Geometry of Teichmüller distance

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Analysis and Geometry of Riemann Surfaces and Related Topics

– Celebrate Professor Hiroshige Shiga's 60th birthday –

Tokyo Institute Technology

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Geometry on Teichmüller space

The geometry of the Teichmüller space in terms of the Teichmüller distance is well-studied.

- (Teichmüller) Teichmüller space with the Teichmüller distance is complete and uniquely geodesic.
- (Royden) The Teichmüller distance is a Finsler distance, and coincides with the Kobayashi distance under the natural complex structure.

The geometry of the Teichmüller distance is also studied as metric space.

- (Masur) The Teichmüller space is not negatively curved in the sense of Busemann. Expecially, it is not a $\mathrm{CAT}(0)$-space.
- (Masur-Wolf, McCauthy-Papadopoulos, Ivanov) The Teichmüller space is not Gromov hyperbolic.

Extremal length geometry after Kerckhoff, Gardiner and Masur

- S. Kerckhoff studies the asymptotic geometry on extremal lengths of simple closed curves.
- Kerckhoff found an excellent formula on the Teichmüller distance, which tells us that the Teichmüller distance is represented by the ratio of the extremal length.
- "Extremal length geometry on the Teichmüller space" is nothing but the geometry on Teichmüller space studied by use of extremal length.

  From Kerckhoff's formula, we often call the geometry on the Teichmuller distance the extremel length geometry of Teichmüller space.

Extremal length geometry after Kerckhoff, Gardiner and Masur

- F. Gardiner and H. Masur formulated a "natural" compactification for the extremal length geometry of the Teichmüller space by assembling asymptotic behaviors of the extremal lengths of simple closed curves, which we call the Gardiner-Masur compactification.

Our naive questions of this study are

Question 1

(1) How natural?

(2) What can we know about the geometry of the Teichmüller distance from the compactification?

(3) What can we know about anything on the moduli of Riemann surfaces from the compactification?

To answer these questions, we attempt to develop

"Thurston theory with extremal length"

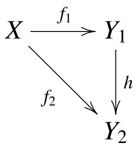
Namely, we will discuss the extremal length geometry on Teichmüller space via intersection number.

Teichmüller theory

Let $T_{g,m}$ be the Teichmüller space of Riemann surfaces of type (g, m) with $2g - 2 + m > 0$. Namely,

$$T_{g,m} = \{(Y, f) \mid f : X \to Y \text{ qc}\}/ \sim$$

where X is a base Riemann surface. Two marked Riemann surfaces (Y_1, f_1) and (Y_2, f_2) are equivalent if there is a conformal mapping $h : Y_1 \to Y_2$ such that $h \circ f_1$ is homotopic to f_2.



The Teichmüller distance is a distance on $T_{g,m}$ defined by

$$d_T((Y_1, f_1), (Y_2, f_2)) = \frac{1}{2} \log \inf\{K(h) \mid h \text{ qc homotipic to } f_2 \circ f_1^{-1}\}$$

where $K(h)$ is the maximal dilatation of h.

The space of measured foliations

Let $\mathcal{S}$ be the set of homotopy classes of non-trivial and non-peripheral simple closed curves on X. We consider the embedding

$$\mathbb{R}_+ \otimes \mathcal{S} \ni t\alpha \mapsto [\mathcal{S} \ni \beta \mapsto t\, i(\beta, \alpha)] \in \mathbb{R}_+^{\mathcal{S}},$$

where $\mathbb{R} = \{t \in \mathbb{R} \mid t \geq 0\}$. Then, the closure $\mathcal{MF} \subset \mathbb{R}_+^{\mathcal{S}}$ of the image is called the space of measured foliations. The quotient space

$$\mathcal{PMF} = (\mathcal{MF} - \{0\})/\mathbb{R}_+ \subset P\mathbb{R}^{\mathcal{S}} = (\mathbb{R}_+^{\mathcal{S}} - \{0\})/\mathbb{R}_{>0}$$

under the action $\mathbb{R}_{>0} \times \mathbb{R}_+^{\mathcal{S}} \ni (t, F) \to tF \in \mathbb{R}_+^{\mathcal{S}}$ is called the space of projective measured foliations.

It is known the following (Thurston).

- $\mathcal{MF}$ is homeomorphic to $\mathbb{R}^{6g-6+2m}$.
- $\mathcal{PMF}$ is homeomorphic to $S^{6g-7+2m}$.

Extremal length

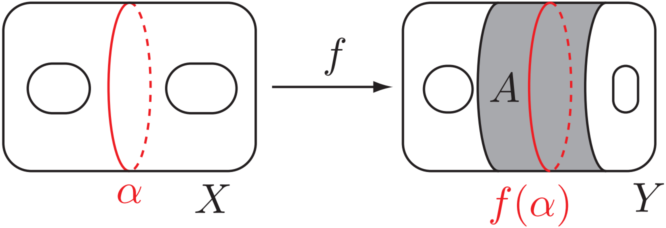
Let $\alpha \in \mathcal{S}$. The extremal length of α on $y = (Y, f) \in T_{g,m}$ is defined by

$$\mathrm{Ext}_y(\alpha) = \inf\{1/\mathrm{Mod}(A) \mid \text{the core of } A \text{ is homotopic to } f(\alpha)\}$$

where

$$\mathrm{Mod}(A) = \frac{1}{2\pi}\log r$$

when $A \cong \{1 < |z| < r\}$.



Properties of extremal length and Kerckhoff's formula

- Kerckhoff showed that

$$\mathbb{R}_+ \otimes \mathcal{S} \ni t\alpha \mapsto \mathrm{Ext}_y(t\alpha) := t^2 \mathrm{Ext}_y(\alpha)$$

  extends continuously on $\mathcal{MF}$.

- The extremal length has the distortion property.

$$e^{-2d_T(y_1,y_2)}\mathrm{Ext}_{y_1}(F) \le \mathrm{Ext}_{y_2}(F) \le e^{2d_T(y_1,y_2)}\mathrm{Ext}_{y_1}(F)$$

  for $F \in \mathcal{MF}$ and $y_1, y_2 \in T_{g,m}$.

Theorem 1 (Kerckhoff's formula)

The Teichmüller distance is represented by the ratio of the extremal length:

$$d_T(y_1,y_2) = \frac{1}{2}\log\sup_{\alpha\in\mathcal{S}} \frac{\mathrm{Ext}_{y_1}(\alpha)}{\mathrm{Ext}_{y_2}(\alpha)}.$$

Gardiner-Masur compactification

Consider a mapping

$$\Phi_{GM} : T_{g,m} \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \mathrm{Ext}_y(\alpha)^{1/2}] \in P\mathbb{R}_+^{\mathcal{S}}.$$

Kerckhoff's formula asserts that Φ_{GM} is injective (due to Gardiner and Masur):
Suppose $\Phi_{GM}(y_1) = \Phi_{GM}(y_2)$. Then, there is a constant $c > 0$ such that $\mathrm{Ext}_{y_2}(\alpha) = c\mathrm{Ext}_{y_1}(\alpha)$ for all $\alpha \in \mathcal{S}$. Hence, we have

$$\begin{aligned}\frac{1}{2}\log c &= \frac{1}{2}\log \sup_{\alpha\in\mathcal{S}} \frac{\mathrm{Ext}_{y_2}(\alpha)}{\mathrm{Ext}_{y_1}(\alpha)} \\ &= d_T(y_2, y_1) = d_T(y_1, y_2) \\ &= \frac{1}{2}\log \sup_{\alpha\in\mathcal{S}} \frac{\mathrm{Ext}_{y_1}(\alpha)}{\mathrm{Ext}_{y_2}(\alpha)} = -\frac{1}{2}\log c,\end{aligned}$$

and hence $c = 1$. This means that $d_T(y_1, y_2) = 0$ and $y_1 = y_2$.

Gardiner-Masur compactification

Gardiner and Masur showed that the closure of the image

$$\mathrm{cl}_{GM}(T_{g,m}) := \overline{\Phi_{GM}(T_{g,m})} \subset P\mathbb{R}_+^{\mathcal{S}}$$

is compact. This compactification of $T_{g,m}$ is called the Gardiner-Masur compactification. We call the boundary

$$\partial_{GM} T_{g,m} = \mathrm{cl}_{GM}(T_{g,m}) - \Phi_{GM}(T_{g,m})$$

the Gardiner-Masur boundary.

The following is a fundamental observation due to Gardiner and Masur.

- $\mathcal{PMF} \subset \partial_{GM} T_{g,m}$ as subsets of $P\mathbb{R}_+^{\mathcal{S}}$. If $3g-3+m \geq 2$, $\mathcal{PMF}$ is a proper subset of $\partial_{GM} T_{g,m}$.

Extremal length geometry via intersection number

Consider cones

$$\begin{aligned} \mathcal{C}_{GM} &= \mathrm{proj}^{-1}(\mathrm{cl}_{GM}(T_{g,m})) \cup \{0\} \subset \mathbb{R}_+^{\mathcal{S}} \\ \mathcal{T}_{GM} &= \mathrm{proj}^{-1}(\Phi_{GM}(T_{g,m})) \cup \{0\} \subset \mathbb{R}_+^{\mathcal{S}} \\ \tilde{\partial}_{GM} &= \mathrm{proj}^{-1}(\partial_{GM} T_{g,m}) \cup \{0\} \subset \mathbb{R}_+^{\mathcal{S}} \end{aligned}$$

where $\mathrm{proj}\colon \mathbb{R}_+^{\mathcal{S}} - \{0\} \to P\mathbb{R}_+^{\mathcal{S}}$ is the projection. Notice that $\mathcal{MF} \subset \tilde{\partial}_{GM}$ since $\mathcal{PMF} \subset \partial_{GM} T_{g,m}$.

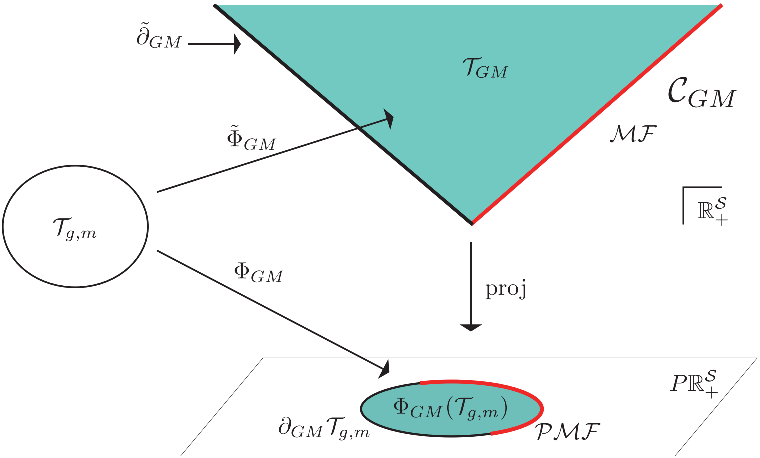
The Gardiner-Masur embedding

$$\Phi_{GM}\colon T_{g,m} \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \mathrm{Ext}_y(\alpha)^{1/2}] \in \Phi_{GM}(T_{g,m}) \subset P\mathbb{R}_+^{\mathcal{S}}$$

has a canonical lift

$$\tilde{\Phi}_{GM}\colon T_{g,m} \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \mathrm{Ext}_y(\alpha)^{1/2}] \in \mathcal{C}_{GM} \subset \mathbb{R}_+^{\mathcal{S}}.$$

Cones


$\tilde{\partial}_{GM}$
$\mathcal{T}_{GM}$
$\mathcal{C}_{GM}$
$\tilde{\Phi}_{GM}$
$\mathcal{MF}$
$\mathcal{T}_{g,m}$
$\mathbb{R}_+^{\mathcal{S}}$
Φ_{GM}
proj
$P\mathbb{R}_+^{\mathcal{S}}$
$\Phi_{GM}(\mathcal{T}_{g,m})$
$\partial_{GM}\mathcal{T}_{g,m}$
$\mathcal{PMF}$

Theorem 2 (Unification of extremal length geometry)

There is a unique continuous function

$$i(\,\cdot\,,\,\cdot\,)\colon \mathcal{C}_{GM}\times\mathcal{C}_{GM}\to\mathbb{R}_+$$

with the following properties.

(i) *For any* $y\in\mathcal{T}_{g,m}$ *and* $\alpha\in\mathcal{S}$,

$$i(\tilde{\Phi}_{GM}(y),\alpha)=\mathrm{Ext}_y(\alpha)^{1/2}\quad \text{for all } \alpha\in\mathcal{S}.$$

(ii) *For* $\mathfrak{a},\mathfrak{b}\in\mathcal{C}_{GM}$ *and* $t,s\geq 0$, $i(t\mathfrak{a},s\mathfrak{b})=ts\,i(\mathfrak{b},\mathfrak{a})$.

(iii) *For any* $y,z\in\mathcal{T}_{g,m}$,

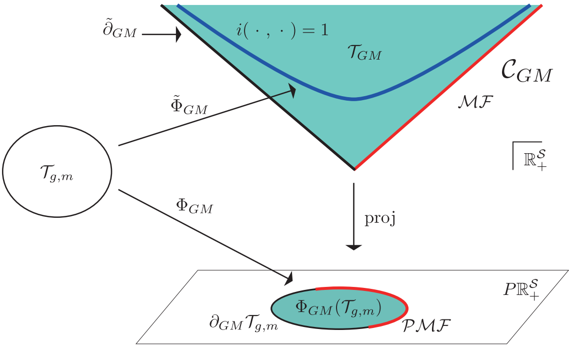
$$i(\tilde{\Phi}_{GM}(y),\tilde{\Phi}_{GM}(z))=\exp(d_T(y,z)).$$

In particular, we have $i(\tilde{\Phi}_{GM}(y),\tilde{\Phi}_{GM}(y))=1$ *for* $y\in\mathcal{T}_{g,m}$.

(iv) *For* $F,G\in\mathcal{MF}\subset\mathcal{C}_{GM}$, *the value* $i(F,G)$ *is equal to the (original) geometric intersection number* F *and* G

Cones and Hyperbolid model

$$\exists!\ i(\,\cdot\,,\,\cdot\,)\colon \mathcal{C}_{GM}\times\mathcal{C}_{GM}\to\mathbb{R}_+$$



From this picture, our hyperboloid model might be expected to be a kind of a counter part of the Bonahon's realization of Teichmüller space into the space of geodesic currents.

Fix $x_0 \in T_{g,m}$. We define a mapping

$$\tilde{\Psi}_{GM} : T_{g,m} \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \mathcal{E}_y(\alpha)] \in C_{GM} \subset \mathbb{R}_+^{\mathcal{S}}$$

where $K_y = \exp(2d_T(x_0, y))$ and $\mathcal{E}_y(F) = (\mathrm{Ext}_y(F)/K_y)^{1/2}$. By definition,

$$\mathrm{proj} \circ \tilde{\Psi}_{GM} = \Phi_{GM} \quad \text{on } T_{g,m}.$$

Theorem 3 (Embedding with basepoint)

The embedding $\tilde{\Psi}_{GM}$ extends homeomorphically to $\mathrm{cl}_{GM}(T_{g,m})$ onto its image.

Notice that

$$\begin{aligned} i(\tilde{\Psi}_{GM}(y_1), \tilde{\Psi}_{GM}(y_2)) &= i\left(K_{y_1}^{-1/2}\tilde{\Phi}_{GM}(y_1), K_{y_2}^{-1/2}\tilde{\Phi}_{GM}(y_2)\right) \\ &= \exp(-d_T(x_0, y_1) - d_T(x_0, y_2) + d_T(y_1, y_2)) \\ &= \exp(-2\langle y_1 \,|\, y_2\rangle_{x_0}). \end{aligned}$$

where

$$\langle y_1 \,|\, y_2\rangle_{x_0} = \frac{1}{2}(d_T(x_0, y_1) + d_T(x_0, y_2) - d_T(y_1, y_2))$$

is the Gromov product.

Since $\tilde{\Psi}_{GM}$ admits a continuous extension to $\mathrm{cl}_{GM}(T_{g,m})$, we have the following.

Corollary 4 (Extension of the Gromov product)

For any $x_0 \in T_{g,m}$, the Gromov product

$$\langle \cdot \mid \cdot \rangle_{x_0} : T_{g,m} \times T_{g,m} \to \mathbb{R}_+$$

extends continuously to $\mathrm{cl}_{GM}(T_{g,m}) \times \mathrm{cl}_{GM}(T_{g,m})$ with values in $\mathbb{R}_+ \cup \{\infty\}$.

Moreover, we can see that

$$\exp(-2\langle [F] \mid [G] \rangle_{x_0}) = \frac{i(F,G)}{\mathrm{Ext}_{x_0}(F)^{1/2}\mathrm{Ext}_{x_0}(G)^{1/2}}$$

where $[F],[G] \in \mathcal{PMF} \subset \partial_{GM} T_{g,m}$.

Sketch of the proof

A rough sketch of the proof is as follows:

(1) Construct systems of “nice neighborhoods” of points of $\mathcal{MF}$ to show that the family $\{\mathcal{E}_y\}_{y\in T_{g,m}}$ is equicontinuous on $\mathcal{MF}$, where we recall

$$\mathcal{E}_y(F) = \left\{\frac{\mathrm{Ext}_y(F)}{K_y}\right\}^{1/2} = e^{-d_T(x_0,y)}\mathrm{Ext}_y(F)^{1/2}.$$

(2) Then, the family $\{\mathcal{E}_y\}_{y\in T_{g,m}}$ is a normal family as a family of continuous functions on $\mathcal{MF}$. Hence, we have a family $\{\mathcal{E}_p \mid p \in \mathrm{cl}_{GM}(T_{g,m})\}$ of continuous functions.
SInce that family is a normal family, we can see that

$$\tilde{\Psi}_{GM}\colon \mathrm{cl}_{GM}(T_{g,m}) \ni p \mapsto [\mathcal{S} \ni \alpha \mapsto \mathcal{E}_p(\alpha)] \in C_{GM} \subset \mathbb{R}_+^{\mathcal{S}}$$

is continuous, which implies Theorem 3.

Notice that $\mathrm{proj}\circ\tilde{\Psi}_{GM} = \Phi_{GM}\colon \mathrm{cl}_{GM}(T_{g,m}) \to \mathrm{cl}_{GM}(T_{g,m})$ is a homeomorphism. Hence, any element in C_{GM} is written as $t\cdot\tilde{\Psi}_{GM}(p)$ with $t\geq 0$ and $p\in \mathrm{cl}_{GM}(T_{g,m})$.

(3) We define the intersection number on $C_{GM}\times\mathcal{MF}$ by

$$i(t\cdot\tilde{\Psi}_{GM}(p),F) = t\mathcal{E}_p(F).$$

We can see that the intersection number coincides with the original intersection number on $\mathcal{MF}\times\mathcal{MF}$.

(4) We define the (new) extremal length of $\mathfrak{a}\in C_{GM}$ on $\mathfrak{y}\in\mathcal{T}_{GM}$ by

$$\mathrm{Ext}_{\mathfrak{y}}(\mathfrak{a}) = t^2 \sup_{F\in\mathcal{MF}} \frac{i(\mathfrak{a},F)^2}{\mathrm{Ext}_y(F)}$$

where $\mathfrak{y} = t\tilde{\Psi}_{GM}(y)$ for $t\geq 0$ and $y\in T_{g,m}$. Notice from Minsky's inequality and Gardiner-Masur's work, when $\mathfrak{a}\in\mathcal{MF}$, the above extremal length coincides with the original extremal length.

(5) We re-define intersection number on $\mathcal{T}_{GM} \times \mathcal{C}_{GM}$ and a function on $\mathcal{C}_{GM}$ by

$$i(\mathfrak{y}, \mathfrak{a}) = \mathcal{E}_{\mathfrak{y}}(\mathfrak{a}) = \left\{ \frac{\mathrm{Ext}_{\mathfrak{y}}(\mathfrak{a})}{K_y} \right\}^{1/2}$$

where $\mathfrak{y} = t\tilde{\Psi}_{GM}(y)$ ($t \geq 0$ and $y \in T_{g,m}$).

(6) We can check that

$$i_{x_0}(y, z) := i(\tilde{\Psi}_{GM}(y), \tilde{\Psi}_{GM}(z)) = \exp(-2\langle y \,|\, z \rangle_{x_0})$$

for $y, z \in T_{g,m}$, which implies our intersection number on $\mathcal{T}_{GM} \times \mathcal{T}_{GM}$ is symmetric.

(7) As before, we construct 'nice neighborhoods" for points in $\mathcal{C}_{GM}$ to show that for any fixed $R > 0$, the family

$$\{\mathcal{E}_{\mathfrak{y}} \mid \mathfrak{y} = t\tilde{\Psi}_{GM}(y) \text{ for } y \in T_{g,m},\ 0 \leq t \leq R\}$$

is a normal family as a family of continuous functions on $\mathcal{C}_{GM}$, and we can see that the intersection number defined in (5) extends on $\mathcal{C}_{GM} \times \mathcal{C}_{GM}$ with desired properties.

Analytic aspect: Insufficiency from composition

We first recall an elementary property of quasiconformal mappings.

Let $f\colon D_0 \to D_1$ and $g\colon D_0 \to D_2$ be quasiconformal mappings. It is known that

$$K(g \circ f^{-1}) \leq K(g)K(f)$$

holds. Then, one may ask

Question 2

How can we measure the difference between $K(g \circ f^{-1})$ and $K(g)K(f)$?

The equality does not hold in general. For instance, we know a trivial (and not interesting) example

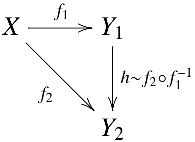
$$1 = K(f \circ f^{-1}) < K(f)K(f^{-1}) = K(f)^2.$$

for any quasiconformal mapping $f\colon D_0 \to D_1$ with $K(f) > 1$.

Recall that the Teichmüller distance is given by

$$d_T(y_1, y_2) = \frac{1}{2} \log \inf\{K(h) \mid h \text{ qc homotipic to } f_2 \circ f_1^{-1}\}$$

where $y_1 = (Y_1, f_1)$ and $y_2 = (Y_2, f_2)$.



Let $f : X \to Y$ be an orientation preserving homeomorphism. Set

$$K^*(f) = \inf\{K(h) \mid h \text{ qc homotipic to } f\}.$$

Then, for $y = (Y, f)$ and $z = (Z, g)$,

$$d_T(x_0, y) = \frac{1}{2} \log K^*(f)$$
$$d_T(y, z) = \frac{1}{2} \log K^*(g \circ f^{-1})$$

where $x_0 = (X, id)$ is the base point.

Since our intersection number satisfies

$$\begin{aligned} i_{x_0}(y,z) &= i(\tilde{\Psi}_{GM}(y), \tilde{\Psi}_{GM}(z)) = \exp(-2\langle y \,|\, z\rangle_{x_0}) \\ &= \exp(-d_T(x_0,y) - d_T(x_0,z) + d_T(y,z)) \\ &= \left\{ \frac{K^*(g\circ f^{-1})}{K^*(g)K^*(f)} \right\}^{1/2}, \end{aligned}$$

we conclude

$$K^*(g\circ f^{-1}) = i_{x_0}(y,z)^2 \cdot K^*(g)K^*(f)$$

Observation 1

For $y_1 = (Y,f), z = (Z,g) \in T_{g,m}$, the intersection number (with base point x_0) satisfies

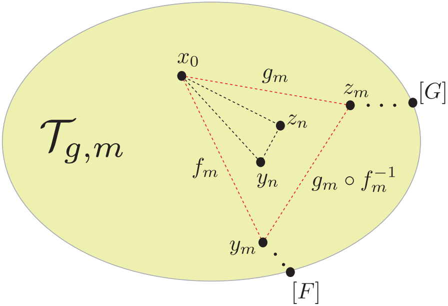
$$\exp(-4\langle y \,|\, z\rangle_{x_0}) = i_{x_0}(y,z)^2 = \frac{K^*(g\circ f^{-1})}{K^*(g)K^*(f)}.$$

Thus, our intersection number on $T_{g,m}$ is noting but the insufficiency or lack arising from the composition.

Asymptotic behavior of insufficiency

For instance, for $[F], [G] \in \mathcal{PMF} \subset \partial_{GM} T_{g,m}$, when $y_n = (Y_n, f_n) \to [F]$ and $z_n = (Z_n, g_n) \to [G]$, the insufficiency behaves eventually as

$$\frac{K^*(g_n \circ f_n^{-1})}{K^*(g_n)K^*(f_n)} = i_{x_0}(y_n, z_n)^2 \to \frac{i(F,G)^2}{\mathrm{Ext}_{x_0}(F)\mathrm{Ext}_{x_0}(G)}.$$



We ask

Problem 1

Give an "effective" result for the insufficiency. For instance, can we estimate the difference

$$\left|\frac{K^*(g_n \circ f_n^{-1})}{K^*(g_n)K^*(f_n)} - \frac{i(F,G)^2}{\mathrm{Ext}_{x_0}(F)\mathrm{Ext}_{x_0}(G)}\right|$$

or the ratio

$$\left\{\frac{K^*(g_n \circ f_n^{-1})}{K^*(g_n)K^*(f_n)}\right\}^{-1} \frac{i(F,G)^2}{\mathrm{Ext}_{x_0}(F)\mathrm{Ext}_{x_0}(G)}$$

in terms of the distances $d_T(x_0, y_n)$ and $d_T(x_0, z_n)$ in the above example?

Asymptotically conservative mappings

As we have seen, the Gromov product gives a strong connection between

- An analytic aspect : Insufficiency of the maximal dilation of the composition
- A topological aspect : Thurston theory (Geometry on simple closed curves with the intersection number).

on Teichmüller space at infinity.

One may ask

Question 3

What can we know the large scale geometry of Teichmüller space by use of the Gromov product?

We say that a sequence $\mathbf{x} = \{x_n\}_{n=1}^{\infty} \subset \mathcal{T}_{g,m}$ is convergent at infinity if

$$\langle x_n \mid x_m \rangle_{x_0} \to \infty \quad (n, m \to \infty).$$

Two sequences convergent at infinity $\mathbf{x} = \{x_n\}_{n=1}^{\infty}, \mathbf{y} = \{y_n\}_{n=1}^{\infty} \subset \mathcal{T}_{g,m}$ are said to be visually indistinguishable if

$$\langle x_n \mid y_m \rangle_{x_0} \to \infty \quad (n, m \to \infty).$$

Remark 1

On any Gromov hyperbolic space, the relation "visually indistinguishable" is an equivalence relation on the set of sequences convergent at infinity. In fact, the equivalence class is recognized as an ideal boundary point (i.e. a point in the Gromov boundary).

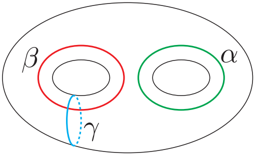
Remark 2 (Answer to Shiga's question at Chiba)

On the other hand, the relation "visually indistinguishable" is NOT an equivalence relation on the set of sequences convergent at infinity on Teichmüller space of complex dimension ≥ 2.

Let $\alpha, \beta, \gamma \in \mathcal{S}$ with $i(\alpha, \beta) = i(\alpha, \gamma) = 0$, but $i(\beta, \gamma) \neq 0$.

Consider sequences $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$, $\mathbf{y} = \{y_n\}_{n\in\mathbb{N}}$ and $\mathbf{z} = \{z_n\}_{n\in\mathbb{N}}$ in $T_{g,m}$ with $x_n \to [\alpha]$, $y_n \to [\beta]$ and $z_n \to [\gamma]$ in $\mathrm{cl}_{GM}(T_{g,m})$. Then,

$$\begin{aligned}
\exp(-2\langle x_n \,|\, y_m\rangle_{x_0}) &= i_{x_0}(x_n, y_m) \to i_{x_0}([\alpha], [\beta]) = 0, \text{ and} \\
\exp(-2\langle x_n \,|\, z_m\rangle_{x_0}) &= i_{x_0}(x_n, z_m) \to i_{x_0}([\alpha], [\gamma]) = 0, \text{ but} \\
\exp(-2\langle y_n \,|\, z_m\rangle_{x_0}) &= i_{x_0}(y_n, z_m) \to i_{x_0}([\beta], [\gamma]) \neq 0.
\end{aligned}$$



In particular, the Teichmüller space is not Gromov-hyperbolic, which was already shown by Masur-Wolf, McCarthy-Papadopoulos, Ivanov.

Let $\mathbf{x} = \{x_n\}_{n=1}^{\infty}$ be a sequence convergent at infinity. Let

$$\mathrm{Vis}(\mathbf{x}) = \{\text{visually indistinguishable sequences from } \mathbf{x}\}$$

Remark 3

For $\mathbf{z} \in \mathrm{Vis}(\mathbf{x})$, any accumulation points p and q of $\mathbf{x}$ and $\mathbf{z}$ in $\partial_{GM} T_{g,m}$ satisfies

$$i_{x_0}(p, q) = 0$$

Conversely, if $\mathbf{z}$ converges to $q \in \partial_{GM} T_{g,m}$ and $i_{x_0}(p, q) = 0$ for all accumulation point p of $\mathbf{x}$, then $\mathbf{z} \in \mathrm{Vis}(\mathbf{x})$.

Remark 4

Obviously, $\mathrm{Vis}(\mathbf{x})$ is defined for a sequence $\mathbf{x}$ convergent at infinity in an arbitrary metric space. In the case of Gromov hyperbolic spaces, $\mathrm{Vis}(\mathbf{x})$ corresponds to a point in the Gromov boundary.

Asymptotically conservative

We will define mappings in an effort to produce a qualitative version of quasi-isometries with caring about the asymptotic behavior of the Gromov product in hyperbolic spaces.

Definition 5 (Asymptotically conservative)

A mapping $\omega\colon T_{g,m} \to T_{g,m}$ is said to be asymptotically conservative if for two sequences $\mathbf{x}$ and $\mathbf{y}$ convergent at infinity, $\mathbf{x}$ and $\mathbf{y}$ are visually indistinguishable if and only if so are $\omega(\mathbf{x})$ and $\omega(\mathbf{y})$.

Remark 5

For two asymptotically conservative mapping ω_1 and ω_2, the composition $\omega_1 \circ \omega_2$ is also asymptotically conservative.

Remark 6

We can also define asymptotically conservative mappings between two metric spaces. In this case, any quasi-isometry between Gromov hyperbolic spaces is asymptotically conservative.

Closeness at infinity and Invertivility

Definition 6 (Close at infinity)

Two asymptotically conservative mappings $\omega_1, \omega_2 : T_{g,m} \to T_{g,m}$ are said to be close at infinity if for two sequences $\mathbf{x}$ and $\mathbf{y}$ convergent at infinity, if Vis$(\mathbf{x})$ = Vis$(\mathbf{y})$, then Vis$(\omega_1(\mathbf{x}))$ = Vis$(\omega_2(\mathbf{y}))$.

Definition 7 (Invertiblity)

An asymptotically conservative mapping $\omega : T_{g,m} \to T_{g,m}$ is called invertible if there is an asymptotically conservative mapping ω' such that both of $\omega \circ \omega'$ and $\omega' \circ \omega$ is close to the identity at infinity.

Let

$$\mathrm{AC}_{\mathrm{inv}}(T_{g,m}) = \{\text{Invertible asymptotically conservative mappings on } T_{g,m}\}.$$

Large scale geometry with the Gromov product

We can see the following.

Theorem 8

The relation "closeness at infinity" is an equivalence relation on $\mathrm{AC}_{\mathrm{inv}}(T_{g,m})$. Furthermore, the relation is a subgroup congruence and the quotient semigroup $\mathfrak{AC}(T_{g,m})$ is a group.

Notice that there is a sequence of monoid homomorphisms

$$\mathrm{Isom}(T_{g,m}) \xrightarrow{\text{inclusion}} \mathrm{AC}_{\mathrm{inv}}(T_{g,m}) \xrightarrow{\text{proj}} \mathfrak{AC}(T_{g,m})$$

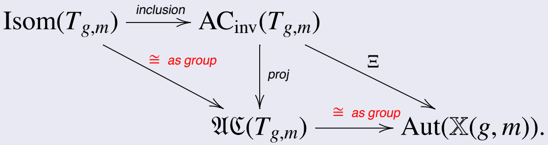
Let $\mathbb{X}(g,m)$ be the complex of curves of the surface of type (g,m). Let $\mathrm{Aut}(\mathbb{X}(g,m))$ be the group of simplicial automorphisms on $\mathbb{X}(g,m)$.

Theorem 9 (Rigidity)

When $3g-3+m \geq 2$*, there is a monoid homomorphism*

$$\Xi\colon \mathrm{AC}_{\mathrm{inv}}(T_{g,m}) \to \mathrm{Aut}(\mathbb{X}(g,m))$$

which induces the following commutative diagram



Namely, any invertible asymptotically conservative mapping is close at infinity to some isometry on $T_{g,m}$.
Hence, we can consider invertible asymptotically conservative mappings as “coarsifications” of isometries on $T_{g,m}$.

No rough homothety with $K \neq 1$

A mapping $\omega\colon T_{g,m} \to T_{g,m}$ is said to be (K, D)-rough homothety if

$$|d_Y(\omega(x), \omega(y)) - K \cdot d_T(x, y)| \leq D \quad (x, y \in T_{g,m}).$$

One can see that any (K, D)-rough homothety is asymptotically conservative.
We also obtain the following hyperbolic characteristic on $T_{g,m}$.

Theorem 10 (No rough homothety with $K \neq 1$)

There is no invertible (K, D)-rough homothety with $K \neq 1$ on $T_{g,m}$.

Namely, Teichmüller space does not admit non-trivial similarity.

Story of the proof of Theorem 9

Let $\omega \in \mathrm{AC}_{\mathrm{inv}}(T_{g,m})$.

1. For $p \in \partial_{GM} T_{g,m}$, let
$$\mathcal{N}(p) = \{q \in \partial_{GM} T_{g,m} \mid i_{x_0}(q, p) = 0\}.$$
2. Let $p_1, p_2 \in \partial_{GM} T_{g,m}$. Let $\mathbf{x}^1 = \{x_n^1\}_{n=1}^{\infty}$ and $\mathbf{x}^2 = \{x_n^2\}_{n=1}^{\infty}$ be sequences in $T_{g,m}$ converging p_1 and p_2 respectively. Let q_i be an accumulation point of $\omega(\mathbf{x}^i)$ for $i = 1, 2$. We can see that if $\mathcal{N}(p_1) \subset \mathcal{N}(p_2)$, then $\mathcal{N}(q_1) \subset \mathcal{N}(q_2)$.
3. We consider a sequence $\{p_1, p_2, \cdots, p_s\}$ with
$$\mathcal{N}(p_1) \supsetneq \mathcal{N}(p_2) \supsetneq \cdots \supsetneq \mathcal{N}(p_s).$$
Then, we can see that p_1 is the initial point of such sequence of length $3g - 3 + m$ if and only if the initial point p_1 is the projective class of a simple closed curve.

4. Then, since ω is invertible, ω "preserves" the length of the sequence and hence we can see that when a sequence $\mathbf{x} = \{x_n\}_{n=1}^{\infty}$ converges to the projective class of a simple closed curve $[\alpha]$, then $\omega(\mathbf{x})$ also converges to the projective class of a simple closed curve $[\beta]$.
5. Then, we can also see that the mapping $\alpha \mapsto \beta$ induces a simplicial automorphism of $\mathbb{X}(g, m)$. Hence, we can find the corresponding isometry ξ_ω on $T_{g,m}$ by Ivanov-Korkmaz-Luo's theorem.
6. Then, by some technical argument, we can see that ω is close to ξ_ω at infinity.

No rough homothety with $K \neq 1$

We may assume that X is not a torus with two holes. Then, ξ_ω is induced by a homeomorphism f_ω on X. Suppose to the contrary that there is an invertible (K, D)-rough homothety ω for some $K \neq 1$.
Then, we have

$$e^{-D_0} i_{x_0}([\alpha], [\beta])^K \leq i_{x_0}([f_\omega(\alpha)], [f_\omega(\beta)]) \leq e^{D_0} i_{x_0}([\alpha], [\beta])^K \tag{1}$$

where D_0 is a constant depending only on D and $d_T(x_0, \omega(x_0))$. On the other hand, let $K_0 = e^{2d_T(x_0, \xi_\omega(x_0))}$. By the quasiconformal invariance of extremal length, we obtain

$$K_0^{-1} i_{x_0}([\alpha], [\beta]) \leq i_{x_0}([f_\omega(\alpha)], [f_\omega(\beta)]) \leq K_0 i_{x_0}([\alpha], [\beta])$$

since f_ω is a homeomorphism on X and $i(f_\omega(\alpha), f_\omega(\beta)) = i(\alpha, \beta)$.
Therefore, we deduce

$$i_{x_0}([\alpha], [\beta])^{1-K} \leq K_0 e^{D_0} \tag{2}$$

$$i_{x_0}([\alpha], [\beta])^{K-1} \leq K_0 e^{D_0} \tag{3}$$

when $i(\alpha, \beta) \neq 0$. Then if $K \neq 1$, we get a contradiction.

Thank you very much for your attention. (^o^)

志賀先生，還暦おめでとうございます.