Extension theory for quasisymmetric embeddings of planar subsets

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Quasisymmetric mapping (qs.)

Definition

Let $\eta : [0, \infty) \to [0, \infty)$ *: homeo. , and* (X, d_X)*,* (Y, d_Y)*: metric spaces.*

An injection $f : X \to Y$ *is* η-**quasisymmetric (qs)** *if*

$$^{\forall}x, y, z \in X \ (x \neq y) \ ; \quad \frac{d_Y(f(x), f(z))}{d_Y(f(x), f(y))} \leq \eta\left(\frac{d_X(x, z)}{d_X(x, y)}\right).$$

Contents

- **The Beurling–Ahlfors theorem.**
- The Väisälä problem.

The Beurling–Ahlfors theorem

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, $S^1 = \{|z| = 1\}$.

FACT

$$F : \mathbb{D} \to \mathbb{D}: \textit{ qc} \quad \Rightarrow \quad F|_{S^1} : S^1 \to S^1: \textit{ qs}$$

Conversely:

Theorem (Beurling–Ahlfors, 1956, Acta Math., 96, pp.125–142)

$${}^{\forall} f : S^1 \to S^1: \textit{ qs}, \ {}^{\exists} F : \mathbb{D} \to \mathbb{D}: \textit{ qc extension.}$$

Conformally natural extension & Harmonic extension

Theorem (Douady–Earle, 1986, Acta Math., 157 pp.23–48)

$$^{\forall} f : S^1 \to S^1 \text{: qs, } ^{\exists} F_f : \mathbb{D} \to \mathbb{D} \text{: qc extension,}$$

$$\text{s.t. } ^{\forall} \gamma, \delta \in Möb(\mathbb{D}); \; F_{\gamma \circ f \circ \delta} = \gamma \circ F_f \circ \delta.$$

Theorem (Schoen conjecture, 1990∼)

$$^{\forall} f : S^1 \to S^1 \text{: qs, } ^{\exists !} F : \mathbb{D} \to \mathbb{D} \text{: qc extension harmonic w.r.t the Poincaré metric } g_{\mathbb{D}}.$$

- Uniqueness: Li–Tam (1993); Ann. Math. 137(1), pp.167–201 & Indiana Univ. Math. J. 42(2), pp.591–635.
- Existence: Markovic (2016); Caltech

Minimal Lagrangian extension

Theorem (Bonsante–Schlenker, 2010, Invent. Math., 182 (2), pp279–333)

$$^{\forall} f : S^1 \to S^1 \text{: } qs, \ ^{\exists !} F : \mathbb{D} \to \mathbb{D} \text{: } qc \text{ extension.}$$

such that

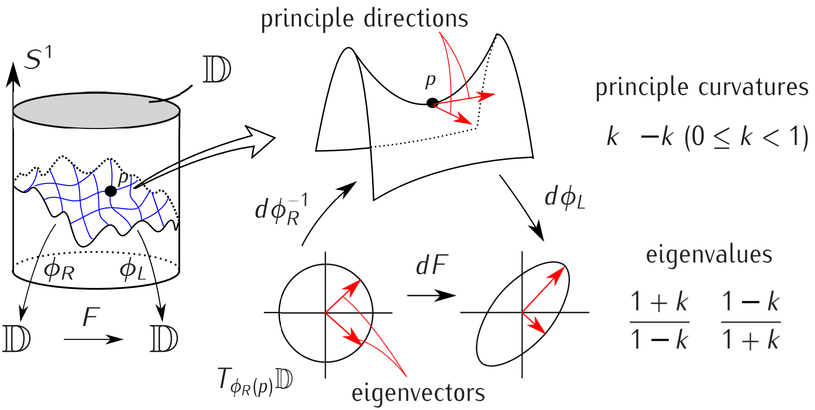
- *Area preserving w.r.t.* $g_{\mathbb{D}}$. *(i.e.* $Area(E) = Area(F(E))$*).*
- $graph(F) \subset \mathbb{D} \times \mathbb{D}$: *minimal (i.e. mean curvature* $H = 0$*).*

Key Observation

- $AdS_3 = \mathbb{D} \times S^1 \ni (z, \theta)$ with $g = g_{\mathbb{D}} - \phi(z)^2 d\theta^2$.
- $u : \mathbb{D} \to S^1$: smooth s.t. $M := graph(u) \subset AdS_3$: spacelike maximal surface.

$\Rightarrow$ There are two canonical projections $\phi_R, \phi_L : M \to \mathbb{D}$.

$\Rightarrow$ $F := \phi_L \circ \phi_R^{-1} : \mathbb{D} \to \mathbb{D}$: diffeo.

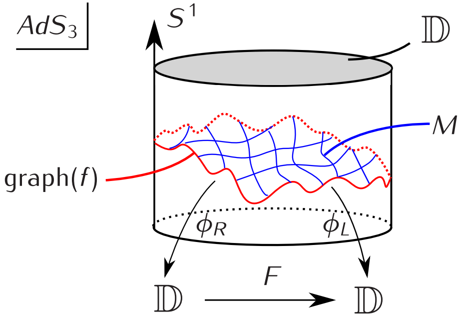


Theorem

If sectional curvature $K_M \leq^{\exists} c < 0 \Rightarrow F$*: qc.*

→ Eventually, Bonsante–Schlenker proved:

$$\{f : S^1 \to S^1 : qs\}$$
$$\updownarrow \quad 1:1$$
$$\{M \subset AdS_3 : \ max.\ graph\ with\ K_M \le c < 0\}$$
$$\updownarrow \quad 1:1$$
$$\{F : \mathbb{D} \to \mathbb{D} : \ area\ pres.\ qc\ with\ graph(F) : \ minimal\}$$



Higher dimensional case

- Beurling–Ahlfors (1956):

  ${}^{\forall} f : \mathbb{R} \to \mathbb{R}$: qs, ${}^{\exists} F : \mathbb{C} \to \mathbb{C}$: qc extension .

- Ahlfors (1964):

  ${}^{\forall} f : \mathbb{R}^2 \to \mathbb{R}^2$: qs, ${}^{\exists} F : \mathbb{R}^3 \to \mathbb{R}^3$: qc extension .

- Carleson (1974):

  ${}^{\forall} f : \mathbb{R}^3 \to \mathbb{R}^3$: qs, ${}^{\exists} F : \mathbb{R}^4 \to \mathbb{R}^4$: qc extension .

- Tukia–Väisälä (1982):

  ${}^{\forall} f : \mathbb{R}^n \to \mathbb{R}^n$: qs, ${}^{\exists} F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$: qc extension .

The Väisälä problem

- The Beurling–Ahlfors theorem.
- **The Väisälä problem.**

The Väisälä problem

Problem (The Väisälä 8th Problem (1995))

Does every η-qs embedding $f : E \hookrightarrow \mathbb{R}^n$ $(E \subset \mathbb{R}^n)$ admit a $K = K(\eta)$-qc extension $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$?

- Trotsenko–Väisälä (1999):

  If $E \subset \mathbb{R}^n$ is **not relatively connected**, then there is a qs embedding $f : E \hookrightarrow \mathbb{R}^n$, which does not admit a qc extension $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$.

Therefore, **the Väisälä 8th problem can not be solved affirmatively for general subsets $E \subset \mathbb{R}^n$ (even if $n = 1$)!**

Theorem (Trotsenko–Väisälä, 1999, Ann. Acad. Sci. Fenn. Math., 24 (2), pp.465–488)

For a metric space X, TFAE:

i. *X: relatively connected.*

ii. *Every qs emb. of X (into any metric spaces) is* **power qs**.

Definition

For $C \geq 1$, and $0 < \alpha \leq 1$,

$$f\colon (C,\alpha)\text{-}\textbf{power qs} :\Leftrightarrow f\colon \eta\text{-qs with } \eta(t) = C\max\{t^{\alpha}, t^{1/\alpha}\}.$$

Definition

*For a metric space X, and $c \geq 1$, X is c-***relatively connected** *if*

$$\forall x \in X, \forall r > 0 \text{ with } \overline{D}(x,r) \neq X;$$
$$\overline{D}(x,r) = \{x\}, \text{ or } \overline{D}(x,r) \setminus D(x,r/c) \neq \emptyset.$$

Recent advances

Theorem (F., 2016)

If $E = \mathbb{Z}$ or $\mathbb{N}$, then

$^{\forall} f : E \to \mathbb{R}$*: η-qs,* $^{\exists} F : \mathbb{C} \to \mathbb{C}$*: $K = K(\eta)$-qc extension.*

Theorem (Vellis, 2016, arXiv:1609.08763v1 [math.MG])

Suppose $E \subset \mathbb{C}$: c-rel. conn. and $\mathbb{C} \setminus \overline{E}$: c-uniform.

$^{\forall} f : E \to \mathbb{C}$*: η-qs which admits a homeo. extension to $\mathbb{C}$,*
$^{\exists} F : \mathbb{C} \to \mathbb{C}$*: $K = K(\eta, c)$-qc extension.*

Questions

Problem (The Väisälä 8th Problem (1995))

Does every η*-qs embedding* $f : E \hookrightarrow \mathbb{R}^n$ $(E \subset \mathbb{R}^n)$ *admit a* $K = K(\eta)$*-qc extension* $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$*?*

Question 1

Suppose $E \subset \mathbb{R}^n$*:* c-**rel. conn.**

Does every η*-qs embedding* $f : E \hookrightarrow \mathbb{R}^n$ $(E \subset \mathbb{R}^n)$ *admit a* $K = K(\eta, c)$*-qc extension* $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$*?*

Question 2

Does every (C, α)-**power qs** *embedding* $f : E \hookrightarrow \mathbb{R}^n$ $(E \subset \mathbb{R}^n)$ *admit a* $K = K(C, \alpha)$*-qc extension* $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$*?*

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