On coincidences of morphisms of closed Riemann surfaces

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1. Outline

- All of the Riemann surfaces are compact and of genera ≥ 1.
- We denote by

$$f_i : X \to Y \quad (i = 1, 2)$$

  two distinct non-constant holomorphic maps of degree d_i between compact Riemann surfaces of genera g and γ, respectively.
- We look at the number of *coincidences*, that is, the number of points $p \in X$ with $f_1(p) = f_2(p)$.

Y. Fuertes and G. González-Diez (1993) gave a sharp bound for the number of coincidences of two holomorphic maps.

Theorem 1.1

Let $f_i : X \to Y$ be as above, and let $L(f_1, f_2)$ denote the number of coincidences appropriately counted. We have
1) $L(f_1, f_2) \leq d_1 + 2\gamma\sqrt{d_1 d_2} + d_2$.
2) In case $\gamma \geq 2$, this bound is attained if and only if Y is hyperelliptic and $f_2 = J \circ f_1$, where J denotes the hyperelliptic involution of Y.

This is a generalization of the well known fact about the number of fixed points of automorphisms, namely, for an arbitrary $T \in \mathrm{Aut}(X)$,

$$L(id., T) \leq 2g + 2.$$

We can map $f_i : X \to Y$ into the set of homomorphisms of Jacobians $\mathrm{Hom}(\mathrm{J}(X), \mathrm{J}(Y))$ which forms a free abelian group of rank $\leq 2g\gamma$. It has an inner product. We showed (T, 2010)

Theorem 1.2

Let $f_i : X \to Y$ be two distinct non-constant holomorphic maps of degree d_i $(i = 1, 2)$ between compact Riemann surfaces of genera g and $\gamma \geq 1$, respectively. Let $\mathfrak{f}_i$ be the homomorphisms of Jacobian varieties induced by f_i $(i = 1, 2)$. Then

$$L(f_1, f_2) = 0 \Longleftrightarrow \begin{cases} d_1 = d_2 \\ \cos(\mathfrak{f}_1, \mathfrak{f}_2) = \gamma^{-1} \end{cases} .$$

Corollary 1.3

Let $f_i : X \to Y$ be two distinct non-constant holomorphic maps of degree $d_i\,(i = 1, 2)$ between compact Riemann surfaces of genera g and $\gamma = 1$, respectively. The following two conditions are equivalent.
1) $L(f_1, f_2) = 0$.
2) The difference between f_1 and f_2 is only a translation on the torus Y.

back We will provide here several results about coincidences afterward. (Most of them are from "A generalization of the Eichler trace formula for morphisms between Riemann surfaces" T, to appear in Ann. Acad. Sci. Fenn. Math.)

2. The Lefschetz trace formula

Let $T \in \mathrm{Aut}(X)$ and let $\Gamma_T = \{(p, T(p))\} \subset X \times X$ be the graph of T.
A fixed point of T corresponds to a point of intersection of the graph Γ_T and the diagonal $\Delta = \{(p, p)\} \subset X \times X$.
We define the Lefschetz number of T to be

$$L(id., T) = \sharp(\Delta \cdot \Gamma_T).$$

By using integral,

$$L(id., T) = \int_{\Gamma_T} \varphi_\Delta = \int_X (id. \times T)^* \varphi_\Delta,$$

where $\varphi_\Delta \in H^2_{DR}(X \times X)$ is a closed form representing the cohomology class Poincaré dual to the class of Δ.

For each q let

$$\{\psi_{q,\mu}\}$$

be a collection of closed q-forms representing a basis for $H^q_{\mathrm{DR}}(X)$, and let

$$\{\psi^*_{2-q,\mu}\}$$

be closed forms representing the dual basis for $H^{2-q}_{\mathrm{DR}}(X)$, i.e., such that

$$\int_X \psi_{q,\mu} \wedge \psi^*_{2-q,\nu} = \delta_{\mu,\nu}.$$

Let π_1 and π_2 denote the two projection maps $X \times X \to X$.
Then

$$\varphi_\Delta = \sum_q (-1)^q \sum_\mu \pi_1^* \psi_{q,\mu} \wedge \pi_2^* \psi^*_{2-q,\mu}.$$

Thus we can evaluate the Lefschetz number by

$$\begin{aligned} L(id., T) &= \int_{\Gamma_T} \varphi_\Delta = \int_X (id. \times T)^* \varphi_\Delta \\ &= \sum_q (-1)^q \sum_\mu \int_X \psi_{q,\mu} \wedge T^* \psi^*_{2-q,\mu} \\ &= \sum_{k=0}^{k=2} (-1)^k \mathrm{trace}\,(T^*|_{H^k_{DR}(X)}), \end{aligned}$$

where $k = 2 - q$. The obtained formula

$$L(id., T) = \sum_{k=0}^{k=2} (-1)^k \mathrm{trace}\,(T^*|_{H^k_{DR}(X)})$$

is so called the Lefschetz trace formula (for two-dimensional case).

Let $f_i : X \to Y$ be as above.
We define the Lefschetz number of f_1 and f_2 as follows.

Definition 2.1

The Lefschetz number of two distinct holomorphic maps $f_i : X \to Y \, (i = 1, 2)$ is defined to be

$$L(f_1, f_2) = \int_X (f_1 \times f_2)^* \varphi_\Delta,$$

where $\varphi_\Delta \in H^2_{DR}(Y \times Y)$ is the Poincaré dual of the diagonal $\Delta \subset Y \times Y$.

Thus denoting

$$\Gamma_{f_1,f_2} = \{(f_1(p), f_2(p))\} \subset Y \times Y,$$

we have

$$L(f_1, f_2) = \sharp(\Delta \cdot \Gamma_{f_1,f_2}) = \int_{\Gamma_{f_1,f_2}} \varphi_\Delta.$$

Definition 2.2

Let $f : X \to Y$ be a holomorphic map between Riemann surfaces. We define a linear map

$$f_* : H^k_{DR}(X) \to H^k_{DR}(Y)$$

by the property

$$\int_Y f_* v \wedge w = \int_X v \wedge f^* w,$$

for any $w \in H^{2-k}_{DR}(Y)$.

Then the analogue to $L(id., T)$ takes the form

$$
\begin{aligned}
L(f_1, f_2) &= \int_X (f_1 \times f_2)^* \varphi_\Delta = \sum_q (-1)^q \sum_\mu \int_X f_1^* \psi_{q,\mu} \wedge f_2^* \psi^*_{2-q,\mu} \\
&= \sum_q (-1)^q \sum_\mu \int_Y f_{2*} \circ f_1^* \psi_{q,\mu} \wedge \psi^*_{2-q,\mu} \\
&= \sum_{q=0}^{q=2} (-1)^q \mathrm{trace}\,(f_{2*} \circ f_1^*|_{H^q_{DR}(Y)}) \\
&= \sum_{k=0}^{k=2} (-1)^k \mathrm{trace}\,(f_1^* \circ f_{2*}|_{H^k_{DR}(X)}),
\end{aligned}
$$

where we use the same symbol $\{\psi_{q,\mu}\}$ and $\{\psi^*_{2-q,\mu}\}$ for the basis for $H^q_{\mathrm{DR}}(Y)$ and for the dual basis for $H^{2-q}_{\mathrm{DR}}(Y)$, respectively. Observing above, we easily have

$$L(f_1, f_2) = L(f_2, f_1).$$

Y. Fuertes and G. González-Diez showed the following lemma.

Lemma 1

i) $f_1^* \circ f_{2*} : H^0(X) \to H^0(X)$ *is multiplication by* d_2.
ii) $f_1^* \circ f_{2*} : H^2(X) \to H^2(X)$ *is multiplication by* d_1.

In our situation, the Lefschetz trace formula is written as

$$\begin{aligned} L(f_1, f_2) &= \sum_{k=0}^{k=2} (-1)^k \mathrm{trace}(f_1^* \circ f_{2*}|_{H^k_{DR}(X)}) \\ &= d_1 - \mathrm{trace}\, f_1^* \circ f_{2*}|_{H^1_{DR}(X)} + d_2. \end{aligned}$$

Definition 2.3

Let $p \in X$ be a coincidence of f_1 and f_2, and let

$$f_1(z) - f_2(z) = c_k z^k + c_{k+1} z^{k+1} + \dots, \quad c_k \neq 0$$

be the Taylor expansion of $f_1 - f_2$ with respect to small parametric discs D around p and D' around $f_i(p)$. We define the multiplicity of f_1 and f_2 at p to be

$$m_p(f_1, f_2) = k.$$

By the definition, $m_p(f_1, f_2)$ is always positive. Furthermore, one can show

$$L(f_1, f_2) = \sum_{\{p \in X; f_1(p) = f_2(p)\}} m_p(f_1, f_2). \tag{1}$$

back Thus $L(f_1, f_2)$ is always greater than or equal to the actual number of coincidences.

$\mathrm{J}(X)$; Jacobian of X.
$\iota_X : X \to \mathrm{J}(X)$; Abel-Jacobi map.

For any holomorphic map $f : X \to Y$, there exists a homomorphism $\mathfrak{f} : \mathrm{J}(X) \to \mathrm{J}(Y)$ with

$$\mathfrak{f} \circ \iota_X = \iota_Y \circ f.$$

Under addition the set of homomorphisms $\mathrm{Hom}(\mathrm{J}(X), \mathrm{J}(Y))$ forms a free abelian group of rank $\leq 2g\gamma$.
We can define an inner product $\langle , \rangle$ such that if $\mathfrak{f}_i$ be the homomorphisms of Jacobian varieties induced by holomorphic maps of Riemann surfaces $f_i \, (i = 1, 2)$, then

$$\langle \mathfrak{f}_1, \mathfrak{f}_2 \rangle = \mathrm{trace} \left(f_1^* \circ f_{2*}|_{H^1_{DR}(X)} \right).$$

For $F, G \in \mathrm{Hom}(\mathrm{J}(X), \mathrm{J}(Y))$, define

$$\cos(F, G) = \frac{\langle F, G \rangle}{\|F\| \cdot \|G\|}$$

as usual.
Partially using the properties of an inner product, we showed Theorem 1.2 and its corollary. Theorem 1.2

New bound for the number of maps ($\gamma = 2$)

Let $f_i : X \to Y (i = 1, 2)$ be distinct non-constant holomorphic maps of degree d between compact Riemann surfaces of genera g and γ.
Then

$$\|\mathfrak{f}_i\| = \langle \mathfrak{f}_i, \mathfrak{f}_i \rangle^{1/2} = \sqrt{2d\gamma}.$$

(On the sphere of radius $\sqrt{2d\gamma}$ in the space of dimension $\leq 2g\gamma$.)
On the other hand,

$$0 \leq L(f_1, f_2) = 2d - \mathrm{trace}\left(f_1^* \circ f_{2*}|_{H^1_{DR}(X)}\right) = 2d - \langle \mathfrak{f}_1, \mathfrak{f}_2 \rangle,$$

and this implies that

$$\|\mathfrak{f}_1 - \mathfrak{f}_2\| \geq \sqrt{4d(\gamma - 1)}.$$

Thus we can give an upper bound for the number of holomorphic maps $X \to Y$ depending only on the genera.

New bound for the number of maps ($\gamma = 2$)

In particular if $\gamma = 2$, we obtain an upper bound depending only on the genus g which is smallest among the known bounds (as order, the best one is cg^{2g}, where c is a constant and it is valid for all $\gamma \geq 2$) .

Theorem 3.1

Let X and Y be compact Riemann surfaces of genera $g > 2$ and $\gamma = 2$. Then the number of non-constant holomorphic maps from X to Y is less than

$$(3^{4g} - 1)(g - 1).$$

The holomorphic Lefschetz number

Let M be a compact Kähler manifold. By the Hodge decomposition,

$$H^r(M,\mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}_{\overline{\partial}}(M)$$

$$H^{p,q}_{\overline{\partial}}(M) = \overline{H^{q,p}_{\overline{\partial}}(M)}.$$

Thus, for a Riemann surface X,

$$H^1(X,\mathbb{C}) \cong H^{1,0}_{\overline{\partial}}(X) \bigoplus H^{0,1}_{\overline{\partial}}(X)$$

holds, where we may identify $H^{1,0}_{\overline{\partial}}(X)$ with the space of holomorphic 1-forms and $H^{0,1}_{\overline{\partial}}(X)$ being the complex conjugate of $H^{1,0}_{\overline{\partial}}(X)$.
H^0 and H^2 are trivial in this case.

Now the Lefschetz number $L(f_1, f_2)$ is written as

$$L(f_1, f_2) = \int_X (f_1 \times f_2)^* \varphi_\Delta = \sum_{p,q} (-1)^{p+q} \mathrm{trace}\left(f_1^* \circ f_{2*}|_{H_{\bar{\partial}}^{p,q}(X)}\right).$$

Let π_1 and π_2 denote the two projection maps $Y \times Y \to Y$. For each p and q let

$$\{\psi_{p,q,\mu}\}$$

be a collection of $\overline{\partial}$-closed (p,q)-forms representing a basis for $H^{p,q}_{\overline{\partial}}(Y)$, and let

$$\{\psi^*_{1-p,1-q,\mu}\}$$

be $\overline{\partial}$-closed forms representing the dual basis for $H^{1-p,1-q}_{\overline{\partial}}(Y)$ under the pairing

$$H^{p,q}_{\overline{\partial}}(Y) \otimes H^{1-p,1-q}_{\overline{\partial}}(Y) \to \mathbb{C}$$

given by

$$\psi \otimes \varphi \mapsto \int_Y \psi \wedge \varphi.$$

A basis for $H^{1,1}_{\overline{\partial}}(Y \times Y)$ is represented by the forms

$$\{\varphi_{p,q,\mu,\nu} = \pi_1^* \psi_{p,q,\mu} \wedge \pi_2^* \psi^*_{1-p,1-q,\nu}\}.$$

The Dolbeault class of the diagonal is represented by the form

$$\varphi_\Delta = \sum_{p,q,\mu} (-1)^{p+q} \varphi_{p,q,\mu,\mu}.$$

Set

$$\varphi_\Delta^0 = \sum_{q,\mu} (-1)^q \varphi_{0,q,\mu,\mu} = \varphi_{0,0} - \varphi_{0,1,\mu,\mu}.$$

Integrating φ_Δ^0 over Γ_{f_1,f_2}, we have

$$\int_{\Gamma_{f_1,f_2}} \varphi_\Delta^0 = \int_X (f_1 \times f_2)^* \varphi_\Delta^0 = \int_X \sum_{q,\mu} (-1)^q f_1^* \psi_{0,q,\mu} \wedge f_2^* \psi_{1,1-q,\mu}^*$$

$$= \int_Y \sum_{q,\mu} (-1)^q f_{2*} \circ f_1^* \psi_{0,q,\mu} \wedge \psi_{1,1-q,\mu}^* = \sum_{q=0}^{1} (-1)^q \,\mathrm{trace} f_{2*} \circ f_1^* |_{H_{\bar\partial}^{0,q}(Y)}$$

$$= \sum_{q=0}^{1} (-1)^q \,\mathrm{trace} f_1^* \circ f_{2*} |_{H_{\bar\partial}^{0,q}(X)}.$$

Definition 4.1

We call the number $\sum_{q=0}^{1}(-1)^q \,\mathrm{trace} f_1^* \circ f_{2*}|_{H_{\overline{\partial}}^{0,q}(X)}$ the holomorphic Lefschetz number of (f_1, f_2) and denote it by $L(f_1, f_2, \mathcal{O})$.

Thus

$$L(f_1, f_2, \mathcal{O}) = d_2 - \mathrm{trace} f_1^* \circ f_{2*}|_{H_{\overline{\partial}}^{0,1}(X)}.$$

An easy calculation shows

$$L(f_1, f_2) = L(f_1, f_2, \mathcal{O}) + L(f_2, f_1, \mathcal{O}). \tag{2}$$

We ask accordingly whether we can evaluate the number $L(f_1, f_2, \mathcal{O})$ in terms of the local behavior of f_1 and f_2 around their coincidences.

Theorem 4.2 (T, 2012@Nagoya)

Let $f_i : X \to Y$ be two distinct morphisms of degree $d_i\,(i = 1, 2)$ between closed Riemann surfaces and let $\{p_\alpha\}$ be their coincidences possibly empty set.
Suppose that all coincidences $p_\alpha \in X$ satisfy $f_2'(p_\alpha) \neq 0$ and $f_2'(p_\alpha) - f_1'(p_\alpha) \neq 0$. Then

$$L(f_1, f_2, \mathcal{O}) = \sum_\alpha \frac{f_2'(p_\alpha)}{f_2'(p_\alpha) - f_1'(p_\alpha)}$$

If $\{p_\alpha\}$ is empty, then we take the right hand side to be 0.

This has been improved as

Theorem 4.3

Let $f_i : X \to Y$ be as above and let $\{p_\alpha\}$ be their coincidences possibly empty set. The holomorphic Lefschetz number of (f_1, f_2) is given by

$$L(f_1, f_2, \mathcal{O}) = \sum_\alpha \mathrm{Res}\left(\frac{f_2'(\xi_\alpha)}{f_2(\xi_\alpha) - f_1(\xi_\alpha)}; p_\alpha\right).$$

If $\{p_\alpha\}$ is empty, then we take the right hand side to be 0.

$$L(f_1, f_2, \mathcal{O}) + L(f_2, f_1, \mathcal{O}) = \sum_\alpha \mathrm{Res}\left(\frac{f_2'(\xi_\alpha) - f_1'(\xi_\alpha)}{f_2(\xi_\alpha) - f_1(\xi_\alpha)}; p_\alpha\right)$$

Def.2.3 $= L(f_1, f_2)$.

Corollary 4.4 (The Eichler trace formula for coincidences)

With the same notation as in Theorem 4.3, we have

$$\mathrm{trace} f_1^* \circ f_{2*}|_{H^{1,0}_{\bar\partial}(X)} = d_2 - \sum_\alpha \overline{\mathrm{Res}\left(\frac{f_2'(\xi_\alpha)}{f_2(\xi_\alpha) - f_1(\xi_\alpha)};\, p_\alpha\right)}.$$

For automorphisms, namely if $X = Y$ and f_2 is the identity map on X and $f_1 \neq \mathrm{id}$. then a coincidence is a fixed point of f_1 and

$$\mathrm{Res}\left(\frac{f_2'(\xi_\alpha)}{f_2(\xi_\alpha) - f_1(\xi_\alpha)};\, p_\alpha\right) = \frac{1}{1 - f_1'(\xi_\alpha(p_\alpha))}.$$

Substituting this into Corollary 4.4, we obtain the Eichler trace formula

$$\begin{aligned} \mathrm{trace} f_1^*|_{H^{1,0}_{\bar\partial}(X)} &= 1 - \sum_\alpha \overline{\frac{1}{1 - f_1'(\xi_\alpha(p_\alpha))}} \\ &= 1 - \sum_\alpha \frac{f_1'(\xi_\alpha(p_\alpha))}{f_1'(\xi_\alpha(p_\alpha)) - 1}, \end{aligned}$$

since $|f_1'(\xi_\alpha(p_\alpha))| = 1$.

It is well known that $H^{1,0}$ carries a hermitian structure given by

$$\langle u, v \rangle = i \int u \wedge \overline{v}.$$

Theorem 4.5 (T, 2012@Nagoya)

Let A be the matrix representation of $f_1^* \circ f_{2*}|_{H^{1,0}}$ with respect to any given orthonormal basis. Then A can be diagonalizable by a unitary matrix if and only if $f_1 \sim f_2$, i.e., there exists an automorphism T on Y with $f_1 = T \circ f_2$.

If the multiplicity of f_1 and f_2 at p_α is 1, then

$$\mathrm{Res}\left(\frac{f_2'(\xi_\alpha)}{f_2(\xi_\alpha) - f_1(\xi_\alpha)};\, p_\alpha\right) = \frac{f_2'(\xi_\alpha(p_\alpha))}{f_2'(\xi_\alpha(p_\alpha)) - f_1'(\xi_\alpha(p_\alpha))}.$$

In this case, we have

Corollary 4.6

Let the conditions of Theorem 4.3 hold, and suppose that every coincidence has multiplicity one. Then the difference in degree between f_1 and f_2 is given by

$$d_2 - d_1 = \sum_\alpha \frac{|f_2'(\xi_\alpha(p_\alpha))|^2 - |f_1'(\xi_\alpha(p_\alpha))|^2}{|f_2'(\xi_\alpha(p_\alpha)) - f_1'(\xi_\alpha(p_\alpha))|^2}.$$

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