On some analytic properties of deformation spaces of Kleinian groups

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Motivation (or a trigger)

Extension problem of holomorphic motions

Let \( \phi : \Delta \times E \to \hat{C} \) be a holomorphic motion of \( E \) over \( \Delta \), where \( K \subseteq \Delta \) is an \( AB \)-removable compact subset of the unit disk \( \Delta \), and \( E \subseteq \hat{C} \) is a closed set. Then, we show the following:

( Beck-Jiang-Mitra-S.)

Let \( \{ \Omega_1, \Omega_2, \ldots \} \) be the set of connected components of \( \hat{C} - E \).

(1) Every \( \Omega_i \) is simply connected, then \( \forall \phi \) can be extended to a holomorphic motion

\[ \hat{\phi} : \Delta \times \hat{C} \to \hat{C} \]

of \( \hat{C} \) over \( \Delta \).
(2) If $\exists \Omega$ is not simply connected, then $\exists \phi : \Delta^* \times E \to \hat{C}$ holomorphic motion of $E$ over $\Delta^* = \{ 0 < |z| < 1 \}$ such that $\phi$ cannot be extended to a holo. motion $\hat{\phi}$ of $\hat{C}$ over $\Delta^*$. 
Sketch of proof

(1)

monodromy $\Theta$ 

$\exists \Theta : \pi_1(\Delta k, x_0)$ 

$\implies QC(\hat{C}/E)$ 

$s.t. \quad \Theta([c])|_E = id.$ 

$\Rightarrow \Theta$ is trivial

($\forall \Omega_i$ is simply connected)

$\Rightarrow$ For any finite subset $E_n \subset \Delta k$, 

we see $\Theta|_{E_n} : \pi_1(\Delta k, x_0) \to \text{Mod}(\hat{C}\setminus E_n)$ 

is also trivial 

$\Rightarrow \exists \Phi : \Delta k \to T(E_n)$ holo.
\[ \exists \Phi_n : \Delta \rightarrow T(\mathbb{C} \setminus \mathbb{E}_n) \text{ holo. map induced by the holo. motion.} \]

\[ \Rightarrow \exists \Phi_n \text{ is extended to } \exists \Phi_n : \Delta \rightarrow T(\mathbb{C} \setminus \mathbb{E}_n) \]
\( E \)

\[
\begin{align*}
(2) & \quad E \\
& a, b, 0 \in E \\
& \lambda \in \Delta^* \quad a \mapsto \lambda a \\
& \quad b \mapsto \lambda b \\
& \quad z \mapsto z \quad \forall z \in E - \{a, b\} \\
\Rightarrow & \quad \text{the monodromy is not trivial.}
\end{align*}
\]
We consider the group equivariant version.

\[ G_0 < \text{PSL}(2, \mathbb{C}) : \text{a Kleinian group (non-elementary)} \]

\[ \text{Hom}(G_0, \text{PSL}(2, \mathbb{C})) = \{ \varphi : G_0 \to \text{PSL}(2, \mathbb{C}) \text{ hom.} \}/\sim \]

\( \mathcal{V} \) : a connected complex mfd \( \exists x_0 \) base point

Consider \( \{ \Theta_x \}_{x \in \mathcal{V}} \subset \text{Hom}(G_0, \text{PSL}(2, \mathbb{C})) \)

satisfying

1. \( \Theta_{x_0} = \text{id} \)

2. For each \( x \in \mathcal{V} \), \( \exists U_x \) : a neighborhood of \( x \)

s.t. \( U_x \ni z \mapsto \Theta_z(g) \in \text{PSL}(2, \mathbb{C}) \) is holomorphic.

for \( \forall g \in G_0 \).

We call \( \mathcal{G} = (G_0, \mathcal{V}, \{ \Theta_x \}_{x \in \mathcal{V}}) \) a holomorphic family of \( G_0 \) over \( \mathcal{V} \).
Thm (Bers, Earle-kra-Krushkal)

\[ G = (G_0, \Delta, \{ \Theta_t \}_{t \in \Delta} ) : \text{a Rolo. family of } G_0 \text{ over } \Delta = \{ |z| < 1 \} \]

Suppose that \( \Theta_t \) is an isomorphism for \( \forall x \in \Delta \), discrete and type preserving. \( \Rightarrow \) \( G \) is a quasi-conformal deformations of \( G_0 \).

Thm 1

\[ K \subset \Delta : \text{AB-removable compact subset of } \Delta \]

\( G_0 : \text{a kleinian group} \)

Suppose that \( \forall \) component of \( \Omega(G_0) \) is simply connected.

Then \( \forall G = (G_0, \Delta_k, \{ \Theta_t \}_{t \in \Delta_k} ) \) \( (\Delta_k = \Delta \setminus K) \)

\[ \text{which is type-preserving,} \]

\( \text{is extended to } \exists \tilde{G} = (G_0, \Delta, \{ \tilde{\Theta}_t \}_{t \in \Delta} ) \text{ over } \Delta. \]
Cor. 1

A generalization of McMullen's disk convexity theorem for $\mathcal{QF}(S)$, the space of $\mathcal{Q}$-Fuchsian groups for $S$.

Disk convexity:

$h : \Delta_k \to \text{Hom}(\mathbb{P}^1, \text{PSL}(2, \mathbb{Q}))$ holds.

Suppose that $h(\partial \Delta) \subset \mathcal{QF}(S)$ is type preserving $\Rightarrow h(\Delta_k) \subset \mathcal{QF}(S)$. 

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{disk.png} \\
\end{array}
\]
Thm 2.

Let $G_0$ be a finitely generated Kleinian group with a non-simply connected component of $\Omega(G_0)$.

Then, there exists $\mathcal{G} = \{G_0, \Delta^*, \{\theta_x\}_{x \in \Delta^*}\}$ ($\Delta^* = \{0 < |z| < 1\}$) such that it cannot be extended to a holomorphic family over $\Delta$.
\[ \gamma : \text{Dehn twist} \rightarrow \varnothing \delta : \text{parabolic} \]

\[ \Delta / \langle \delta \rangle = \Delta^* \]
The holomorphic convexity of the deformation space of a Kleinian group.

$M$: a connected complex manifold

$O(M)$: the space of holomorphic functions on $M$

$O \subset O(M)$

$M$ is $O$-convex or convex for $O$

if for all $K \subset M$, $\widehat{K}_O$ is compact in $M$, where

$$\widehat{K}_O := \{ p \in M \mid |f(p)| \leq \|f\|_{\infty,K} \text{ for all } f \in O \}.$$

$M$ is called holomorphically convex if it is $O(M)$-convex.

If $O_1 \subset O_2 \subset O(M)$, then $O_1$-convex $\Rightarrow$ $O_2$-convex.
$D \subset \mathbb{C}^n$ domain

$D$ is called polynomially convex if it is convex for the space of polynomials in $\mathbb{C}^n$.

**Fact**

- $(Oka)$ $D \subset \mathbb{C}^n$ is hol. convex iff $D$ is a domain of holomorphy.
- $\Gamma_0$: a finitely generated Fuchsian group.
- $T(\Gamma_0)$: the Teichmüller space of $\Gamma_0$

$\Rightarrow T(\Gamma_0)$ is holomorphically convex (Bers-Ehrenpreis)

" is $H^\infty$-convex (Krushkal)

Bers embedding $T(\Gamma_0) \subset \mathbb{C}^n$ is polynomially convex (S.)
the space of bounded holomorphic functions
• $G_0$: a finitely generated Kleinian group

$\hat{D}(G_0)$: the space of quasi-conformal deformations of $G_0$

$\hat{\text{Hom}}(G_0, \text{PSL}(2,\mathbb{C}))$

$D(G_0)$ is holomorphically convex (Kra-Maskit).

• If the Carathéodory distance of $M$ is complete, then $M$ is $H^\infty$-convex.

$$C_m(p, q) := \sup_{f \in \text{Hd}(M, \Delta)} P_{\Delta}(f(p), f(q))$$

Poincaré distance on $\Delta$.\]
\textbf{Prop.}

$\Gamma_0$ : a finitely generated Fuchsian group

$Q_F(\Gamma_0) := D(\Gamma_0)$

$\Rightarrow Q_F(\Gamma_0)$ is $H^\infty$-convex.

$\therefore Q_F(\Gamma_0) \cong T(\Gamma_0) \times T(\overline{\Gamma_0})$

$C_{\text{max}} \geq \max(C_M, C_N)$
Thm 3

Let $G_0$ be a finitely generated Kleinian group with non-simply connected component of $\Omega(G_0)$.

Then $D(G_0)$ is not $H^\infty$-convex and the Carathéodory distance is not complete.
On $D(G_0)$, we can define the Teichmüller distance $d_T^{D(G_0)}$:

$$d_T^{D(G_0)}(p_1, p_2) = \inf_{W_1, W_2} \log K(W_1 \circ W_2^{-1}) \quad (W_i \leftrightarrow p_i)$$

**Thm 4**

$d_T^{D(G_0)} = \text{the Kobayashi distance on } D(G_0)$

**Cor.**

In $D(G_0)$, the Kobayashi distance and the Carathéodory distance do not coincide if $G_0$ has a non-simply connected component of $\Omega(G_0)$. 