Universal Holomorphic Motions in Motion Theory

Yunping Jiang

City University of New York

(A talk given in "Analysis and Geometry of Riemann Surfaces and Related Topics" in celebration of Professor Shiga's 60th birthday, Tokyo, Japan.

June 21-23, 2013

X a (path) connected Hausdorff space

Y a Riemann surface

V a simply connected complex Banach manifold

W a connected complex Banach manifold

Δ the open unit disk

$\hat{\mathbb{C}}$ the Riemann sphere

$E \subseteq Y$ a subset

$x_0 \in X$ or W or V a basepoint

Definition 1. A motion $\varphi : X \times E \to Y$ of E over X is a map satisfying

(i) $\varphi(x_0, z) = z$, $\forall z \in E$;

(ii) $\forall x \in X$, $\varphi_x(z) = \varphi(x, z) : E \to Y$ is injective.

when $Y = \hat{\mathbb{C}}$, we call a motion a normalized motion if $0, 1, \infty \in E$, and $\varphi(x, 0) = 0$, $\varphi(x, 1) = 1$, $\varphi(x, \infty) = \infty$, $\forall x \in X$.

Definition 2. A motion $\varphi : X \times E \to Y$ is called a continuous motion if (i) +

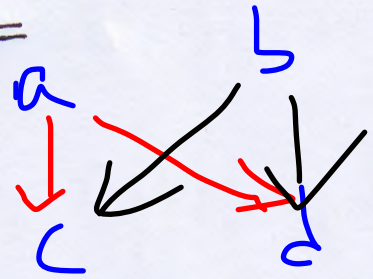
(ii)' $\forall x \in X$, $\varphi_x(z) = \varphi(x,z) : E \to \varphi_x(E)$ is a homeomorphism.

(iii) $\varphi : X \times E \to Y$ is continuous.

$Q = \{a, b, c, d\}$ is a quadruplet in $\hat{\mathbb{C}}$. We use

$$\varphi_y(Q) = cr(\varphi_y(a), \varphi_y(b), \varphi_y(c), \varphi_y(d)) =$$

$$= \frac{(\varphi_y(a) - \varphi_y(c))}{(\varphi_y(a) - \varphi_y(d))} \frac{(\varphi_y(b) - \varphi_y(d))}{(\varphi_y(b) - \varphi_y(c))}$$



Let $\rho(\cdot,\cdot)$ be the hyperbolic metric on $\mathbb{C}_{0,1} = \mathbb{C} - \{0, 1\}$

Definition 3. A normalized motion $\varphi : X \times E \to \hat{\mathbb{C}}$ is called a quasiconformal motion if (i) +

(iv) $\forall x \in X$, $\forall \varepsilon > 0$, $\exists$ a neighborhood U_x of x in X such that $\forall Q = \{a, b, c, d\} \subset E$

$$\rho(\varphi_y(Q), \varphi_{y'}(Q)) < \varepsilon, \quad \forall y, y' \in U_x .$$

Definition 4. A motion $\varphi : W \times E \to Y$ is called a holomorphic motion if (i) + (v) $\forall z \in E$, $\varphi_z(x) = \varphi(x, z) : W \to Y$ is holomorphic.

Definition 4 $\Rightarrow$ Definition 3 $\Rightarrow$ Definition 2.

Let $M(\mathbb{C})$ be the unit ball of $\mathcal{L}^\infty(\mathbb{C})$. Let $T(E) = \{ [\mu]_E \mid \mu \in M(\mathbb{C}) \}$ be the space of all equivalence classes of μ, where $\mu \sim \nu$ if

$$(w^\nu)^{-1} \circ w^\mu \sim id \text{ rel } E$$

$T(E)$ is called the Teichmüller space of the set E and is a simply connected complex Banach manifold

$P_E : M(\mathbb{C}) \to T(E)$; $P_E(\mu) = [\mu]_E$ is a holomorphic split submersion.

Definition 5. $\Phi_E : T(E) \times E \to \hat{\mathbb{C}}$; $\Phi_E([\mu]_E, z) = w^\mu(z)$ is called a universal holomorphic motion.

The reason we call it a universal holomorphic motion is that

Theorem 1 (Mitra, 2000, J d'Analyse)

If $\varphi : V \times E \to \hat{\mathbb{C}}$ is a holomorphic motion, then $\exists$! holomorphic basepoint preserving map $f : V \to T(E)$ such that $f^*(\Phi_E) = \varphi$.

Previously, this statement has been also proved in

Bers-Royden, 1986, Acta math, $V = \Delta$, E = a finite set

Lieb, 1990, Thesis, Cornell Univ. $V = \Delta$, E = any set.

see also J-Mitra, 2012, contemp. math. Theorem 4.2.

Theorem 2 (Mitra, 2007, Isreal J., J-Mitra, 2012 Contemp. Math. Theorems 10)

Suppose $\varphi : W \times E \to \hat{\mathbb{C}}$ is a holomorphic motion. Then the following are equivalent:

(1) $\exists$ a continuous motion $\mathring{\varphi} : W \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ s.t. $\mathring{\varphi}$ extends φ.

(2) $\exists$ a quasiconformal motion $\hat{\varphi} : W \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ s.t. $\hat{\varphi}$ extends φ.

(3) $\exists$ a basepoint preserving holomorphic map $f : V \to T(E)$ s.t. $f^*(\Phi_E) = \varphi$.

Our original purpose was to prove Theorem 2 as well as Theorem 1 under the assumption that $\varphi : X \times E \to \hat{\mathbb{C}}$ is a quasiconformal motion. However, we finally got a very unexpect outcome by first disproving a theorem in Sullivan–Thurston's 1986 Acta math paper and developing a full theory for local quasiconformal motions.

Before presenting our work, we first see some applications and implementations of universal holomorphic motions.

Theorem 3 (Earle, 1969, Duke Math J., Hubbard, 1976. memo AMS, J-Mitra, 2006, Kodai Math)

$\bar{\Phi}_E : T(E) \times E$ can not be extended to a holomorphic motion of $\hat{\mathbb{C}}$ over $T(E)$ as long as $\#(E) \geq 5$.

Note that from Deaudy-Earle section $S : T(E) \to M(\mathbb{C})$

$\bar{\Phi}_E : T(E) \times E \to \hat{\mathbb{C}}$ can be extended to a quasiconformal motion $\hat{\varphi} : T(E) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, refer to J-Mitra, 2012 Contemp Math. Theorem 6.4.

Theorem 4 (Slodkowski, 1991, Proc AMS)

Every holomorphic motion $\varphi: \Delta \times E \to \hat{\mathbb{C}}$ can be extended to a holomorphic motion $\tilde{\varphi}: \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Chirka, 2004, Doklady math ✓

also also Gardiner-J-Wang, 2010, London Math for a complete survey plus a new result about infinitesimal versions of holomorphic motions.

By using the universal holomorphic motion, Theorem 4 is equivalent to

Theorem 5 (Lifting, J-Mitra-Wang, 2009, Kodai Math J)

For any holomorphic map $f: \Delta \to T(E)$, $f(0) = [0]_E$, $\exists$ a holomorphic $\tilde{f}: \Delta \to M(\mathbb{C})$ s.t.

$$\begin{array}{ccc} & \tilde{f} \nearrow & M(\mathbb{C}) \\ & & \downarrow P_E \\ \Delta & \xrightarrow{f} & T(E) \end{array}$$ commutes.

Definition 6 (Guiding Isotopy)

Suppose $\varphi: X \times E \to Y$ is a continuous motion.

A guiding isotopy for φ is a continuous motion $\dot{\varphi}: X \times Y \to Y$ which extends φ. It is called a quasiconformal guiding isotopy and denote as $\hat{\varphi}: X \times Y \to Y$ if $\forall y \in X$, $\hat{\varphi}_y(z) = \hat{\varphi}(y, z): Y \to Y$ is a quasiconformal homeomorphism. whose M_y depends on y continuous.

1) Note that when $Y = \hat{\mathbb{C}}$ and φ is a normalized quasiconformal motion, then a quasiconformal guiding isotopy is a quasiconformal motion extension of φ.

2) If a quasiconformal guiding isotopy exists, then it is unique up to Botopy rel E.

Theorem 6 (Canonical Replacement, Gardiner-J, 2013)

Suppose X is a hyperbolic Riemann surface. Suppose $3 < \#(E) < \infty$. Let $\varphi : X \times E \to Y$ is a holomorphic motion. Suppose we have a quasiconformal guiding isotopy $\hat{\varphi} : X \times Y \to Y$. Then we can replace $\hat{p}(x) = \hat{\varphi}(x, p) : X \to Y$ for any $p \in Y|E$ by a holomorphic map $\tilde{p}(x) : X \to Y$ s.t. $\tilde{p}(x) \notin Y|E_x$, $E_x = \varphi(x, E)$. Thus

$$\tilde{\varphi}(x, z) = \begin{cases} \varphi(x, z), & z \in E, \ x \in X \\ \tilde{p}(x), & z = p, \ x \in X \end{cases}$$

is a holomorphic motion which extends φ.

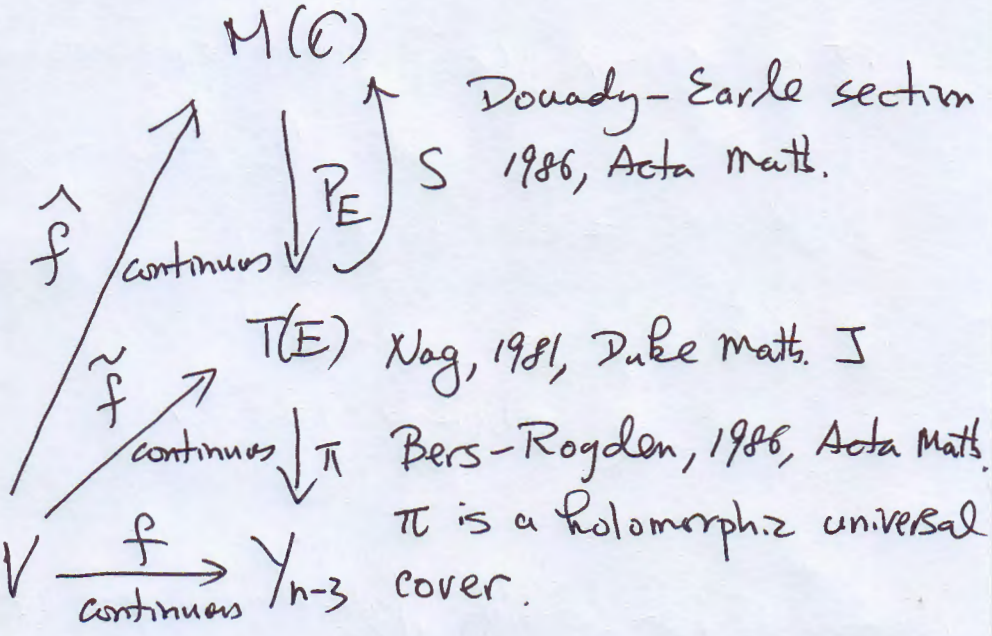
Ingredient in the proof: Bers' embedding, Ahlfors-Weill extension, Heights Mapping Theorem, Quadratic Differential with double poles.

Theorem 6 implies Slodkowski's theorem (Theorem 4) because any holomorphic motion of a finite subset $E \subseteq \hat{\mathbb{C}}$ over Δ can be extended to a quasiconformal motion of $\hat{\mathbb{C}}$ over Δ, refer to Bers-Royden, 1986 Sullivan-Thurston, 1986 Acta Matts, Gardiner-J, 2013. More general, we have

Theorem 7 (J-Mitra-Shiga-Wang, 2013)

If $3 < \#(E) < \infty$, then any normalized quasiconformal motion $\varphi: V \times E \to \hat{\mathbb{C}}$ can be extended to a normalized quasiconformal motion $\hat{\varphi}: V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Proof. $Y_{n-3} = \{ (z_1, \cdots, z_{n-3}) \in \mathbb{C}^{n-3} \mid z_i \neq z_j,\ z_i \neq 0, 1 \}$



$\hat{\varphi} = \hat{f}^{\sharp}(\bar{\Phi}_{\hat{\mathbb{C}}})$ extends φ and is a normalized quasiconformal motion.

What is happening if $\#(E) = \infty$?

Theorem 8 (J-Mitra-Shiga-Wang, 2015)

$\exists$ a closed set $E \subseteq \hat{\mathbb{C}}$, $\#(E) = \infty$ and a normalized quasiconformal motion $\varphi: I \times E \to \hat{\mathbb{C}}$, $I = (-1, 1)$, s.t.

(i) $\exists$ a continuous motion $\hat{\varphi}: I \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which extends φ.

(ii) φ can not be extended to any quasiconformal motion of $\hat{\mathbb{C}}$ over ~~I~~ any nbhd U of 0.

$\Rightarrow$ $\Phi_E: T(E) \times E \to \hat{\mathbb{C}}$ is NOT a universal quasiconformal motion.

Theorem 2 is invalid under the assumption that $\varphi: X \times E \to \hat{\mathbb{C}}$ is a quasiconformal motion. Actually, (i) $\not\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

This also disproved a statement in Sullivan-Thurston, 1986, Acta Math:

Theorem: Any quasiconformal motion $\varphi: I \times E \to \hat{\mathbb{C}}$ can be extended to a quasiconformal motion $\hat{\varphi}: I \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Definition 7 (local quasiconformal motions)

A motion $\varphi: X \times E \to \hat{\mathbb{C}}$ is called a local quasiconformal motion if $\forall x \in X$, $\exists$ a neighborhood U_x of x in X and a continuous map $y \to \mu_y : U_x \to M(\mathbb{C})$ s.t. $\varphi_y(z) = \varphi(y, z) = w^{\mu_y}(z)$, $\forall z \in E$, $y \in U_x$.

Theorem 9 (J-Mitra-Shiga-Wang, 2013)

Any local quasiconformal motion $\varphi: X \times E \to \hat{\mathbb{C}}$, where X is a simply connected Hausdorff space, can be extended to a quasiconformal motion $\hat{\varphi}: X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Theorem 10 (J-Mitra-Shiga-Wang, 2013)

Let $\varphi: X \times E \to \hat{\mathbb{C}}$ be a local quasiconformal motion. Then the following are equivalent:

(i) $\exists$ a continuous motion $\tilde{\varphi}: X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which extends φ

(ii) $\exists$ a quasiconformal motion $\hat{\varphi}: X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which extends φ

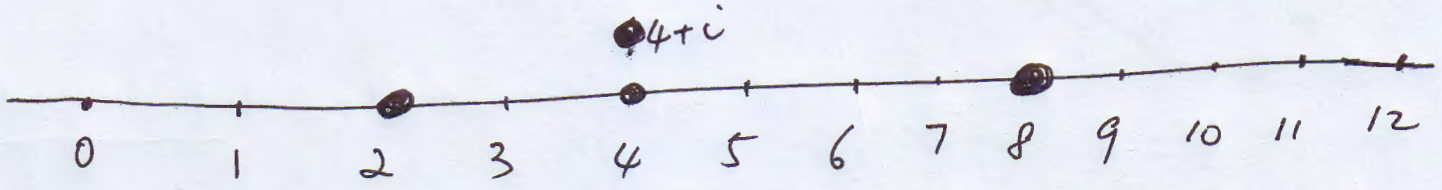
(iii) $\exists!$ a basepoint preserving continuous map $f: X \to T(E)$ s.t. $f^*(\Phi_E) = \varphi$.

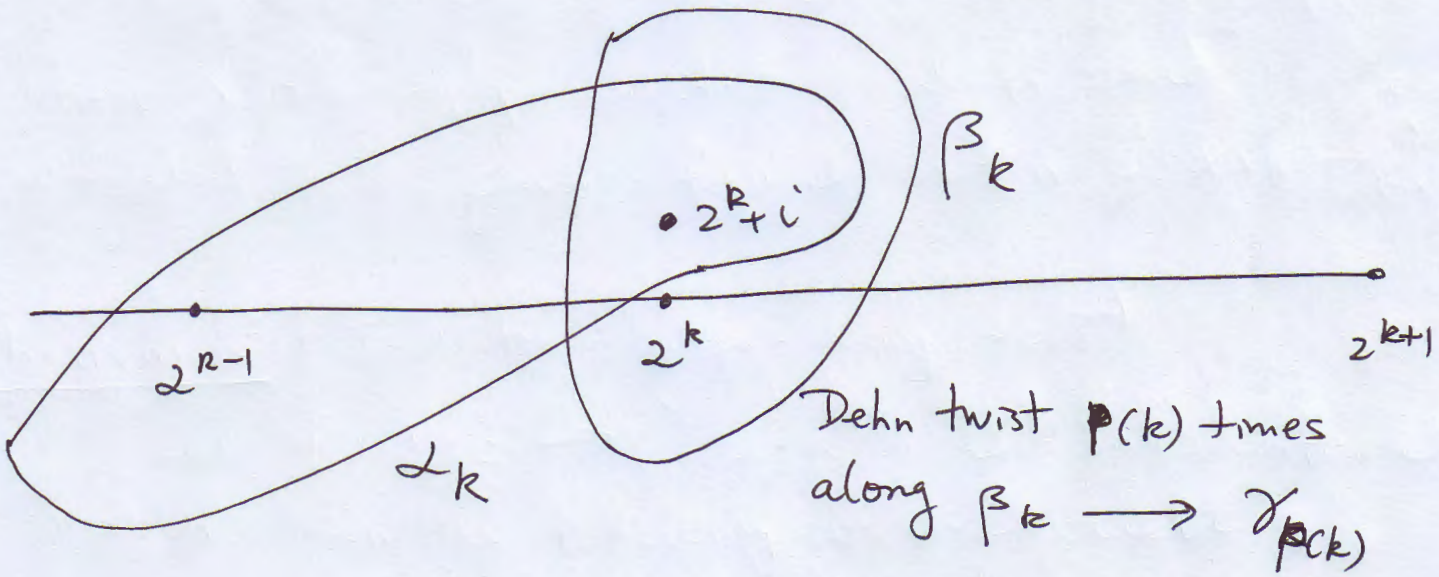
Outline of the proof of Theorem 8 :

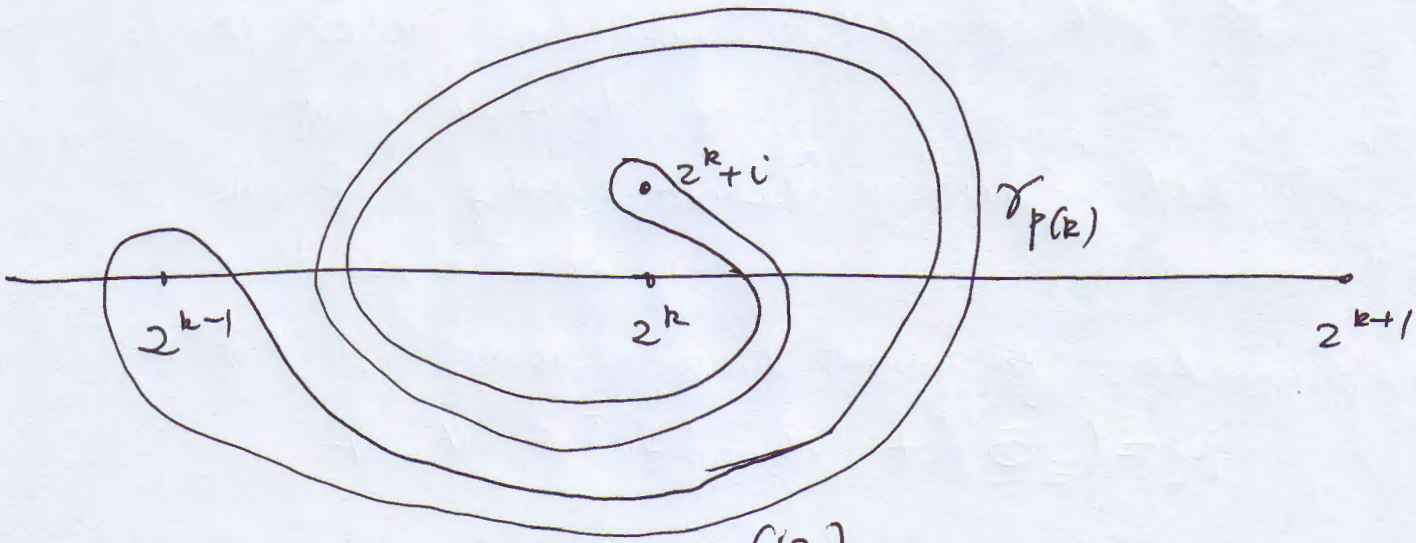
$I = (-1, 1)$

Take $T_k = \{ 2^{k-1}, 2^k, 2^k + i, 2^{k+1} \}$, $k \geq 2$.

$E = \bigcup_{k \geq 2} T_k \cup \{ 0, 1, \infty \}$







Here $\dfrac{\ell(\gamma_{p(k)})}{\ell(\alpha_k)} \to \infty$ as $k \to \infty$, where

$\ell(c)$ be the hyperbolic length of a unique geodesic in $\hat{\mathbb{C}} \setminus E \sim c$.

Define $\varphi_k(t, 2^k+i) = e^{2\pi i p(k) k^{(k+1)} (t - \frac{1}{k+1})} + 2^k$

$t \in [\frac{1}{k+1}, \frac{1}{k}]$

$\varphi_k(t, z) = z, \quad t \in [\frac{1}{k+1}, \frac{1}{k}], \quad z \neq 2^k + i \in E.$

$t \in I \setminus [\frac{1}{k+1}, \frac{1}{k}], \quad \forall z \in E.$

Extend φ_k to a continuous map $\mathring{\varphi}_k$ which is φ_k on $I \times E$ and Id on $I \times (\hat{\mathbb{C}} \setminus D_{3/2}(2^k))$

$$\mathring{\varphi}(t, z) = \lim_{n \to \infty} \varphi_n \circ \cdots \circ \varphi_1(t, z). \quad (t, z) \in I \times \hat{\mathbb{C}}.$$

It is clear $\mathring{\varphi}$ is continuous at (t, z), $z \neq \infty$.

But it is also continuous at (t, ∞) since

$\operatorname{diam}_{sp}(D_{3/2}(2^k)) \to 0$ as $k \to \infty$ and $\mathring{\varphi}(t, \infty) = \infty$.

Let $E_t = \varphi(t, E)$, $\exists\, K(r) \to 1$ as $r \to 0$ and a quasiconformal map $w_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ s.t. $w_t(E) = E_t$.

$K(w_t) \le K(r) \Rightarrow \varphi$ is a quasiconformal motion.

φ can not be extended to a quasiconformal motion $\hat{\varphi} : U \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ for any neighborhood U of 0.

Suppose we have a quasiconformal motion extension $\hat{\varphi} : U \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Suppose $\exists\, k \ge 1$ s.t. all

$$\hat{\varphi}_t(z) = \varphi(t, z) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

are k-quasiconformal for all $t \in U$.

Let $\tilde{\gamma}_k \subset \hat{\mathbb{C}} \setminus E$ s.t. $\hat{\varphi}_{\frac{1}{k-1}}(\tilde{\gamma}_k) = \alpha_k$

$$\Rightarrow \quad \hat{\varphi}_{\frac{1}{k}} \circ \hat{\varphi}^{-1}_{\frac{1}{k-1}}(\alpha_k) = \tilde{\tilde{\gamma}}_k \sim \gamma_{p(k)}$$

$$\Rightarrow \quad l\left(\hat{\varphi}_{\frac{1}{k}} \circ \hat{\varphi}^{-1}_{\frac{1}{k-1}}(\alpha_k)\right) = l(\gamma_{p(k)})$$

$$\Rightarrow \quad \frac{l\left(\hat{\varphi}_{\frac{1}{k}} \circ \hat{\varphi}^{-1}_{\frac{1}{k-1}}(\alpha_k)\right)}{l(\alpha_k)} \to \infty \text{ as } k \to \infty.$$

But this contradicts that $K\left(\hat{\varphi}_{\frac{1}{k}} \circ \hat{\varphi}^{-1}_{\frac{1}{k-1}}\right) \le k^2$ *.

Wolpert 1979, Shiga 2005

Definition 8 (Differentiable quasiconformal motions)

Suppose $\varphi: \Delta \times E \to \hat{\mathbb{C}}$ is a normalized quasiconformal motion. We say it is differentiable if $\forall\, t_0 \in \Delta$,

$$\varphi_t(z) = \varphi_{t_0}(z) + (t-t_0)\, V_{t_0}(z) + o(|t-t_0|)$$

where $V_{t_0}(z)$ is a normalized continuous vector field on E, i.e., $V_{t_0}(0) = V_{t_0}(1) = V_{t_0}(\infty) = 0$

Then we have two norms on $V = V_{t_0}$.

Teichmüller norm: $\|V\|_T = \inf_{\tilde{V}} \|\bar{\partial}\tilde{V}\|_\infty$ where $\tilde{V}$ is a continuous extension of V on $\hat{\mathbb{C}}$.

Cross-ratio norm: For any quadraplate $Q = \{z_1, z_2, z_3, z_4\}$ in E, $\|V\|_{cr} = \sup_{Q \subset E} |Cr(Q)\, \rho(Cr(Q))\, V(Q)|$

where $Cr(Q) = \dfrac{z_1 - z_3}{z_1 - z_4} \dfrac{z_2 - z_4}{z_2 - z_3} \neq 0, 1, \infty$, $\rho(\cdot)$ is the infinitesimal hyperbolic metric on $\mathbb{C}_{0,1}$,

$$V(Q) = \frac{V(z_1) - V(z_4)}{z_1 - z_4} - \frac{V(z_1) - V(z_3)}{z_1 - z_3} + \frac{V(z_2) - V(z_3)}{z_2 - z_3} - \frac{V(z_2) - V(z_4)}{z_2 - z_4}$$

Proposition 1 (Earle–Gardiner–Lakic, 1997, Contemp Math)

$\exists$ a constant $C>0$ s.t. for any continuous vector field V on E (closed) with bounded $\|V\|_{cr}$, then $\|V\|_T \le C\|V\|_{cr}$.

Theorem II (J–Mitra–Shiga–Wang 2013)

Suppose $E \subseteq \hat{\mathbb{C}}$ is a closed subset, $0, 1, \infty \in E$. Then any normalized differentiable quasiconformal motion $\varphi : \Delta \times E \to \hat{\mathbb{C}}$ with $\|\varphi\|_{cr} = \sup_{t_0 \in \Delta} \|V_{t_0}\|_{cr} < \infty$ is a local quasiconformal motion.

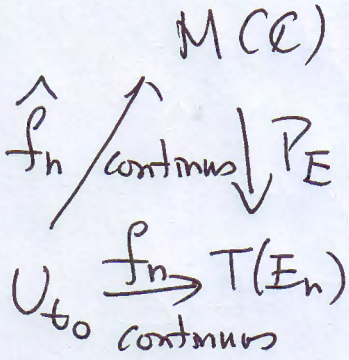
Proof. Let $E_n \uparrow$, $\#(E_n) < \infty$, $E = \overline{\bigcup_{n=1}^{\infty} E_n}$. $0, 1, \infty \in E_n$.

Let $\varphi_n : \Delta \times E_n \to \hat{\mathbb{C}}$, $\varphi_n = \varphi|_{\Delta \times E_n}$.

$\Rightarrow$ φ_n is a normalized quasiconformal motion.

$\Rightarrow \exists$ $\hat{\varphi}_n : \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal motion which extends φ_n. $\Rightarrow \exists f_n : \Delta \to T(E_n)$ s.t. $f_n^*(\Phi_{E_n}) = \varphi_n$. Since $\|V_{t_0}\|_T < \infty$, $\Rightarrow$

$\exists\ U_{t_0}$ above t_0 in Δ,



$\{\hat{f}_n\}$ is a normal family on U_{t_0}, the limiting $\hat{f}: U_{t_0} \to M(\mathbb{C})$ is a continuous map s.t.

$$\varphi(t,z) = w^{\hat{f}(t)}(z).\ \forall z \in E,\ t \in U_{t_0}.$$

$\Rightarrow \varphi$ is a local quasiconformal motion.

Definition 9 (local quasiconformal guiding)

Suppose $\varphi : X \times E \to \hat{\mathbb{C}}$ is a normalized quasiconformal motion. We say φ has local quasiconformal guidings if

(1) $\exists$ a continuous motion $\mathring{\varphi} : X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which extends φ.

(2) $\forall$ quadruplate $Q = \{z_1, z_2, z_3, z_4\} \subset E$, $\exists$ a quasiconformal motion $\hat{\varphi}_Q : X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which extends $\varphi | X \times Q$.

(3) $\forall x \in X$, $\hat{\varphi}_{Qx} \sim \mathring{\varphi}_x$ on $\hat{\mathbb{C}} | Q$.

(4) $\exists K(x) \geq 1$, $K(x_0) = 1$, continuous s.t.

$$d_{T(Q)}([\varphi_{Qx}]_Q, [o]_Q) \leq K(x), \quad \forall Q \subset E.$$

Conjecture 1 If φ has local quasiconformal guidings (in particular, if φ only satisfies (*)) then φ is a local quasiconformal motion.

Definition 4 A motion $\varphi : W \times E \to Y$ is called a holomorphic motion if

(v) $\forall z \in E$, $\varphi_z(x) = \varphi(x, z) : W \to Y$ is holomorphic.

Theorem 1. If $\varphi : W \times E \to \hat{\mathbb{C}}$ is a normalized holomorphic motion, then it is also a quasiconformal motion.

Proof. For any quadruplet $a, b, c, d \in E$,

$\varphi_y(a, b, c, d) : W \to \mathbb{C}_{0,1}$ is a holomorphic map.

So it weakly decreases the Kobayashi's metrics on W and on $\mathbb{C}_{0,1}$. The condition (iv) is satisfied clearly.

Theorem 2. If $\varphi : X \times E \to \hat{\mathbb{C}}$ is a quasiconformal motion, then it is also a continuous motion.

Proof. We first show that $\forall x \in X$, $\varphi_x(z) : E \to \hat{\mathbb{C}}$ is continuous. Let $l(t) : [0, 1] \to X$ be a path connecting 0 and x. Then its image l is a compact subset in X. Let $\varepsilon_0 > 0$ be a fixed number. For any $\tilde{x} \in l$, we have a nbhd $U_{\varepsilon_0}(\tilde{x}) \subset X$ such that

$$\rho\left(\varphi_y(a, b, c, d), \varphi_{y'}(a, b, c, d)\right) < \varepsilon_0$$

$\forall$ quadruplet $a, b, c, d \in E$.

Then we have finit number of points $x_0, x_1 \cdots x_n \in \ell$ such that $\{U_{\varepsilon_0}(x_i)\}_{i=0}^{n}$ covers ℓ and $x_n = x$.

This implies

$$\rho(\varphi_{x_0}(a,b,c,d), \varphi_x(a,b,c,d)) < n\varepsilon_0 = \delta_0.$$

If $z \neq 0, 1$, consider the quadruplate $z, 1, 0, \infty \in E$.

we have $$\rho(z, \varphi_y(z)) \leq \delta_0, \quad \forall y \in \bigcup_{i=0}^{n} U_{\varepsilon_0}(x_i) = U$$

This implies we have a constant $0 < m < M < \infty$ such that $$m \leq |\varphi_y(z)| \leq M, \quad \forall y \in U$$

Now for any $z, w \neq 0, 1$, consider the quadruplate $w, 0, z, \infty \in E$, we have

$$\rho\left(\frac{w-z}{z}, \frac{\varphi_y(w) - \varphi_y(z)}{\varphi_y(z)}\right) \leq \delta_0, \quad y \in U$$

$\Rightarrow$ We have a constant $C = C(\delta_0, m, M) > 0$ s.t.

$$|\varphi_y(w) - \varphi_y(z)| \leq C|w - z|.$$

This implies φ_y is uniformly continuous on ~~the annulus~~ any annulus $A_{mM} = \{z \in \mathbb{C} \mid m \leq |z| \leq M\}$. Interchanging $0, 1, \infty$, A_{mM} covers $\hat{\mathbb{C}}$. Thus $\varphi_y(z): E \to \hat{\mathbb{C}}$ is continuous.

Similarly, $\forall \varepsilon > 0$, we have a nbhd $U_x \subset U$ s.t.

$$\rho(\varphi_y(z), \varphi_{y'}(z)) < \varepsilon, \quad \forall y, y' \in U_x$$

This implies $|\varphi_y(z) - \varphi_x(z)| < \varepsilon'$, where $\varepsilon' \to 0$ as $\varepsilon \to 0$.

$$\Rightarrow \quad |\varphi_y(w) - \varphi_x(z)| \le |\varphi_y(w) - \varphi_y(z)| + |\varphi_y(z) - \varphi_x(z)|$$

$$\le C|w-z| + \varepsilon'$$

This implies $\varphi(y, w): X \times E \to \hat{\mathbb{C}}$ is continuous on any annulus A_{mM}. Interchanging $0, 1, \infty$, A_{mM} covers $\hat{\mathbb{C}}$. Thus $\varphi(y, w): X \times E \to \hat{\mathbb{C}}$ is continuous.

This implies condition (ii)'.

Since the inverse $\varphi^{-1}(y, w) = \varphi_y(w)^{-1}: X \times E \to \hat{\mathbb{C}}$ is also a quasiconformal motion, we get $\varphi_y(w)^{-1}: E \to \hat{\mathbb{C}}$ is also continuous, so $\varphi_y(w): E \to \varphi_y(E)$ is a homeomorphism. This is the condition (ii)'. We proved Theorem 2.

Corollary 1. Every quasiconformal motion $\varphi: X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a continuous motion.

Theorem 3 (Gardiner-J-Wang, 2010, London Math)

Let $\varphi : \Delta \times E \to \hat{\mathbb{C}}$ be a normalized holomorphic motion. Let $\varphi(t,z) = z + tV(z) + o(t^2)$.

Then $V(z) = o(|z|^2)$ as $z \to \infty$, $V(0)=0$, $V(1)=0$ and $\exists\, C>0$ depending on $R>0$ s.t. $\forall\, |z_1|<R$, $|z_2|<R$ and $|z_1 - z_2| < \delta$,

$$|V(z_2) - V(z_1)| \le |z_1 - z_2| \left(2 + \frac{C}{\log 1/\delta}\right) \left(\log \frac{1}{|z_2 - z_1|}\right).$$

Corollary 2 (Gardiner-J-Wang, 2010, London Math)

Suppose $0<r<1$ and $R>0$. If $|t| \le r$, $|z_1(t)| \le R$ $|z_2(t)| \le R$, and $|z_1(t) - z_2(t)| < \delta$, then

$$\left|V(z_1(t)) - V(z_2(t))\right| \le \frac{2+\varepsilon}{1-|c|^2} |z_2(t) - z_1(t)| \log \frac{1}{|z_2(t) - z_1(t)|}.$$

where $\varepsilon \le \dfrac{M}{\log 1/\delta}$ and $\delta \ge |z_1(0) - z_2(0)|$. Moreover,

$\exists$ a constant $C>0$ s.t.

$$|z_1(t) - z_2(t)| \le C\, |z_1 - z_2|^{\frac{1-|c|}{1+|c|}}.$$