Weil-Petersson metric on the square integrable Teichmüller space

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Teichmüller space

$\longrightarrow$ Deformation space of Fuchsian groups (Riemann surfaces)

- $\mathbb{H} = \{\mathrm{Im}\, z > 0\}$: upper half-plane
- Γ: Fuchsian group on $\mathbb{H}$ (discrete subgroup of Möb($\mathbb{H}$))
- $L^\infty(\mathbb{H}, \Gamma) = \{\mu$: measurable (-1, 1)-differential on $\mathbb{H} \mid \|\mu\|_\infty < \infty\}$

$$\mu(\gamma(z))(\gamma'(z))^{-1}\overline{\gamma'(z)} = \mu(z) \quad \forall z \in \mathbb{H}$$

$$\|\mu\|_\infty = \operatorname*{ess\,sup}_{z \in N} |\mu(z)| \quad (N: \text{ fundamental region})$$

$\mathrm{Bel}(\mathbb{H}, \Gamma) = \{\mu \in L^\infty(\mathbb{H}, \Gamma) | \|\mu\|_\infty < 1\}$: space of Beltrami coefficients

Fact (measurable Riemann mapping theorem)

For every $\mu \in \mathrm{Bel}(\mathbb{H}, \Gamma)$, $\exists f^\mu : \mathbb{H} \to \mathbb{H}$ sense-pres. homeo. s.t.

$$\bar{\partial} f^\mu(z) = \mu(z) \partial f^\mu(z) \quad \text{a.e. } z \in \mathbb{H}$$

f^μ is called a quasiconformal mapping. We normalize f^μ by fixing 0, 1, ∞.

Teichmüller space of Γ

$$T(\Gamma) = \mathrm{Bel}(\mathbb{H}, \Gamma)/\sim_T$$

- $\mu \sim_T \nu \overset{\text{def}}{\Longleftrightarrow} f^\mu|_{\mathbb{R}} = f^\nu|_{\mathbb{R}}$
- if $\mu \sim_T \nu$, then $\Gamma^\mu (= f^\mu \Gamma (f^\mu)^{-1}) = \Gamma^\nu$.

$[\mu]$: Teichmüller equivalence class represented by $\mu \in \mathrm{Bel}(\mathbb{H}, \Gamma)$
$0 := [0]$: base point of $T(R)$
$\Gamma^\tau := \Gamma^\mu$ for $\tau = [\mu] \in T(\Gamma)$

Basic fact

Theorem

For every Fuchsian group Γ, the Teichmüller space $T(\Gamma)$ has a complex structure modeled on the Banach space $B(\mathbb{H}^*, \Gamma)$.

- $\mathbb{H}^* = \{\operatorname{Im} z < 0\}$: lower half-plane
- $\rho_{\mathbb{H}^*}(z) = (-2 \operatorname{Im} z)^{-1}$: Poincaré metric on $\mathbb{H}^*$
- $B(\mathbb{H}^*, \Gamma) = \{\varphi :$ hol. (2, 0)-diff. on $\mathbb{H}^* | \|\varphi\|_\infty < \infty\}$

$$\varphi(\gamma(z))\gamma'(z)^2 = \varphi(z) \quad \forall z \in \mathbb{H}^*$$

$$\|\varphi\|_\infty = \sup_{z \in N^*} |\varphi(z)| \rho_{\mathbb{H}^*}(z)^{-2} (N^* : \text{ fundamental region})$$

$$\beta : T(\Gamma) \to B(\mathbb{H}^*, \Gamma) \text{ homeo. (Bers embedding)}$$

p-integrable Teichmüller space

p-integrable Teichmüller space ($p \geq 1$)

$$T^p(\Gamma) = \{\tau \in T(\Gamma) | \exists \mu \in \tau \text{ s.t. } \mu \in \mathrm{Ael}^p(\mathbb{H}, \Gamma)\}$$

- $\rho_{\mathbb{H}}(z) = (2\,\mathrm{Im}\,z)^{-2}$: Poincaré metric on $\mathbb{H}$
- $L^p(\mathbb{H}, \Gamma) = \{\mu :$ measurable (-1, 1)-differential$\,|\, \|\mu\|_p < \infty\}$

$$\|\mu\|_p = \left(\iint_N |\mu(z)|^p \rho_{\mathbb{H}}(z)^2 dxdy \right)^{\frac{1}{p}} < \infty$$

- $\mathrm{Ael}^p(\mathbb{H}, \Gamma) = \mathrm{Bel}(\mathbb{H}, \Gamma) \cap L^p(\mathbb{H}, \Gamma)$: space of p-integrable Beltrami coefficients

p-integrable Teichmüller space

Fact

If Γ is cofinite (i.e. $\mathbb{H}/\Gamma$ has a finite hyperbolic area), then

$$T^p(\Gamma) = T(\Gamma) \quad \forall p \geq 1.$$

$\downarrow$ Hence

This study is significant for coinfinite type.

Remark. $T(\Gamma)$ is an infinite dimensional manifold for Γ of coinfinite type.

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Remark. $T(\Gamma)$ is an infinite dimensional manifold for Γ of coinfinite type.

History

2000 G. Cui considered $T^2(1)$ ($1 = \{id_{\mathbb{H}}\}$: trivial group).

- Introduction of a complex structure on $T^2(1)$
- Completeness of Weil-Petersson metric on $T^2(1)$

2000 L. A. Takhtajan, L.-P. Teo also considered $T^2(1)$.

- Kählerity of Weil-Petersson metric on $T^2(1)$
- Curvatures of Weil-Petersson metric on $T^2(1)$

2013 S. Tang extended the arguments of Cui to $p \geq 2$.

2014 D. Radnell, E. Schippers, W. Staubach introduced a complex structure of $T^2(\Gamma)$ for $\mathbb{H}/\Gamma$: (g, n)-type bordered

Purpose of this study

We extend their arguments to more general Fuchsian groups.

History

Theorem (Cui 2000, Takhtajan-Teo 2000, Tang 2013, Y. 2014)

Let Γ be a Fuchsian group with Lehner's condition and $p \geq 2$.
Then the p-integrable Teichmüller space $T^p(\Gamma)$ has a complex structure modeled on the Banach space $A^p(\mathbb{H}^*, \Gamma)$.

$$A^p(\mathbb{H}^*, \Gamma) = \{\varphi : \text{ hol. (2, 0)-diff.} \mid \|\varphi\|_p < \infty\}$$

$$\|\varphi\|_p = \left(\iint_{N^*} |\varphi(z)|^p \rho_{\mathbb{H}^*}(z)^{2-2p} dxdy \right)^{\frac{1}{p}}$$

$\rightarrow$ We can consider the complex analytic structure of p-integrable Teichmüller spaces.

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History

Lehner's condition

A Fuchsian group Γ satisfies Lehner's condition

$$\overset{\text{def}}{\Leftrightarrow} \inf\{\text{length of simple closed geodesics on } \mathbb{H}/\Gamma\} > 0.$$

Proposition (Lehner 1973, Rao 1974, Niebur-Sheingorn 1977)

A Fuchsian group Γ satisfies Lehner's condition if and only if

$$A^p(\mathbb{H}^*, \Gamma) \subset B(\mathbb{H}^*, \Gamma) \quad \exists(\forall) p \geq 1.$$

Holomorphic tangent vector space on $T^p(\Gamma)$

Harmonic Beltrami differential

For $\mu \in L^{p,\infty}(\mathbb{H}, \Gamma^\tau)$,

$$H^\tau[\mu](z) = -2\rho_{\mathbb{H}}(z)^{-2} D\Phi_\tau(0)\mu(\bar{z}) \quad (z \in \mathbb{H})$$

is called the harmonic Beltrami differential for μ.

Φ_τ: lift of $\beta_\tau : T^p(\Gamma^\tau) \to A^p(\mathbb{H}^*, \Gamma^\tau)$ for the Teichmüller projection

$$\varpi_\tau : \mathrm{Ael}^p(\mathbb{H}, \Gamma^\tau) \to T^p(\Gamma^\tau)$$

Set $\mathrm{HB}^p(\mathbb{H}, \Gamma^\tau) = H^\tau(L^{p,\infty}(\mathbb{H}, \Gamma^\tau))$.

$$H^\tau : L^{p,\infty}(\mathbb{H}, \Gamma) \to L^{p,\infty}(\mathbb{H}, \Gamma) \text{ bdd. lin. operator}$$

Holomorphic tangent vector space on $T^p(\Gamma)$

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For $\tau \in T^p(\Gamma)$,

$$T_\tau T^p(\Gamma) \simeq A^p(\mathbb{H}^*, \Gamma^\tau) \simeq \mathrm{HB}^p(\mathbb{H}, \Gamma^\tau).$$

Hereafter, let $p = 2$.

Hence, $T_\tau T^2(\Gamma)$ has the Hermitian inner product on $\mathrm{HB}^2(\mathbb{H}, \Gamma^\tau)$ as

$$h^\tau(\mu, \nu) = \iint_{N^\tau} \mu(z)\overline{\nu(z)}\rho_{\mathbb{H}}(z)^2 dxdy \quad (\mu, \nu \in \mathrm{HB}^2(\mathbb{H}, \Gamma^\tau))$$

Weil-Petersson metric

For a nbd. $\tau \in U_0$ of the base point 0 and $\mu, \nu \in \mathrm{HB}^2(\mathbb{H}, \Gamma)$,

$$h_{\mathrm{WP}}(\tau)(\mu, \nu) = h^\tau(H^\tau \circ L^{\varsigma(\tau)}[\mu], H^\tau \circ L^{\varsigma(\tau)}[\nu]).$$

- $\varsigma(\tau) \in \varpi_0^{-1}(U_0) \subset \mathrm{HB}^2(\mathbb{H}, \Gamma)$
- $L^{\varsigma(\tau)} : L^{2,\infty}(\mathbb{H}, \Gamma) \to L^{2,\infty}(\mathbb{H}, \Gamma^\tau)$ certain Banach isomorphism

Remark. For every $\eta \in T^2(\Gamma)$, h_{WP} can be defined on a nbd. U_η of η similarly.

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Kähler metric

Hermitian metric

M: complex Hilbert manifold
h_p: Hermitian inner product on T_pM $(p \in M)$

For every C^∞-vector fields ξ, η on M, if the function

$$\tilde{h}(p) = h_p(\xi(p), \eta(p))$$

is of class C^∞, $h = \{h_p\}_{p \in M}$ is called a Hermitian metric on M.

Kähler metric

(M, h): Hermitian manifold
The function $\omega = -2\,\mathrm{Im}\,h$ is a real 2-form on M.

$$h \text{ is a Kähler metric } \overset{\text{def}}{\Longleftrightarrow} d\omega \equiv 0$$

Remark. (Darboux's theorem)
If M is a n-dimensional Kähler manifold, for every $p \in M$, there exists a C^∞-function f on a neighborhood of p s.t.

$$\omega = \partial\bar{\partial} f = \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k.$$

History

Theorem (Ahlfors 1961)

Let Γ be an arbitrary cocompact Fuchsian group.

(i) The Weil-Petersson metric h_{WP} on $T(\Gamma)$ is Kähler;

(ii) The holomorphic sectional curvature and Ricci curvature of h_{WP} are negative.

Takhtajan-Teo showed this result is also valid for $T^2(1)$.

$\longrightarrow$ Can we extend this theorem to Fuchsian groups with Lehner's condition?

If we can do that, $T^2(\Gamma)$ has fruitful properties as the Kähler manifold.

Can we apply Ahlfors's proofs?

Charecterization of Kählerity

Proposition (finite dim. version)

Let (M, h) be a Hermitian manifold ($n = \dim M < \infty$) and $h_{k\bar{\ell}} = h(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_\ell})$. Then the following conditions are equivalent:

(i) h is a Kähler metric on M;

(ii) For $\forall j, k, \ell = 1, \cdots, n$,

$$\frac{\partial h_{k\bar{\ell}}}{\partial z_j} = \frac{\partial h_{j\bar{\ell}}}{\partial z_k};$$

(iii) For $\forall p \in M$, $\exists (w_1, \cdots, w_n)$: local coordinate s.t. $w(p) = 0$, $h_{k\bar{\ell}}(0) = \delta_{k\ell}$ and

$$\frac{\partial h_{k\bar{\ell}}}{\partial w_j}(0) = 0 \quad \forall j, k, \ell = 1, \cdots, n.$$

Charecterization of Kählerity

Theorem (infinite dim. version, Y. 2015?)

Let (M, h) be a Hermitian manifold. Then the following conditions are equivalent:

(i) h is a Kähler metric on M;

(ii) For every $\xi, \eta, \zeta \in \Gamma(TM)$,

$$D'h\xi(\eta, \zeta) = D'h\eta(\xi, \zeta);$$

(iii) For $\forall p \in M$, $\exists(w_1, \cdots, w_n)$: local coordinate s.t. $w(p) = 0$ and

$$h(w) = \langle \cdot, \cdot \rangle + o(\|w\|).$$

Remark. In this talk, we use the representation of finite dim. version for simplicity of explanation.

Rough sketch of proof of Kählerity

The computation itself is same as Ahlfors's one.

$$\begin{aligned}\frac{\partial h_{k\bar{\ell}}}{\partial t_j} &= -\frac{24}{\pi^2}\iint_{N(t)}\left\{\iint_{\mathbb{H}}\nu_k(t)(\zeta)T_j(z,\zeta)K(\zeta,\bar{z})d^2\zeta\right\}\overline{\nu_\ell(t)(z)}d^2z \\ &= -\frac{24}{\pi^2}\iint_{N(t)}\left\{\iint_{\mathbb{H}}\nu_j(t)(w)T_k(z,w)K(w,\bar{z})d^2w\right\}\overline{\nu_\ell(t)(z)}d^2z \\ &= \frac{\partial h_{j\bar{\ell}}}{\partial t_k},\end{aligned}$$

Here, $d^2z = dxdy$ and

$$\begin{aligned}T_j(z,\zeta) &= \iint_{\mathbb{H}}\nu_j(t)(w)K(w,\zeta)K(w,\bar{z})d^2w, \\ \nu_j(t) &= L^{\nu(t)}[\nu_j] \quad \left(t\in\ell^2(\mathbb{C}),\nu(t)=\textstyle\sum_{j=1}^\infty t_j\nu_j\in \mathrm{HB}^2(\mathbb{H},\Gamma)\right), \\ N(t) &= f^{\nu(t)}(N), \quad K(\zeta,z)=(\zeta-z)^{-2}.\end{aligned}$$

Rough sketch of proof of Kählerity

Obstruction

We have to check the commutativity between the signs of differentiation and integration.

It is necessary to show the expression

$$\iint_{N(t)} \iint_{\mathbb{H}} \left| \nu_k(t)(\zeta)\overline{\nu_\ell(t)(z)} T_j(z,\zeta) K(\zeta,\bar{z}) \right| d^2\zeta d^2 z$$

converges uniformly on t in a nbd. of 0.

Ahlfors used the finiteness of the hyperbolic area of $\mathbb{H}/\Gamma$.

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Rough sketch of proof of Kählerity

For $r > e$, let $u(r) = \log\log r$ and

$$D_1(t,r) = N(t) \cap B_h(i, u(r)), \quad D_2(r) = \mathbb{H} \cap \{|z| < r\}.$$

Here, $B_h(i, u(r))$ is the hyperbolic disk centered at i of radius $u(r)$.

Then,

$$(N(t) \times \mathbb{H}) \setminus (D_1(t,r) \times D_2(r)) = \underbrace{(D_1(t,r)^c \times \mathbb{H})}_{(1)} \sqcup \underbrace{(D_1(t,r) \times D_2(t)^c)}_{(2)}.$$

$\longrightarrow$ It is sufficeint to show both of the integrations on (1) and (2) converges to 0 as $r \to \infty$ uniformly on t.

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Rough sketch of proof of Kählerity

(1) integration on $D_1(t,r)^c \times \mathbb{H}$

It follows from Ahlfors's computation and the lemma as follows:

Lemma A (Takhtajan-Teo 2006)

Let $0 < a < 1$. Then, there exists a constant $C(a) > 0$ s.t. for every $z \in \mathbb{H}$ and every $\mu \in H(L^\infty(\mathbb{H},\Gamma))$ with $\|\mu\|_\infty < a$,

$$|\partial f^\mu(z)|^2 \rho_{\mathbb{H}}(f^\mu(z))^2 \le C(a)\rho_{\mathbb{H}}(z)^2.$$

$$\iint_{D_1(t,r)^c} |\nu_j(t)(z)|^2 \rho_{\mathbb{H}}(z)^2 d^2z \le \frac{C(a)}{(1-a^2)^2} \|\nu_j|_{N\setminus(f^{\nu(t)})^{-1}(B_h(i,u(r)))}\|_2^2$$
$$\to 0 \quad (r \to \infty).$$

Rough sketch of proof of Kählerity

(2) integration on $D_1(t,r) \times D_2(r)^c$,

Ahlfors's computation, Lemma A and the lemma as follows:

Lemma B (Y. 2014)

$$F(r) = \iint_{B_h(i,u(r))} \iint_{D_2(r)} \frac{d^2\zeta d^2 z}{|\zeta - \bar{z}|^4} \to 0 \quad (r \to \infty).$$

$$(\text{integration on } D_1(t,r) \times D_2(r)^c) \leq \frac{\pi^2 C(a)^{\frac{1}{2}} \|\nu_\ell\|_2 \|\nu_j\|_\infty \|\nu_k\|_\infty}{(1-a^2)^3} F(r)$$
$$\to 0 \quad (r \to \infty).$$

Rough sketch of proof of negativity

Negativity of curvature

The hol. sec. curvature (resp. the Ricci curvature) is negative if

$$\mathrm{HSec}(0)(\nu(t)) < 0 \text{ (resp. } \mathrm{Ric}(0)(\nu(t)) < 0)$$

for every $t \in \ell^2(\mathbb{C}) \setminus \{0\}$.

Rough sketch of proof of negativity

From condition (iii),

$$h_{k\bar{\ell}}(0) = \delta_{k\ell}, \quad \frac{\partial h_{k\bar{\ell}}}{\partial t_j}(0) = 0 \quad \forall j, k, \ell = 1, \cdots .$$

Then for $\nu(t) = \sum_{j=1}^{\infty} t_j \nu_j \in \mathrm{HB}(\mathbb{H}, \Gamma) \setminus \{0\}$,

$$\mathsf{HSec}(0)(\nu(t)) = \sum_{j,k,\ell,m=1}^{\infty} \frac{\partial^2 h_{j\bar{k}}}{\partial t_\ell \partial \bar{t}_m}(0) t_j \bar{t}_k t_\ell \bar{t}_m,$$

$$\mathsf{Ric}(0)(\nu(t)) = \sum_{\ell=1}^{\infty} \left(\sum_{j,k=1}^{\infty} \frac{\partial^2 h_{j\bar{\ell}}}{\partial t_\ell \partial \bar{t}_k}(0) t_j \bar{t}_k \right).$$

Rough sketch of proof of negativity

$$\mathsf{HSec}(0)(\nu(t)) = -\frac{3}{\pi^5}\int_{N\times\mathbb{H}} G d^2\zeta d^2 z \le 0.$$

Here,

$$G = |\sum_{j,k=1}^{\infty}(L_{j\ell}(\bar{\zeta},\bar{z}) + L_{\ell j}(\bar{\zeta},\bar{z}))t_j t_\ell|^2 + 2|\sum_{j,k=1}^{\infty} L_{j\bar{k}}(\zeta,\bar{z})t_j\bar{t}_k|^2,$$

$$L_{jk}(\bar{\zeta},\bar{z}) = \int_{\mathbb{H}^2} K(z_1,\bar{\zeta})K(z_1,w_1)K(w_1,\bar{z})\nu_j(z_1)\nu_k(w_1)d^2z_1 d^2w_1,$$

$$L_{j\bar{k}}(\zeta,\bar{z}) = \int_{\mathbb{H}^2} K(z_2,\zeta)K(z_2,\bar{w}_2)K(\bar{w}_2,\bar{z})\nu_j(z_2)\overline{\nu_k(w_2)}d^2z_2 d^2w_2.$$

Rough sketch of proof of negativity

Suppose $\mathrm{HSec}(0)(\nu(t)) = 0$. Then for every $\zeta, z \in \mathbb{H}$,

$$\sum_{j,k=1}^{\infty} L_{j\bar{k}}(\zeta, \bar{z}) t_j \bar{t}_k = 0 \quad \forall j, k = 1, \cdots .$$

Hence,

$$0 = \iint_N \sum_{j,k=1}^{\infty} L_{j\bar{k}}(\zeta, \bar{\zeta}) t_j \bar{t}_k d^2\zeta = -\frac{1}{3} \|\nu(t)\|_2.$$

This contradicts the assumption $\nu(t) \neq 0$.

Similarly, it follows $\mathrm{Ric}(0)(\nu(t)) < 0$.

Rough sketch of proof of negativity

Obstruction

Is this computation valid for infinite dim. manifolds?

- Hol. sec. curvature $\cdots$ Ahlfors's computation + Lemma A, B
- Ricci curvature $\cdots$ These tools + Theory of reproducing kernel Hilbert space

Rough sketch of proof of negativity

For $\nu(t) = \sum_{j=1}^{\infty} t_j \nu_j \in \mathrm{HB}(\mathbb{H}, \Gamma) \setminus \{0\}$,

$$\mathrm{HSec}(0)(\nu(t)) = \sum_{j,k,\ell,m=1}^{\infty} \frac{\partial^2 h_{j\bar{k}}}{\partial t_\ell \partial \bar{t}_m}(0) t_j \bar{t}_k t_\ell \bar{t}_m$$
$$= -D' D'' h(0)(\nu(t), \nu(t))(\nu(t), \nu(t)),$$

$$\mathrm{Ric}(0)(\nu(t)) = \sum_{\ell=1}^{\infty} \left(\sum_{j,k=1}^{\infty} \frac{\partial^2 h_{j\bar{\ell}}}{\partial t_\ell \partial \bar{t}_k}(0) t_j \bar{t}_k \right)$$
$$= -\mathrm{Re} \sum_{\ell=1}^{\infty} D' D'' h(0)(\nu_\ell, \nu(t))(\nu(t), \nu_\ell).$$

Rough sketch of proof of negativity

Fact

The Hilbert space $\mathrm{HB}(\mathbb{H}, \Gamma)$ has the reproducing kernel

$$Q_\Gamma(z,\zeta) = \frac{3}{\pi}\rho_{\mathbb{H}}(z)^{-2}\rho_{\mathbb{H}}(\zeta)^{-2}\sum_{\gamma\in\Gamma}\frac{\gamma'(\zeta)^2}{(\bar{z}-\gamma(\zeta))^4},$$

that is,

$$\nu(z) = \iint_N \nu(\zeta) Q_\Gamma(z,\zeta) d^2\zeta$$

for every $\nu \in \mathrm{HB}^2(\mathbb{H}, \Gamma)$.

It follows that

$$Q_\Gamma(z,\zeta) = \sum_{\ell=1}^{\infty} \nu_\ell(z)\overline{\nu_\ell(\zeta)}.$$

Rough sketch of proof of negativity

Proposition (Lehner 1973, Rao 1974, Niebur-Sheingorn 1977)

A Fuchsian group satisfies Lehner's condition if and only if

$$\sup_{z\in\mathbb{H}} |Q_\Gamma(z,z)| < \infty.$$

The convergence of $\mathrm{Ric}(0)(\nu(t))$ is shown from this proposition.

Conclusion

Theorem (Ahlfors 1961, Takhtajan-Teo 2000, Y. 2014)

Let Γ be a Fuchsian group with Lehner's condition.

(i) The Weil-Petersson metric h_{WP} on $T^2(\Gamma)$ is Kähler;

(ii) The holomorphic sectional curvature and Ricci curvature of h_{WP} are negative.

$\uparrow$ because

We can apply Ahlfors's method.

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Thank you for your kind attention!