Weil-Petersson metric on the square integrable Teichmüller space

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1. $p$-integrable Teichmüller space

2. Weil-Petersson metric

3. Main results
Teichmüller space

\[ \mathbb{H} = \{ \text{Im} z > 0 \}: \text{upper half-plane} \]
\[ \Gamma: \text{Fuchsian group on } \mathbb{H} \text{ (discrete subgroup of } \text{Möb}(\mathbb{H})) \]
\[ L^\infty(\mathbb{H}, \Gamma) = \{ \mu: \text{measurable } (-1, 1)-\text{differential on } \mathbb{H} \mid \| \mu \|_\infty < \infty \} \]

\[ \mu(\gamma(z))(\gamma'(z))^{-1}\gamma'(z) = \mu(z) \quad \forall z \in \mathbb{H} \]

\[ \| \mu \|_\infty = \text{ess sup}_{z \in N} |\mu(z)| \quad (N: \text{fundamental region}) \]
Teichmüller space

\[ \text{Bel}(\mathbb{H}, \Gamma) = \{ \mu \in L^\infty(\mathbb{H}, \Gamma) ||\mu||_\infty < 1 \}: \text{space of Beltrami coefficients} \]

Fact (measurable Riemann mapping theorem)

For every \( \mu \in \text{Bel}(\mathbb{H}, \Gamma) \), \( \exists f^\mu : \mathbb{H} \to \mathbb{H} \) sense-pres. homeo. s.t.

\[ \bar{\partial} f^\mu(z) = \mu(z) \partial f^\mu(z) \quad \text{a.e. } z \in \mathbb{H} \]

\( f^\mu \) is called a quasiconformal mapping. We normalize \( f^\mu \) by fixing 0, 1, \( \infty \).
Teichmüller space

**Teichmüller space of \( \Gamma \)**

\[
T(\Gamma) = \frac{\text{Bel}(\mathbb{H}, \Gamma)}{\sim_T}
\]

- \( \mu \sim_T \nu \iff f^\mu|_\mathbb{R} = f^\nu|_\mathbb{R} \)
- if \( \mu \sim_T \nu \), then \( \Gamma^\mu(= f^\mu \Gamma (f^\mu)^{-1}) = \Gamma^\nu \).

\([\mu] : \) Teichmüller equivalence class represented by \( \mu \in \text{Bel}(\mathbb{H}, \Gamma) \)

0 := [0] : base point of \( T(R) \)

\( \Gamma^\tau := \Gamma^\mu \) for \( \tau = [\mu] \in T(\Gamma) \)
Teichmüller space

Basic fact

Theorem

For every Fuchsian group $\Gamma$, the Teichmüller space $T(\Gamma)$ has a complex structure modeled on the Banach space $B(\mathbb{H}^*, \Gamma)$.

- $\mathbb{H}^* = \{\text{Im } z < 0\}$: lower half-plane
- $\rho_{\mathbb{H}^*}(z) = (-2 \text{Im } z)^{-1}$: Poincaré metric on $\mathbb{H}^*$
- $B(\mathbb{H}^*, \Gamma) = \{\varphi : \text{hol. (2, 0)-diff. on } \mathbb{H}^* | \|\varphi\|_\infty < \infty\}$

$$\varphi(\gamma(z))\gamma'(z)^2 = \varphi(z) \quad \forall z \in \mathbb{H}^*$$

$$\|\varphi\|_\infty = \sup_{z \in N^*} |\varphi(z)|\rho_{\mathbb{H}^*}(z)^{-2}(N^* : \text{fundamental region})$$

$$\beta : T(\Gamma) \to B(\mathbb{H}^*, \Gamma) \text{ homeo. (Bers embedding)}$$
$p$-integrable Teichmüller space

$p$-integrable Teichmüller space ($p \geq 1$)

$T^p(\Gamma) = \{ \tau \in T(\Gamma) | \exists \mu \in \tau \text{ s.t. } \mu \in \text{Ael}^p(\mathbb{H}, \Gamma) \}$

- $\rho_\mathbb{H}(z) = (2 \text{Im } z)^{-2}$: Poincaré metric on $\mathbb{H}$
- $L^p(\mathbb{H}, \Gamma) = \{ \mu : \text{measurable (-1, 1)-differential} \|\| \mu \|_p < \infty \}$

$$\|\mu\|_p = \left( \int_{\mathbb{H}} |\mu(z)|^p \rho_\mathbb{H}(z)^2 dx dy \right)^{\frac{1}{p}} < \infty$$

- $\text{Ael}^p(\mathbb{H}, \Gamma) = \text{Bel}(\mathbb{H}, \Gamma) \cap L^p(\mathbb{H}, \Gamma)$: space of $p$-integrable Beltrami coefficients
Fact

If \( \Gamma \) is cofinite (i.e. \( \mathbb{H}/\Gamma \) has a finite hyperbolic area), then

\[
T^p(\Gamma) = T(\Gamma) \quad \forall p \geq 1.
\]

Hence

This study is significant for coinfinite type.

Remark. \( T(\Gamma) \) is an infinite dimensional manifold for \( \Gamma \) of coinfinite type.
$p$-integrable Teichmüller space

Fact

If $\Gamma$ is cofinite (i.e. $\mathbb{H}/\Gamma$ has a finite hyperbolic area), then

$$T^p(\Gamma) = T(\Gamma) \quad \forall p \geq 1.$$ 

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This study is significant for coinfinite type.

Remark. $T(\Gamma)$ is an infinite dimensional manifold for $\Gamma$ of coinfinite type.
History

2000  G. Cui considered $T^2(1)$ ($1 = \{id_H\}$: trivial group).

- Introduction of a complex structure on $T^2(1)$
- Completeness of Weil-Petersson metric on $T^2(1)$

2000  L. A. Takhtajan, L.-P. Teo also considered $T^2(1)$.

- Kählerity of Weil-Petersson metric on $T^2(1)$
- Curvatures of Weil-Petersson metric on $T^2(1)$

2013  S. Tang extended the arguments of Cui to $p \geq 2$.

2014  D. Radnell, E. Schippers, W. Staubach introduced a complex structure of $T^2(\Gamma)$ for $\mathbb{H}/\Gamma$: $(g, n)$-type bordered

Purpose of this study

We extend their arguments to more general Fuchsian groups.
Let $\Gamma$ be a Fuchsian group with Lehner’s condition and $p \geq 2$. Then the $p$-integrable Teichmüller space $T^p(\Gamma)$ has a complex structure modeled on the Banach space $A^p(\mathbb{H}^*, \Gamma)$.

\[ A^p(\mathbb{H}^*, \Gamma) = \{ \varphi : \text{hol. (2, 0)-diff.} \| \varphi \|_p < \infty \} \]

\[ \| \varphi \|_p = \left( \int \int_{N^*} |\varphi(z)|^p \rho_{\mathbb{H}^*}(z)^{2-2p} \, dx \, dy \right)^{\frac{1}{p}} \]

→ We can consider the complex analytic structure of $p$-integrable Teichmüller spaces.
History


Let $\Gamma$ be a Fuchsian group with Lehner’s condition and $p \geq 2$. Then the $p$-integrable Teichmüller space $T^p(\Gamma)$ has a complex structure modeled on the Banach space $A^p(\mathbb{H}^*, \Gamma)$.

$$A^p(\mathbb{H}^*, \Gamma) = \{ \varphi : \text{hol. } (2, 0)\text{-diff.} \| \varphi \|_p < \infty \}$$

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→ We can consider the complex analytic structure of $p$-integrable Teichmüller spaces.
Lehner’s condition

A Fuchsian group $\Gamma$ satisfies Lehner’s condition

$$\text{def} \iff \inf\{\text{length of simple closed geodesics on } \mathbb{H}/\Gamma\} > 0.$$ 


A Fuchsian group $\Gamma$ satisfies Lehner’s condition if and only if

$$A^p(\mathbb{H}^*, \Gamma) \subset B(\mathbb{H}^*, \Gamma) \exists \forall p \geq 1.$$
1. $p$-integrable Teichmüller space

2. Weil-Petersson metric

3. Main results
Holomorphic tangent vector space on $T^p(\Gamma)$

Harmonic Beltrami differential

For $\mu \in L^{p,\infty}(\mathbb{H}, \Gamma^\tau)$,

$$H^\tau[\mu](z) = -2\rho_\mathbb{H}(z)^{-2}D\Phi_\tau(0)\mu(\bar{z}) \quad (z \in \mathbb{H})$$

is called the harmonic Beltrami differential for $\mu$.

$\Phi_\tau$: lift of $\beta_\tau : T^p(\Gamma^\tau) \rightarrow A^p(\mathbb{H}^*, \Gamma^\tau)$ for the Teichmüller projection

$\varpi_\tau : \text{Ael}^p(\mathbb{H}, \Gamma^\tau) \rightarrow T^p(\Gamma^\tau)$

Set $\text{HB}^p(\mathbb{H}, \Gamma^\tau) = H^\tau(L^{p,\infty}(\mathbb{H}, \Gamma^\tau))$.

$H^\tau : L^{p,\infty}(\mathbb{H}, \Gamma) \rightarrow L^{p,\infty}(\mathbb{H}, \Gamma)$ bdd. lin. operator
Holomorphic tangent vector space on $T^p(\Gamma)$

For $\tau \in T^p(\Gamma)$,

$$T_\tau T^p(\Gamma) \simeq A^p(\mathbb{H}^*, \Gamma^\tau) \simeq HB^p(\mathbb{H}, \Gamma^\tau).$$

Hereafter, let $p = 2$.

Hence, $T_\tau T^2(\Gamma)$ has the Hermitian inner product on $HB^2(\mathbb{H}, \Gamma^\tau)$ as

$$h^\tau(\mu, \nu) = \iint_{N^\tau} \mu(z)\overline{\nu(z)}\rho_{\mathbb{H}}(z)^2 dx dy \quad (\mu, \nu \in HB^2(\mathbb{H}, \Gamma^\tau))$$
Weil-Petersson metric

For a nbd. $\tau \in U_0$ of the base point $0$ and $\mu, \nu \in \text{HB}^2(\mathbb{H}, \Gamma)$,

$$h_{WP}(\tau)(\mu, \nu) = h^\tau (H^\tau \circ L^\varsigma(\tau)[\mu], H^\tau \circ L^\varsigma(\tau)[\nu]).$$

- $\varsigma(\tau) \in \omega^{-1}_0(U_0) \subset \text{HB}^2(\mathbb{H}, \Gamma)$
- $L^\varsigma(\tau) : L^{2,\infty}(\mathbb{H}, \Gamma) \rightarrow L^{2,\infty}(\mathbb{H}, \Gamma^\tau)$ certain Banach isomorphism

Remark. For every $\eta \in T^2(\Gamma)$, $h_{WP}$ can be defined on a nbd. $U_\eta$ of $\eta$ similarly.
Weil-Petersson metric

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$$h_{WP}(\tau)(\mu, \nu) = h^\tau(H^\tau \circ L^{\varsigma(\tau)}[\mu], H^\tau \circ L^{\varsigma(\tau)}[\nu]).$$

- $\varsigma(\tau) \in \mathcal{W}_0^{-1}(U_0) \subset \text{HB}^2(\mathbb{H}, \Gamma)$
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**Remark.** For every $\eta \in T^2(\Gamma)$, $h_{WP}$ can be defined on a nbd. $U_\eta$ of $\eta$ similarly.
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3. Main results
Kähler metric

**Hermitian metric**

\( M \): complex Hilbert manifold

\( h_p \): Hermitian inner product on \( T_p M \) (\( p \in M \))

For every \( C^\infty \)-vector fields \( \xi, \eta \) on \( M \), if the function

\[ \tilde{h}(p) = h_p(\xi(p), \eta(p)) \]

is of class \( C^\infty \), \( h = \{h_p\}_{p \in M} \) is called a **Hermitian metric** on \( M \).
Kähler metric

(M, h): Hermitian manifold

The function \( \omega = -2 \text{Im} \ h \) is a real 2-form on \( M \).

\[
h \text{ is a Kähler metric } \iff d\omega \equiv 0
\]

Remark. (Darboux’s theorem)

If \( M \) is a \( n \)-dimensional Kähler manifold, for every \( p \in M \), there exists a \( C^\infty \)-function \( f \) on a neighborhood of \( p \) s.t.

\[
\omega = \partial \bar{\partial} f = \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k.
\]
**Theorem (Ahlfors 1961)**

Let $\Gamma$ be an arbitrary cocompact Fuchsian group.

(i) The Weil-Petersson metric $h_{\text{WP}}$ on $T(\Gamma)$ is Kähler;

(ii) The holomorphic sectional curvature and Ricci curvature of $h_{\text{WP}}$ are negative.

Takhtajan-Teo showed this result is also valid for $T^2(1)$. 
Can we extend this theorem to Fuchsian groups with Lehner’s condition?

If we can do that, $T^2(\Gamma)$ has fruitful properties as the Kähler manifold.
How should we overcome the problem?

Can we apply Ahlfors’s proofs?
characterization of Kählerity

proposition (finite dim. version)

let \((M, h)\) be a hermitian manifold \((n = \dim M < \infty)\) and 
\[ h_{k\ell} = h\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_{\ell}}\right). \]
then the following conditions are equivalent:

(i) \(h\) is a Kähler metric on \(M\);

(ii) for \(\forall j, k, \ell = 1, \ldots, n,\)
\[
\frac{\partial h_{k\ell}}{\partial z_j} = \frac{\partial h_{j\ell}}{\partial z_k};
\]

(iii) for \(\forall p \in M, \exists (w_1, \ldots, w_n)\): local coordinate s.t. \(w(p) = 0,\)
\[ h_{k\ell}(0) = \delta_{k\ell} \quad \text{and} \]
\[
\frac{\partial h_{k\ell}}{\partial w_j}(0) = 0 \quad \forall j, k, \ell = 1, \ldots, n.
\]

M. Yanagishita (Waseda Univ.)
Theorem (infinite dim. version, Y. 2015?)

Let \((M, h)\) be a Hermitian manifold. Then the following conditions are equivalent:

(i) \(h\) is a Kähler metric on \(M\);
(ii) For every \(\xi, \eta, \zeta \in \Gamma(TM)\),
\[
D'h\xi(\eta, \zeta) = D'h\eta(\xi, \zeta);
\]
(iii) For \(\forall p \in M\), \(\exists (w_1, \cdots, w_n)\): local coordinate s.t. \(w(p) = 0\) and
\[
h(w) = \langle \cdot, \cdot \rangle + o(\|w\|).
\]

Remark. In this talk, we use the representation of finite dim. version for simplicity of explanation.
Rough sketch of proof of Kählerity

The computation itself is same as Ahlfors’s one.

\[
\frac{\partial h_{k\bar{\ell}}}{\partial t_j} = -\frac{24}{\pi^2} \int \int_{N(t)} \left\{ \int \int_{\mathbb{H}} \nu_k(t)(\zeta)T_j(z, \zeta)K(\zeta, \bar{z})d^2\zeta \right\} \nu_\ell(t)(z)d^2z
\]

\[
= -\frac{24}{\pi^2} \int \int_{N(t)} \left\{ \int \int_{\mathbb{H}} \nu_j(t)(w)T_k(z, w)K(w, \bar{z})d^2w \right\} \nu_\ell(t)(z)d^2z
\]

\[
= \frac{\partial h_{j\bar{\ell}}}{\partial t_k},
\]

Here, \(d^2z = dx dy\) and

\[
T_j(z, \zeta) = \int \int_{\mathbb{H}} \nu_j(t)(w)K(w, \zeta)K(w, \bar{z})d^2w,
\]

\[
\nu_j(t) = L^{\nu(t)}[\nu_j] \quad (t \in \ell^2(\mathbb{C}), \nu(t) = \sum_{j=1}^{\infty} t_j \nu_j \in \text{HB}^2(\mathbb{H}, \Gamma)),
\]

\[
N(t) = f^{\nu(t)}(N), \quad K(\zeta, z) = (\zeta - z)^{-2}.
\]
We have to check the commutativity between the signs of differentiation and integration.

It is necessary to show the expression

$$\int \int_{N(t)} \int \int_{\mathbb{H}} |\nu_k(t)(\zeta)\nu_\ell(t)(z)T_j(z, \zeta)K(\zeta, \bar{z})|d^2\zeta d^2z$$

converges uniformly on $t$ in a nbd. of 0.

Ahlfors used the finiteness of the hyperbolic area of $\mathbb{H}/\Gamma$. 
We have to check the commutativity between the signs of differentiation and integration. It is necessary to show the expression

\[
\int \int_{N(t)} \int \int_{\mathbb{H}} |\nu_k(t)(\zeta)\nu_\ell(t)(z)T_j(z, \zeta)K(\zeta, \bar{z})| \, d^2\zeta \, d^2z
\]

converges uniformly on \( t \) in a nbd. of 0.

Ahlfors used the finiteness of the hyperbolic area of \( \mathbb{H}/\Gamma \).
For $r > e$, let $u(r) = \log \log r$ and

\[ D_1(t, r) = N(t) \cap B_h(i, u(r)), \quad D_2(r) = \mathbb{H} \cap \{|z| < r\}. \]

Here, $B_h(i, u(r))$ is the hyperbolic disk centered at $i$ of radius $u(r)$.

Then,

\[ (N(t) \times \mathbb{H}) \setminus (D_1(t, r) \times D_2(r)) = (D_1(t, r)^c \times \mathbb{H}) \sqcup (D_1(t, r) \times D_2(t)^c). \]

($1$) \hspace{3cm} ($2$)

\[ \rightarrow \text{It is sufficient to show both of the integrations on (1) and (2) converges to 0 as } r \to \infty \text{ uniformly on } t. \]
For $r > e$, let $u(r) = \log \log r$ and

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Here, $B_h(i, u(r))$ is the hyperbolic disk centered at $i$ of radius $u(r)$. Then,

$$(N(t) \times \mathbb{H}) \setminus (D_1(t, r) \times D_2(r)) = (D_1(t, r)^c \times \mathbb{H}) \sqcup (D_1(t, r) \times D_2(t)^c).$$

$\rightarrow$ It is sufficient to show both of the integrations on (1) and (2) converges to 0 as $r \to \infty$ uniformly on $t$. 


Rough sketch of proof of Kählerity

(1) integration on $D_1(t, r)^c \times \mathbb{H}$

It follows from Ahlfors’s computation and the lemma as follows:

**Lemma A (Takhtajan-Teo 2006)**

Let $0 < a < 1$. Then, there exists a constant $C(a) > 0$ s.t. for every $z \in \mathbb{H}$ and every $\mu \in H(L^\infty(\mathbb{H}, \Gamma))$ with $\|\mu\|_\infty < a$,

$$|\partial f^\mu(z)|^2 \rho_H(f^\mu(z))^2 \leq C(a) \rho_H(z)^2.$$ 

$$\int \int_{D_1(t, r)^c} |\nu_j(t)(z)|^2 \rho_H(z)^2 d^2 z \leq \frac{C(a)}{(1 - a^2)^2} \|\nu_j|_{N \setminus (f^\nu(t))^{-1}(B_h(i, u(r)))}\|^2 \rightarrow 0 \quad (r \rightarrow \infty).$$
(2) integration on $D_1(t, r) \times D_2(r)^c$, 

Ahlfors’s computation, Lemma A and the lemma as follows:

**Lemma B (Y. 2014)**

\[
F(r) = \int \int_{B_h(i,u(r))} \int \int_{D_2(r)} \frac{d^2 \zeta d^2 z}{|\zeta - \bar{z}|^4} \to 0 \quad (r \to \infty).
\]

(integration on $D_1(t, r) \times D_2(r)^c$) \leq \frac{\pi^2 C(a) \frac{1}{2} \|\nu_\ell\|_2 \|\nu_j\|_\infty \|\nu_k\|_\infty}{(1 - a^2)^3} F(r) \to 0 \quad (r \to \infty).
Rough sketch of proof of negativity

Negativity of curvature

The hol. sec. curvature (resp. the Ricci curvature) is negative if

$$\text{HSec}(0)(\nu(t)) < 0 \quad \text{resp.} \quad \text{Ric}(0)(\nu(t)) < 0$$

for every $t \in \ell^2(\mathcal{C}) \setminus \{0\}$. 
Rough sketch of proof of negativity

From condition (iii),

$$h_{k\ell}(0) = \delta_{k\ell}, \quad \frac{\partial h_{k\ell}}{\partial t_j}(0) = 0 \quad \forall j, k, \ell = 1, \ldots.$$  

Then for $\nu(t) = \sum_{j=1}^{\infty} t_j \nu_j \in \operatorname{HB}(\mathbb{H}, \Gamma) \setminus \{0\}$,

$$\operatorname{HSec}(0)(\nu(t)) = \sum_{j,k,\ell,m=1}^{\infty} \frac{\partial^2 h_{j\bar{k}}}{\partial t_\ell \partial \bar{t}_m}(0) t_j \bar{t}_k t_\ell \bar{t}_m,$$

$$\operatorname{Ric}(0)(\nu(t)) = \sum_{\ell=1}^{\infty} \left( \sum_{j,k=1}^{\infty} \frac{\partial^2 h_{j\bar{k}}}{\partial t_\ell \partial \bar{t}_k}(0) t_j \bar{t}_k \right).$$
Rough sketch of proof of negativity

\[ \text{HSec}(0)(\nu(t)) = -\frac{3}{\pi^5} \int_{N \times \mathbb{H}} G d^2 \zeta d^2 z \leq 0. \]

Here,

\[ G = \left| \sum_{j,k=1}^{\infty} (L_{j\ell}(\bar{\zeta}, \bar{z}) + L_{\ell j}(\bar{\zeta}, \bar{z})) t_j t_\ell \right|^2 + 2 \left| \sum_{j,k=1}^{\infty} L_{j\bar{k}}(\zeta, \bar{z}) t_j \bar{t}_k \right|^2, \]

\[ L_{j\bar{k}}(\zeta, \bar{z}) = \int_{\mathbb{H}^2} K(z_1, \bar{\zeta}) K(z_1, w_1) K(w_1, \bar{z}) \nu_j(z_1) \nu_k(w_1) d^2 z_1 d^2 w_1, \]

\[ L_{j\bar{k}}(\zeta, \bar{z}) = \int_{\mathbb{H}^2} K(z_2, \zeta) K(z_2, \bar{w}_2) K(\bar{w}_2, \bar{z}) \nu_j(z_2) \overline{\nu_k(w_2)} d^2 z_2 d^2 w_2. \]
Suppose $\text{HSec}(0)(\nu(t)) = 0$. Then for every $\zeta, z \in \mathbb{H}$,

$$
\sum_{j,k=1}^{\infty} L_{j,k}(\zeta, \bar{\zeta}) t_j \bar{t}_k = 0 \quad \forall j, k = 1, \ldots.
$$

Hence,

$$
0 = \int_{\mathbb{H}} \int_{\mathbb{H}} \sum_{j,k=1}^{\infty} L_{j,k}(\zeta, \bar{\zeta}) t_j \bar{t}_k d^2 \zeta = -\frac{1}{3} \|\nu(t)\|_2.
$$

This contradicts the assumption $\nu(t) \neq 0$.

Similarly, it follows $\text{Ric}(0)(\nu(t)) < 0$. 
Rough sketch of proof of negativity

Obstruction

Is this computation valid for infinite dim. manifolds?

- Hol. sec. curvature \( \cdots \) Ahlfors’s computation + Lemma A, B
- Ricci curvature \( \cdots \) These tools + Theory of reproducing kernel Hilbert space
Rough sketch of proof of negativity

For \( \nu(t) = \sum_{j=1}^{\infty} t_j \nu_j \in \text{HB}(\mathbb{H}, \Gamma) \setminus \{0\}, \)

\[
\text{HSec}(0)(\nu(t)) = \sum_{j,k,\ell,m=1}^{\infty} \frac{\partial^2 h_{j\bar{k}}}{\partial t_\ell \partial \bar{t}_m}(0) t_j \bar{t}_k t_\ell \bar{t}_m
\]

\[
= -D' D'' h(0)(\nu(t), \nu(t))(\nu(t), \nu(t)),
\]

\[
\text{Ric}(0)(\nu(t)) = \sum_{\ell=1}^{\infty} \left( \sum_{j,k=1}^{\infty} \frac{\partial^2 h_{j\bar{\ell}}}{\partial t_\ell \partial \bar{t}_k}(0) t_j \bar{t}_k \right)
\]

\[
= -\text{Re} \sum_{\ell=1}^{\infty} D' D'' h(0)(\nu_\ell, \nu(t))(\nu(t), \nu_\ell).
\]
Rough sketch of proof of negativity

Fact

The Hilbert space $\mathcal{H}_B(\mathbb{H}, \Gamma)$ has the reproducing kernel

$$Q_\Gamma(z, \zeta) = \frac{3}{\pi} \rho_\mathbb{H}(z)^{-2} \rho_\mathbb{H}(\zeta)^{-2} \sum_{\gamma \in \Gamma} \frac{\gamma'(\zeta)^2}{(\bar{z} - \gamma(\zeta))^4},$$

that is,

$$\nu(z) = \int \int_N \nu(\zeta) Q_\Gamma(z, \zeta) d^2\zeta$$

for every $\nu \in \mathcal{H}_B^2(\mathbb{H}, \Gamma)$.

It follows that

$$Q_\Gamma(z, \zeta) = \sum_{\ell=1}^{\infty} \nu_\ell(z) \overline{\nu_\ell(\zeta)}.$$
A Fuchsian group satisfies Lehner’s condition if and only if

\[ \sup_{z \in \mathbb{H}} |Q_{\Gamma}(z, z)| < \infty. \]

The convergence of Ric(0)(\nu(t)) is shown from this proposition.
Let $\Gamma$ be a Fuchsian group with Lehner’s condition.

(i) The Weil-Petersson metric $h_{WP}$ on $T^2(\Gamma)$ is Kähler;
(ii) The holomorphic sectional curvature and Ricci curvature of $h_{WP}$ are negative.

↑ because

We can apply Ahlfors’s method.


References


Thank you for your kind attention!