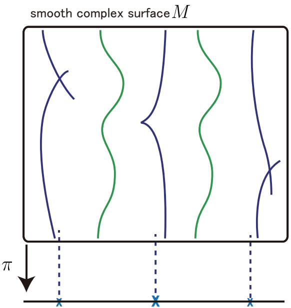
Splittings of singular fibers into Lefschetz fibers



Takayuki OKUDA

the University of Tokyo

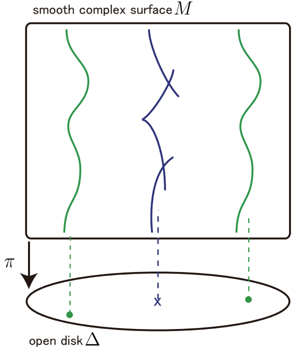
Tohoku University

January 8, 2017

Introduction

1. **Splitting of singular fibers**
2. **Topological monodromy**
3. **Results**
4. **Applications**

   (Dehn-twist expression)



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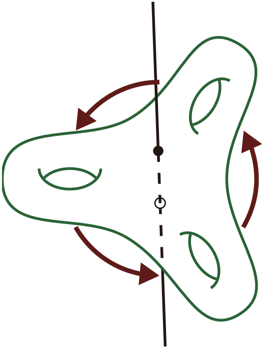

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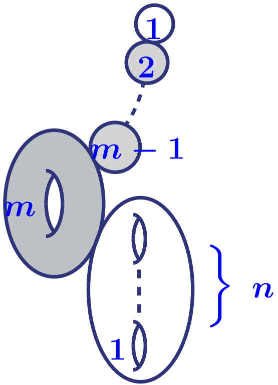

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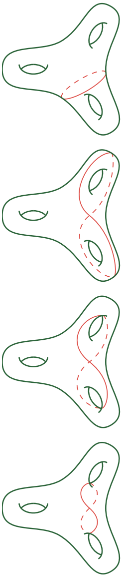


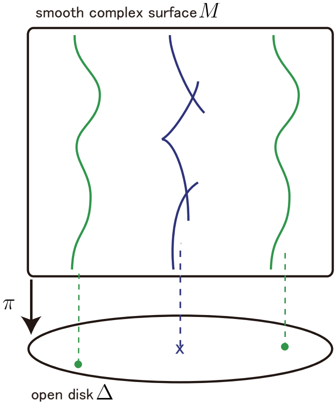
Splitting of singular fibers

Degeneration of Riemann surfaces

M : a smooth complex surface Δ : the unit disk in $\mathbb{C}$
$\pi : M \to \Delta$: a proper surjective holomorphic map s.t.

- $X_s := \pi^{-1}(s)$ $(s \neq 0)$ are smooth curves of genus g.
- $X_0 := \pi^{-1}(0)$ is a singular fiber.

($\Longleftrightarrow$ 0 is a unique critical value.)



$\pi : M \to \Delta$ is called
a **degeneration** (or, degenerating family)
of Riemann surfaces of genus g.

Regard X_0 as the divisor defined by π:

$$X_0 = \sum m_i \Theta_i,$$

where Θ_i is an irreducible component
with **multiplicity** m_i.

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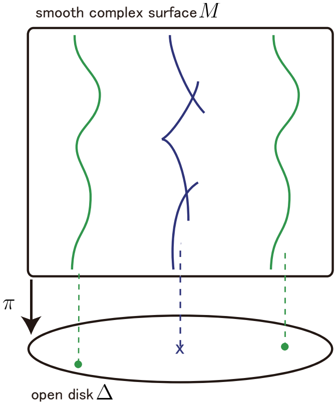

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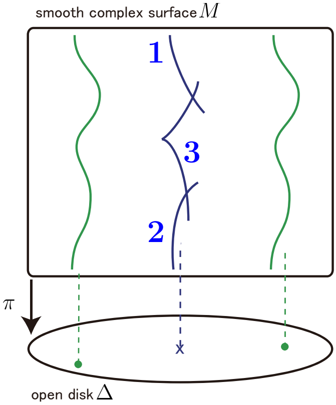

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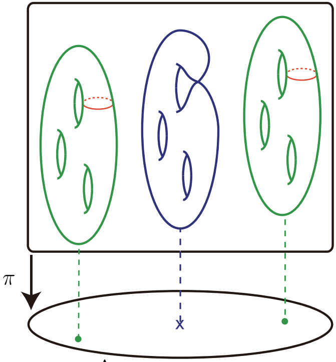

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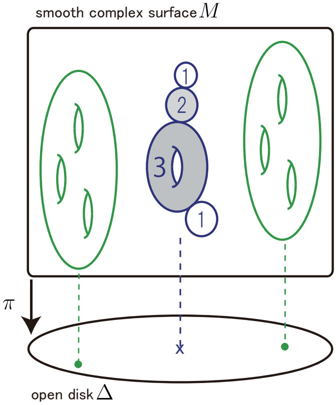

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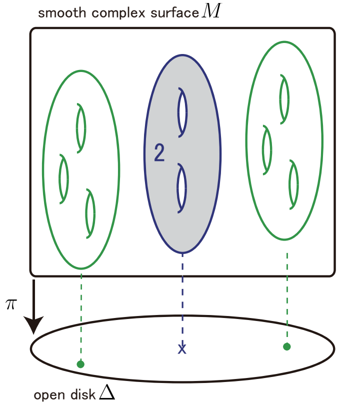

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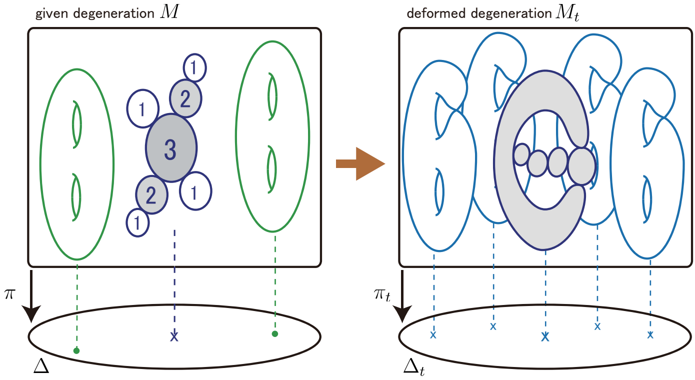
Splitting of singular fibers

$\pi : M \to \Delta$: a degeneration w/ **singular fiber X_0**

$\{\pi_t : M_t \to \Delta\}$: a family of deformations of $\pi : M \to \Delta$
i.e. $\pi_0 : M_0 \to \Delta$ coincides with $\pi : M \to \Delta$.



If π_t $(t \neq 0)$ has **k singular fibers $X_{s_1}, \ldots, X_{s_k}$**, $k \geq 2$,
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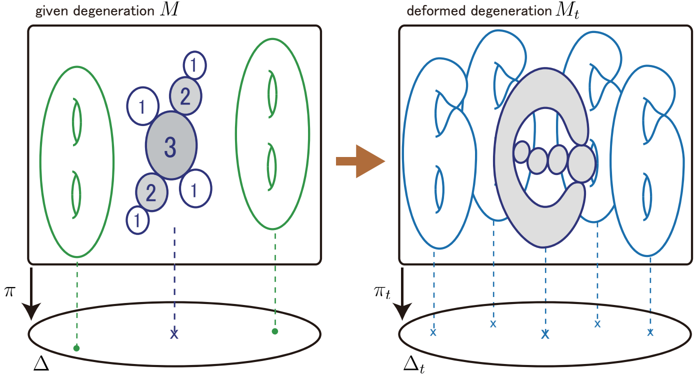

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Fact
(1) A **Lefschetz fiber** and (2) a **multiple smooth fiber**
admit no splittings (i.e. any deformation is equisingular).

Such fibers are said to be **atomic**.

Conjecture (from topological aspect)
Every singular fiber can split into singular fibers
each of which is (1) or (2), in finite steps of deformations.

How to construct splitting families

- **Double covering method**
  - Moishezon (genus 1 case), Horikawa (genus 2 case), Arakawa-Ashikaga (hyperelliptic case)
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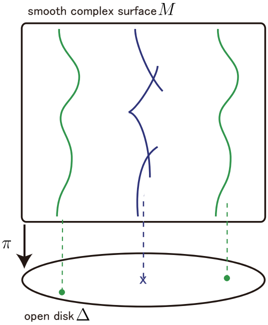
Topological monodromy

Monodromy of degenerations

$\pi : M \to \Delta$: a degeneration of Riemann surfaces of genus g
with **singular fiber X_0**

A loop C in Δ^* turning around 0 once in positive direction
induces a self-homeomorphism φ of a general fiber X_s.
(**monodromy homeomorphism**)



Identifying X_s with Σ_g,
its isotopy class $[\varphi] \in \mathrm{MCG}(\Sigma_g)$
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Theorem
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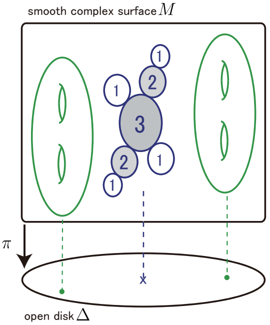

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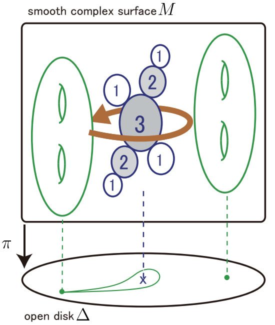

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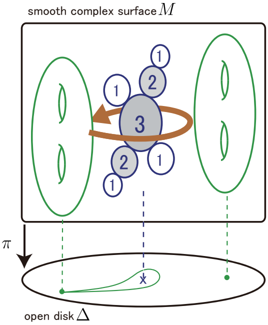

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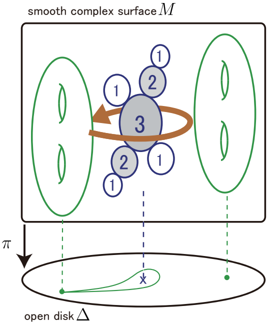

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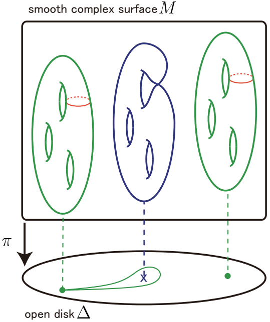
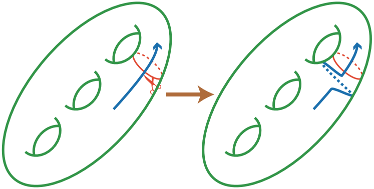
via **topological monodromy**, for $g \geq 2$.

Lefschetz fiber



Right-handed Dehn twist



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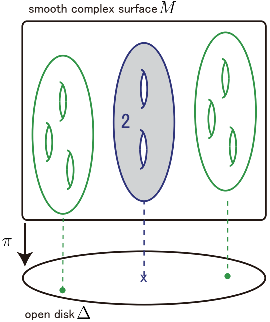
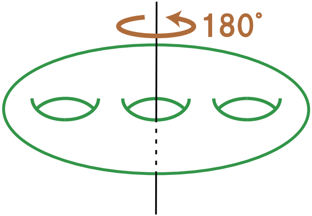
Multiple smooth fiber



Periodic mapping class w/o multiple points



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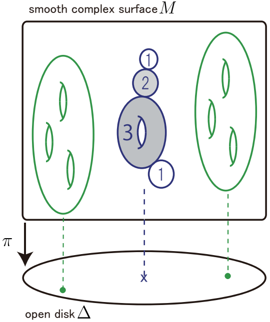
Stellar fiber



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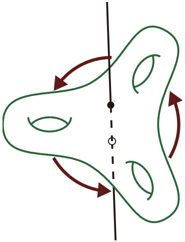

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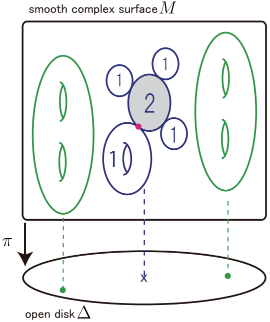
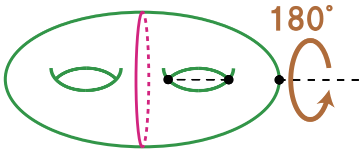
Singular fiber



Pseudo-periodic mapping class of negative twist

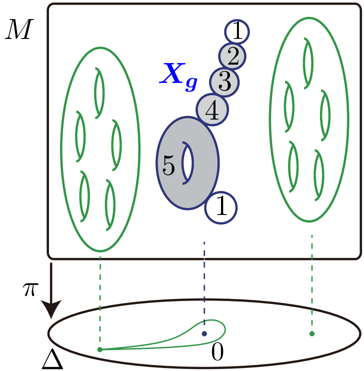


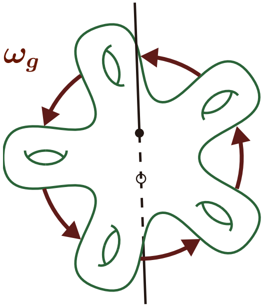
Degeneration of propeller surfaces

A **propeller surface** is a Riemann surface Σ_g of genus $g \geq 2$ equipped with $\mathbb{Z}_g$-action s.t. $\Sigma_g/\mathbb{Z}_g$ has genus 1.

ω_g : a **propeller automorphism**

X_g : the singular fiber with monodromy ω_g





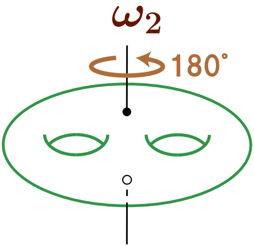
Splittability into Lefschetz fibers

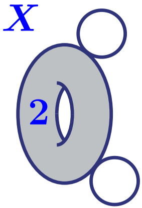
Theorem (Y. Matsumoto)

$\pi : M \to \Delta$: the degeneration of Riemann surfaces of genus 2 with monodromy ω_2

Then its singular fiber X can split into four Lefschetz fibers.

Moreover, their vanishing cycles are as depicted below.





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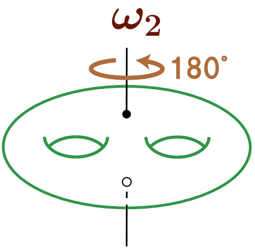
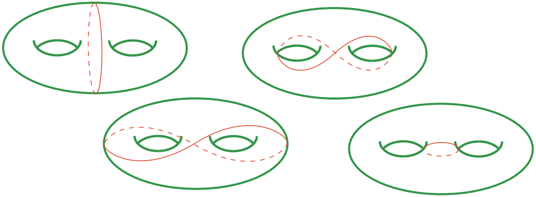
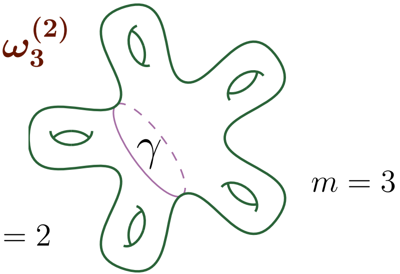
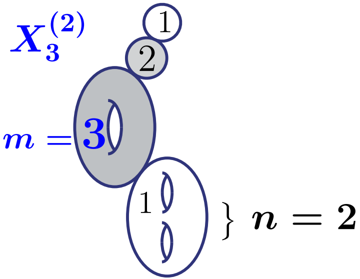




$$\omega_2 = \tau_0 \circ \tau_a \circ \tau_b \circ \tau_c$$





γ : a separating simple loop on Σ_g
s.t. $\Sigma_g \setminus \gamma = \Sigma_{m,1} \coprod \Sigma_{n,1}$ $\quad (g = m + n, m \geq 1, n \geq 0)$

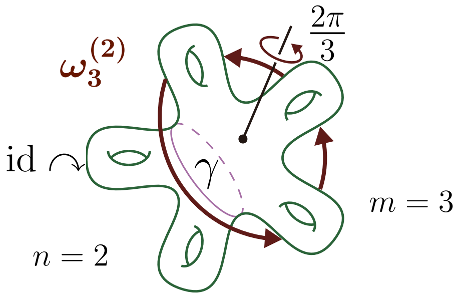
$\omega_m^{(n)}$: a psudo-periodic map satisfying

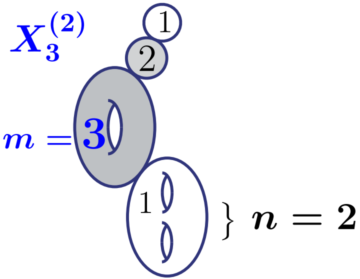
- $\omega_m^{(n)}|_{\Sigma_{m,1}} \sim$ a periodic map with a fixed pt of rot angle $\frac{2\pi}{m}$.
- $\omega_m^{(n)}|_{\Sigma_{n,1}} \sim \mathrm{id}.$

NOTE: $(\omega_m^{(n)})^m = \tau_\gamma$

$X_m^{(n)}$: the singular fiber with monodromy $\omega_m^{(n)}$

Psudo-propeller maps





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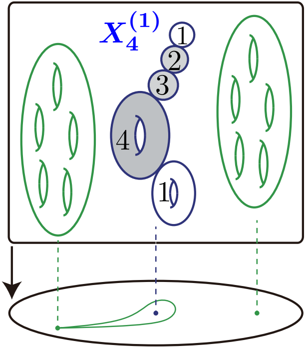
$\omega_m^{(n)}$: a psudo-periodic map satisfying

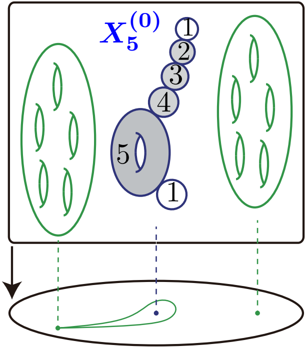
- $\omega_m^{(n)}|_{\Sigma_{m,1}} \sim$ a periodic map with a fixed pt of rot angle $\frac{2\pi}{m}$.
- $\omega_m^{(n)}|_{\Sigma_{n,1}} \sim \mathrm{id}$.

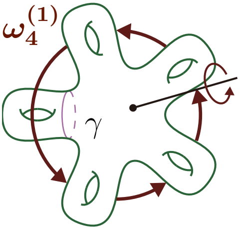
NOTE: $(\omega_m^{(n)})^m = \tau_\gamma$

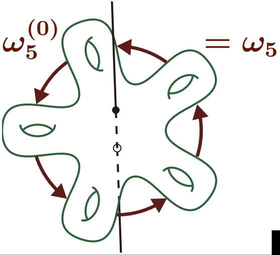
$X_m^{(n)}$: the singular fiber with monodromy $\omega_m^{(n)}$

Psudo-propeller maps









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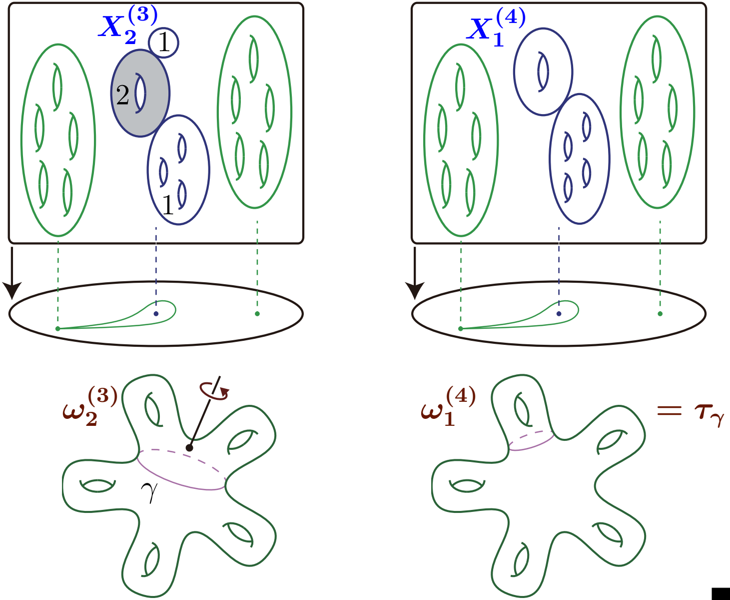
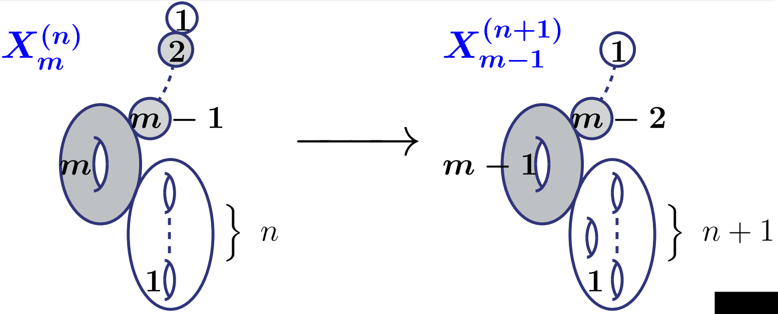


Theorem (O-Takamura)

1. For any $m \geq 2$, $n \geq 0$, the singular fiber $X_m^{(n)}$ can split into $X_{m-1}^{(n+1)}$ and three Lefschetz fibers.
2. For any $g \geq 2$, we have the following sequence:
$$X_g^{(0)} \longrightarrow X_{g-1}^{(1)} \longrightarrow \cdots \longrightarrow X_2^{(g-2)} \longrightarrow X_1^{(g-1)},$$
where "$A \to B$" means "A splits into B and 3 Lefschetz fibers".



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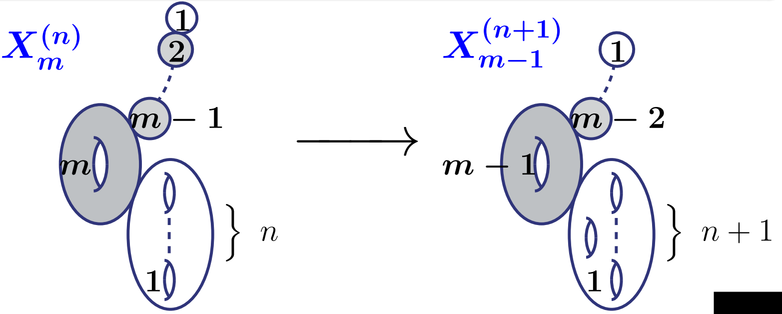
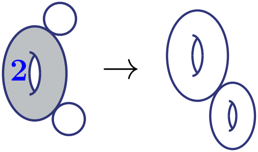


$$\boldsymbol{X_m^{(n)}} \longrightarrow \boldsymbol{X_{m-1}^{(n+1)}}$$

$g = 2$



$g = 3$

$g = 4$

$$X_m^{(n)} \longrightarrow X_{m-1}^{(n+1)}$$

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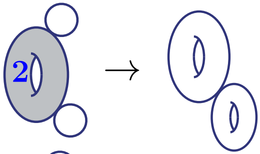

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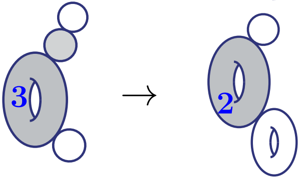

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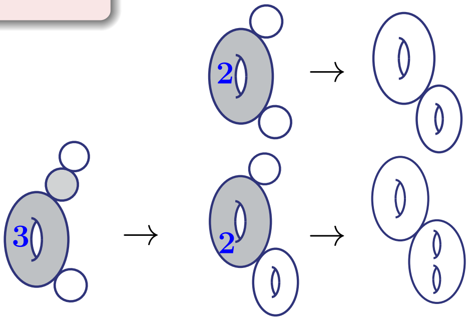

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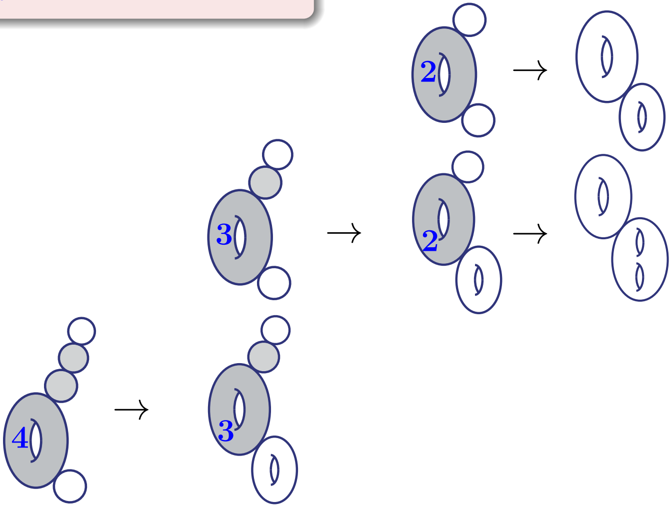

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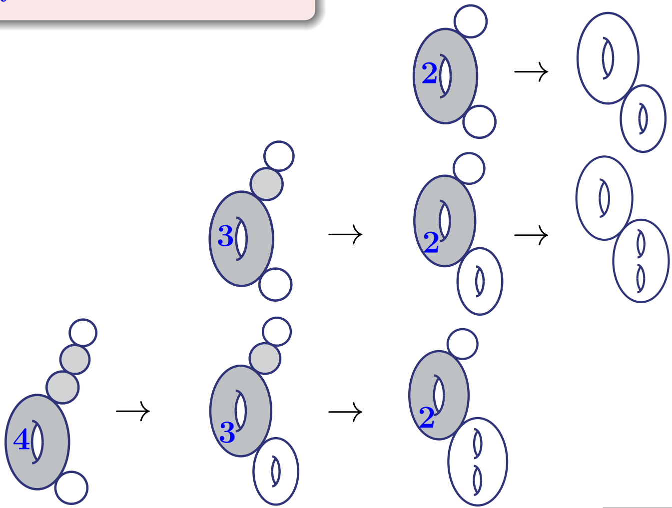

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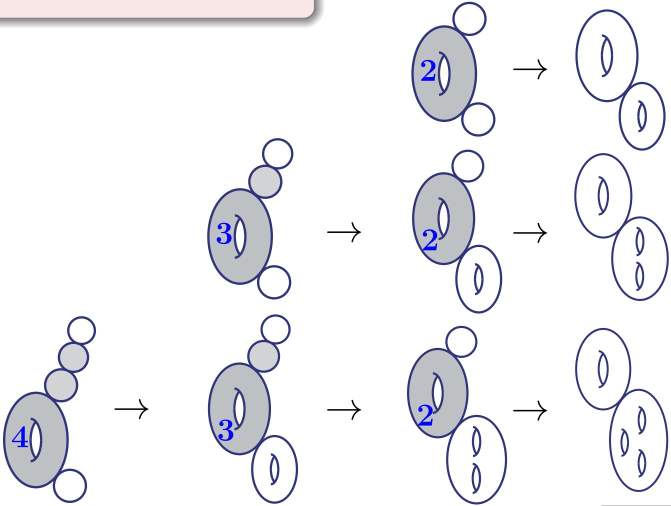


Sketch proof of Theorem

Θ_0 : the core of $\boldsymbol{X_m^{(n)}}$ (the component with multiplicity m)
p_1, p_2 : the intersection points of Θ_0 with other components.
N_0 : a tubular nbhd of Θ_0 in M
(embedded in the normal bundle on Θ_0 in M)

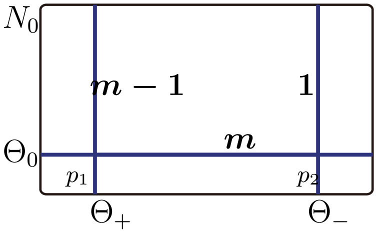
1. Realize the restriction of π to N_0 as a holomorphic function

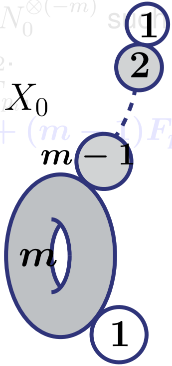
$$\pi(z,\zeta) = \zeta^m \sigma(z), \quad (z,\zeta) \in N_0$$

where σ is a holomorphic section of $N_0^{\otimes(-m)}$ such that

$$\operatorname{div}(\sigma) = (m-1)p_1 + p_2.$$

Denoting the fiber of N_0 over p_i by F_{p_i},
then $\boldsymbol{X_m^{(n)}}|_{N_0}$ $(= \operatorname{div}(\pi))$ $= \boldsymbol{m\Theta_0 + (m-1)F_{p_1} + F_{p_2}}$,





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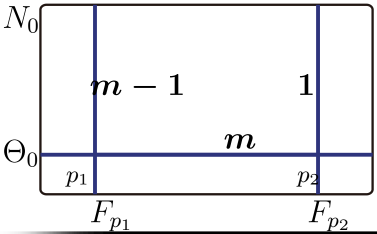

Sketch proof of Theorem

2. Find a meromorphic section ρ of $N_0^{\otimes(-1)}$ such that
$$\mathrm{div}(\rho) = p_1 + p_2 - q \quad (q \neq p_i),$$
so that a holomorphic function π_t on N_0 can be defined by
$$\pi_t(z,\zeta) = \zeta^m \sigma(z)\left(1 - \frac{t}{\zeta\rho(z)}\right)$$
$$= \zeta^{m-1}\frac{\sigma(z)}{\rho(z)}(\rho(z)\zeta - t) \quad \text{near } z = p_i$$
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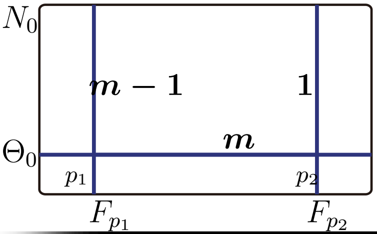
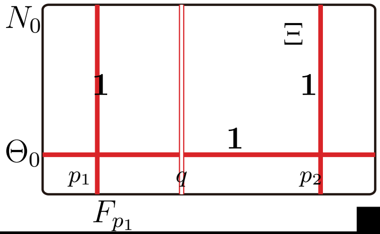
Then $\pi_t^{-1}(0) = (m-1)\Theta_0 + (m-2)F_{p_1} + \Xi$,
where Ξ is an irreducible component intersecting Θ_0 at q.





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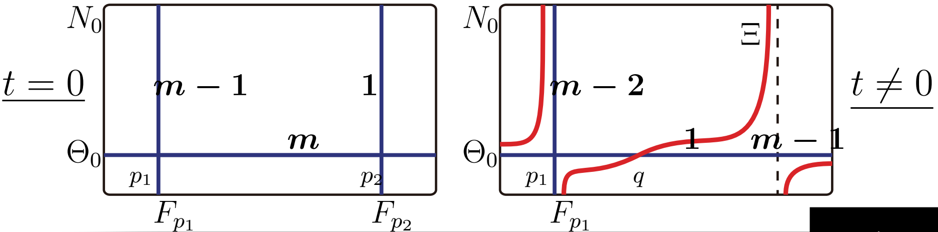
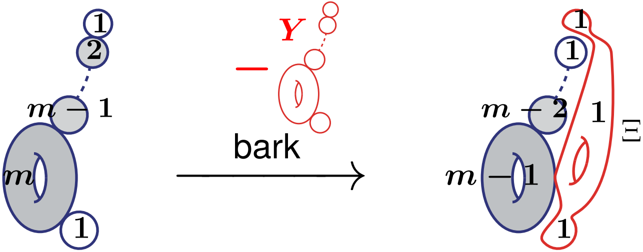
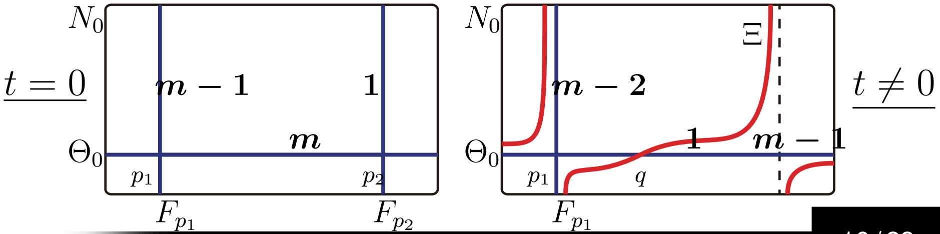
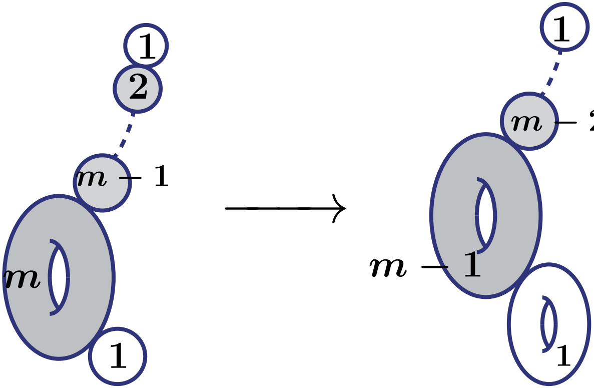


3. Propagate the deformation π_t on N_0
to obtain a deformation $\pi_t : M_t \to \Delta$ with $X^{(n+1)}_{m-1}$
(by deforming tubular nbhds of some irred components).



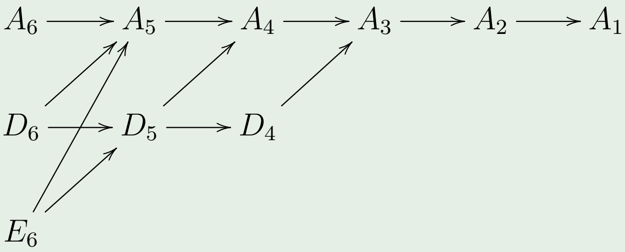


4. By taking a “good” section ρ, we can assume that $\pi_t|_{N_0}$ has Lefschetz singularities, and they lie distinct fibers.



Remark of Theorem

1. Generalization of Matsumoto's splitting for genus 2.
2. Analogous to adjacency diagrams of singularities:



3. Splitting of a singular fiber into Lefschetz fibers gives a **Dehn-twist expression** of its topological monodromy.

Dehn-twist expression of propeller automorphisms

Theorem (Dehn)

The mapping class group of a closed Riemann surface Σ

$$\mathrm{MCG}(\Sigma) = \mathrm{Homeo}^+(\Sigma)/\mathrm{isotopy}$$

is generated by finitely many Dehn twists.

Find **Dehn-twist expressions** of automorphisms

$$\varphi = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_k \text{ (up to isotopy).}$$

Ex. **Hyperelliptic involution** (J. Birman and H. Hilden)

$$\iota = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_{2g} \circ \tau_{2g+1} \circ \tau_{2g+1} \circ \tau_{2g} \circ \cdots \circ \tau_2 \circ \tau_1$$

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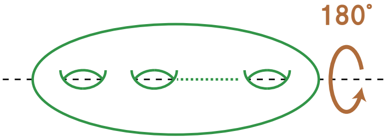

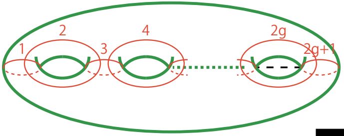

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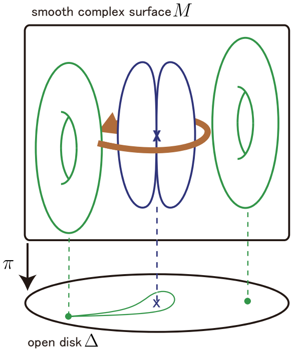

split →

— Geometric expressions —

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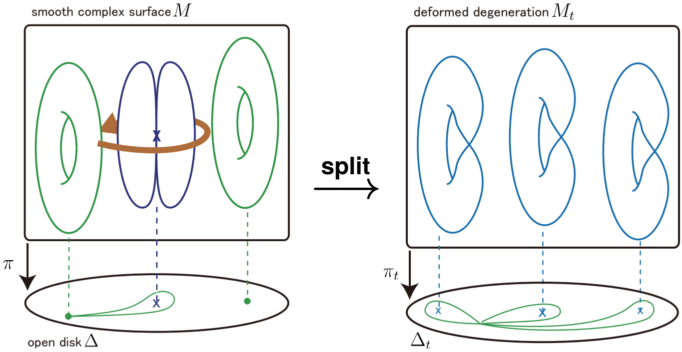

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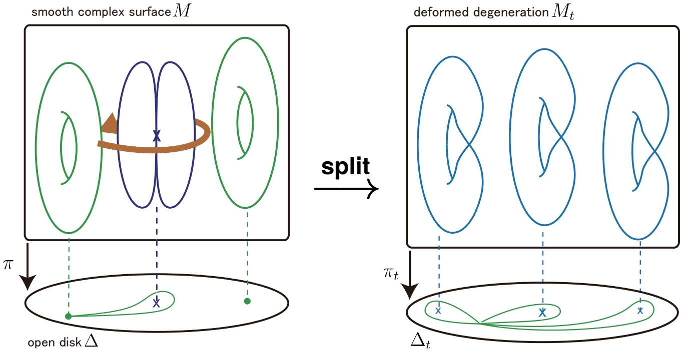


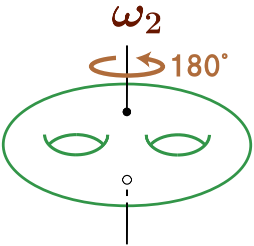
— Geometric expressions —

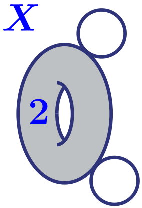
Theorem (Y. Matsumoto)

$\pi : M \to \Delta$: the degeneration of Riemann surfaces of genus 2 with monodromy ω_2

Then its singular fiber X can split into four Lefschetz fibers.

Moreover, their vanishing cycles are as depicted below.





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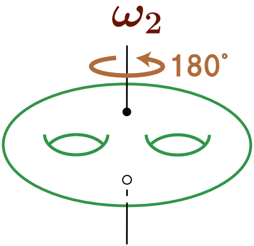
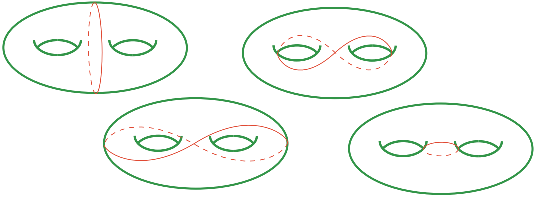
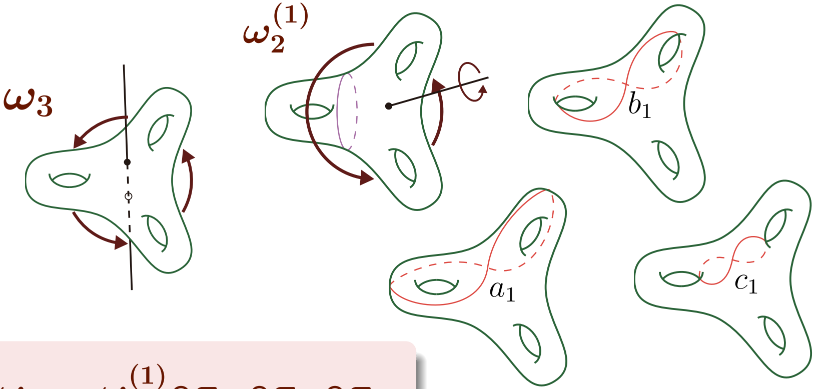




$$\omega_2 = \tau_0 \circ \tau_a \circ \tau_b \circ \tau_c$$

Lemma

$X_3^{(0)}$ can split into $X_2^{(1)}$ and three Lefshetz fibers, and their vanishing cycles are as depicted below.



$$\omega_3 = \omega_2^{(1)} \circ \tau_{a1} \circ \tau_{b1} \circ \tau_{c1}.$$

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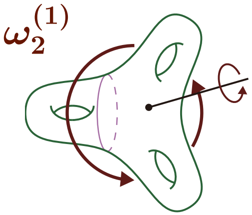
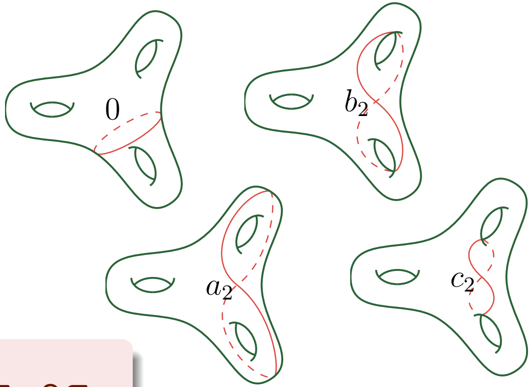
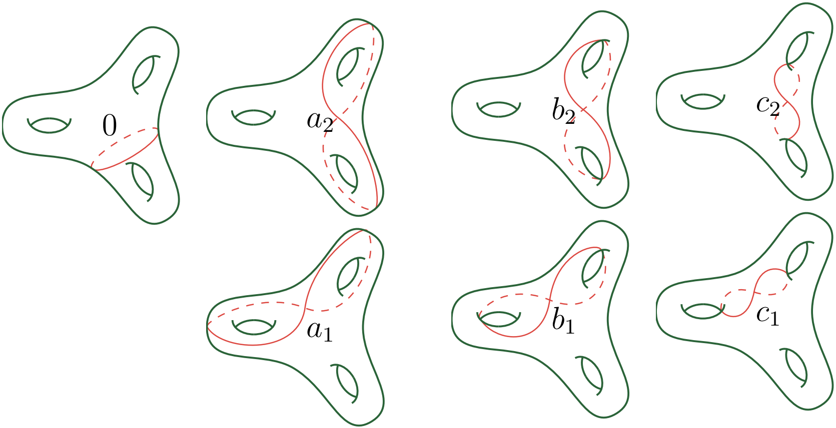




$$\omega_2^{(1)} = \tau_0 \circ \tau_{a2} \circ \tau_{b2} \circ \tau_{c2}.$$

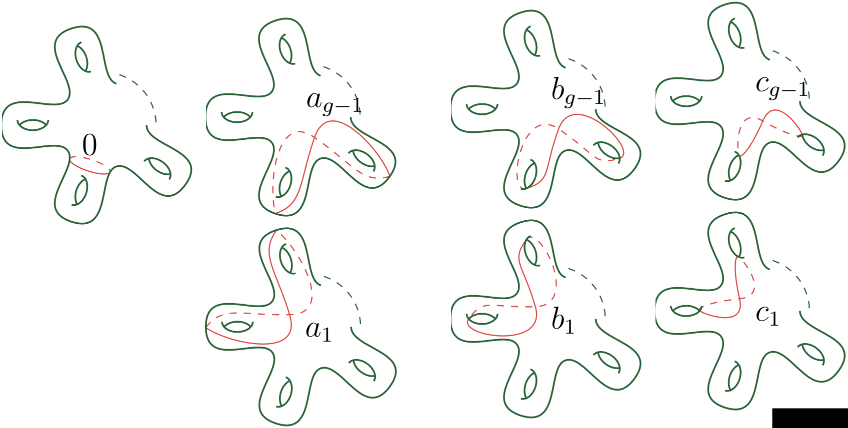
Proposition

$\omega_3 = \tau_0 \circ \tau_{a2} \circ \tau_{b2} \circ \tau_{c2} \circ \tau_{a1} \circ \tau_{b1} \circ \tau_{c1}$.



Theorem

$$\omega_g = \tau_0 \circ (\tau_{a_{g-1}} \circ \tau_{b_{g-1}} \circ \tau_{c_{g-1}}) \circ \cdots \circ (\tau_{a1} \circ \tau_{b1} \circ \tau_{c1}).$$



Thank you for your attention.