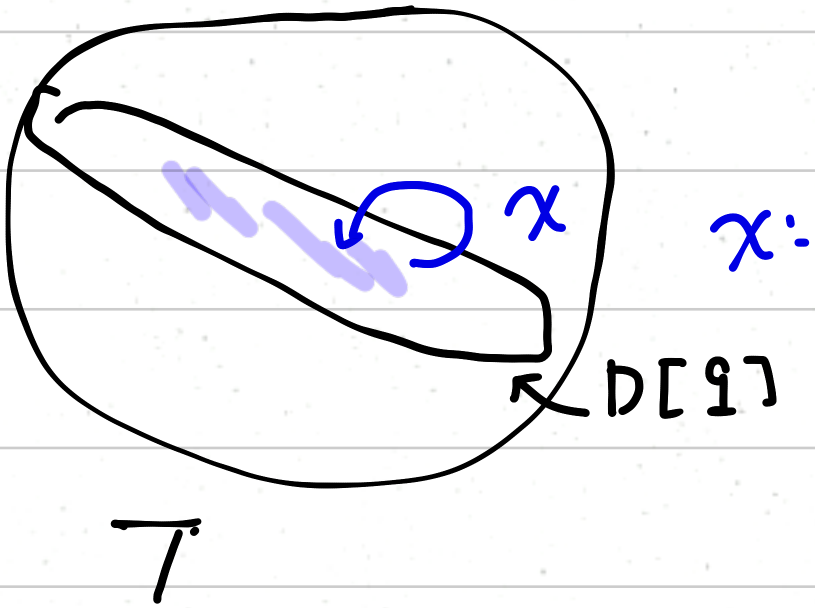
G_0 : a finitely generated Kleinian group with a **non-simply connected** component of $\Omega(G_0)$.

Then, $\exists\ \mathcal{G} = \{G_0, \Delta^*, \{\theta_x\}_{x\in\Delta^*})$ $(\Delta^* = \{0<|z|<1\})$

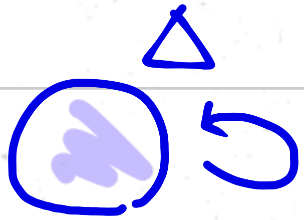
such that it cannot be extended to a holo. family over Δ.







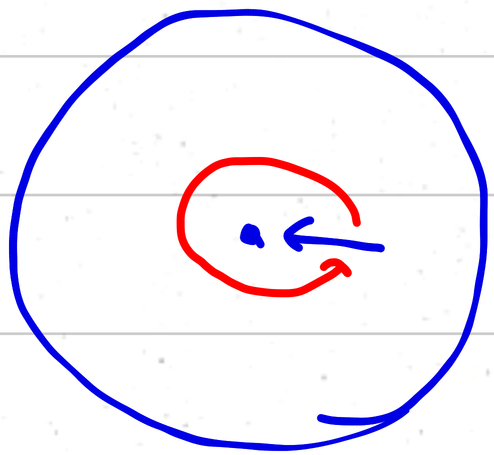
x : Dehn twist $\longrightarrow$



γ : parabolic

$\Delta / \langle \gamma \rangle = \Delta^*$

Δ^*



The holomorphic convexity of the deformation space of a Kleinian group.

M : a connected complex manifold

$\mathcal{O}(M)$: the space of holo. functions on M

$\mathcal{O} \subset \mathcal{O}(M)$

M is $\mathcal{O}$-convex or convex for $\mathcal{O}$

if for $\forall K \Subset M$, $\hat{K}_{\mathcal{O}}$ is compact in M, where

$$\hat{K}_{\mathcal{O}} := \{ p \in M \mid |f(p)| \leq \|f\|_{\infty, K} \text{ for } \forall f \in \mathcal{O} \}.$$

M is called holomorphically convex if it is $\mathcal{O}(M)$-convex.

If $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}(M)$, then $\mathcal{O}_1$-convex $\Rightarrow$ $\mathcal{O}_2$-convex.

$D \subset \mathbb{C}^n$ domain

D is called polynomially convex if it is convex for the space of polynomials in $\mathbb{C}^n$.

FACT

- (Oka) $D \subset \mathbb{C}^n$ is holo. convex iff D is a domain of holomorphy.
- Γ_0 : a finitely generated Fuchsian group.

$T(\Gamma_0)$: the Teichmüller space of Γ_0

$\Rightarrow$ $T(\Gamma_0)$ is holomorphically convex (Bers-Ehrenpreis)

" is H^∞-convex (Krushkal)

Bers embedding $T(\Gamma_0) \subset \mathbb{C}^n$ is polynomially convex (S.)

the space of bounded holo. functions

- G_0 : a finitely generated Kleinian group

$D(G_0)$: the space of quasi-conformal deformations of G_0

$\cap$

$\mathrm{Hom}(G_0, PSL(2,\mathbb{C}))$

$D(G_0)$ is holomorphically convex (Kra-Maskit).

- If the Carathéodory distance of M is complete, then M is H^∞-convex.

c_M

$$c_M(p,q) := \sup_{f \in \mathrm{Hol}(M,\Delta)} \rho_\Delta(f(p), f(q))$$

Poincaré distance on Δ.

Prop.

Γ_0 : a finitely generated Fuchsian group

$QF(\Gamma_0) := D(\Gamma_0)$

$\Rightarrow QF(\Gamma_0)$ is H^∞-convex.

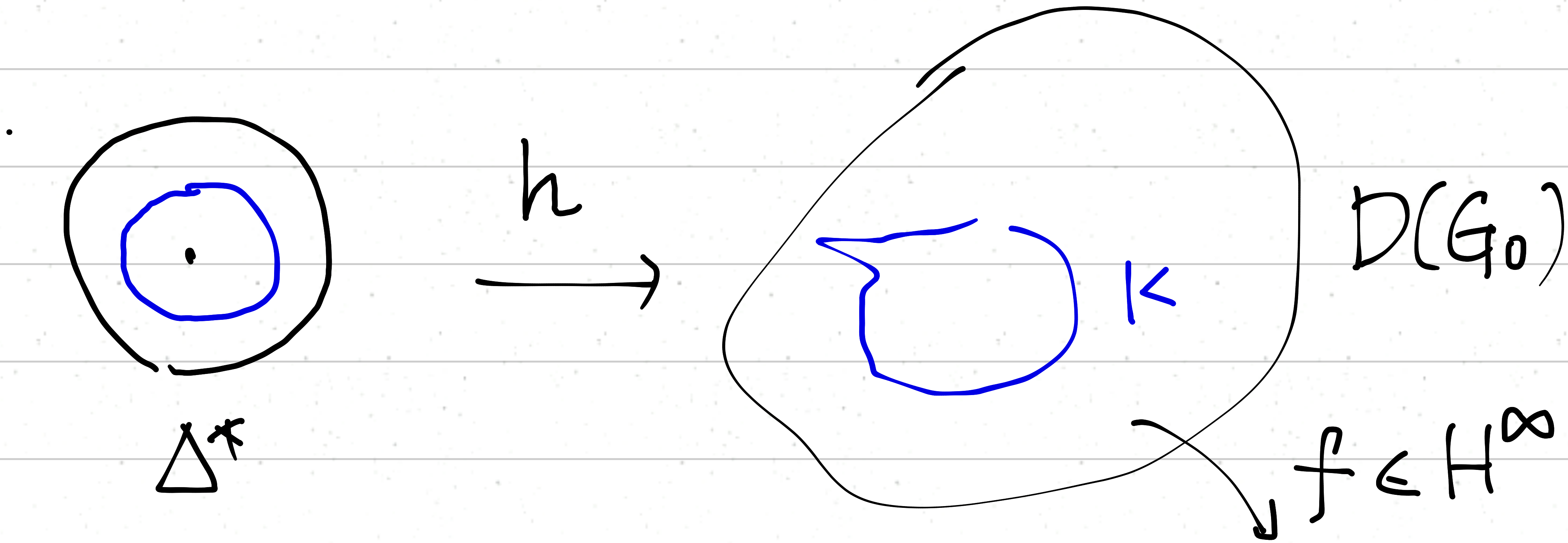
$\because$) $QF(\Gamma_0) \cong T(\Gamma_0) \times T(\overline{\Gamma_0})$

$$c_{M\times N} \geqq \max(c_M, c_N)$$

Thm 3

G_0 : a finitely generated Kleinian group with non-simply connected component of $\Omega(G_0)$.

Then $D(G_0)$ is not H^∞-convex and the Carathéodory distance is not complete.



On $D(G_0)$, we can define the Teichmüller distance $d_T^{D(G_0)}$:

$$d_T^{D(G_0)}(p_1, p_2) = \inf_{w_1, w_2} \log K(w_1 \circ w_2^{-1}) \quad (w_i \leftrightarrow p_i)$$

Thm 4

$d_T^{D(G_0)}$ = the Kobayashi distance on $D(G_0)$

Cor.

In $D(G_0)$, the Kobayashi distance and the Carathéodory distance do **not** coincide if G_0 has a non-simply connected component of $\Omega(G_0)$.