Relations between holomorphic 1-forms and holomorphic 1-cochains on Riemann surfaces

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Contents

1. Combinatorial Hodge theory

Notation

- M : a closed oriented Riemannian n-manifold.
- $\Omega^j(M)$: the space of smooth j-forms on M with an inner product $\langle , \rangle_\Omega$
- $d : \Omega^j(M) \to \Omega^{j+1}(M)$: the exterior derivative
- K : a triangulation of M
- $C^j(K)$: the simplicial cochains of degree j of K with values in $\mathbb{R}$

Now we fix an ordering of vertices of K and then we have a coboudary operator

- $\delta : C^j(K) \to C^{j+1}(K)$

Since M is compact, we may identify the cochains and chains of K and $\sigma \in C^j(K)$ is written by

$$\sigma = \sum_{\tau} c_{\tau} \cdot \tau$$

where $c_{\tau} \in \mathbb{R}$ and the sum is over all j-simplices of K.

Now suppose that cochains $C(K) = \oplus_j C^j(K)$ are equipped with a non-degenarate inner product $\langle , \rangle_C$ such that $C^i(K)$ and $C^j(K)$ are orthogonal for distict i, j.

Then we define the adjoint δ^* of δ by

$$\langle \delta^* \sigma, \tau \rangle_C = \langle \sigma, \delta \tau \rangle_C.$$

Using δ and δ^*, we define the space $\mathcal{H}C^j(K)$ of harmonic j-cochains by

$$\mathcal{H}C^j(K) = \left\{\sigma \in C^j(K) | \delta\sigma = \delta^*\sigma = 0\right\}.$$

Eckmann showed the following.

Theorem (Eckmann, 1945)

Let (C, δ) be a finite dimensional complex with an inner product. There is an orthogonal direct sum decomposition

$$C^j(K) = \delta C^{j-1}(K) \oplus \mathcal{H}C^j(K) \oplus \delta^* C^{j+1}(K)$$

and $\mathcal{H}C^j \cong H^j(K)$, the cohomology of (K, δ) in degree j.

Clealy, this decomposition depends upon the choice of inner product.

In 1957, Whitney defined a map from cochains into forms.

Notation

- p_i : the i-th vertex of K
- μ_i : the barycentric coordinate corresponding to p_i

We write a j-simplix τ of K by

$$\tau = [p_0, \cdots, p_j]$$

with the vertices in an increasing sequence with respect to the ordering of vertices in K.

- $\mathcal{L}^2\Omega^j(M)$: the completion of $\Omega^j(M)$ with respect to $\langle , \rangle_\Omega$

Definition (the Whitney map)

For a j-simplex $\tau = [p_0, \cdots, p_j]$, we define $W\tau \in \mathcal{L}^2\Omega^j(M)$ by

$$W\tau = j! \sum_{i=0}^{j} (-1)^i \mu_i d\mu_0 \wedge \cdots \wedge \widehat{d\mu_i} \wedge \cdots \wedge d\mu_j.$$

We define W on all of $C^j(K)$ by extending linearly.

On the other hand, we define a map in the converse direction, the de Rham map from differential forms to $C(K)$.

Definition (de Rham map)

For any differential j-form ω and chain c, we define $R\omega \in C^j(K)$ by

$$R\omega(c) = \int_c \omega.$$

Theorem (Whitney, 1957)

The following holds:

$$RW = Id.$$

In general $WR \neq Id$, but Dodziuk and Patodi showed that WR is approximately equal to the identity.

Definition

Let K be a triangulation. The mesh $\eta(K)$ of K is defined by

$$\eta(K) = \sup r(p, q),$$

where r means the geodesic distance in M and the supremum is taken over all the pair of vertices p, q of a 1-simplex in K.

Theorem (Dodziuk and Patodi, 1976)

There exist a positive constant C and positive integer m, independent of K, such that

$$\|\omega - WR\omega\|_\Omega \leq C \cdot \|(Id + \Delta)^m \omega\|_\Omega \cdot \eta(K)$$

for all C^∞ differential forms ω on M, where Δ is the Laplacian.

In 2007, Wilson defined a star operator $\star$ on cochains.

Definition

We define $\cup : C^j(K) \otimes C^k(K) \to C^{j+k}(K)$ by

$$\sigma \cup \tau = R(W\sigma \wedge W\tau).$$

Definition (the combinatorial Hodge star operator)

Let $\langle , \rangle_C$ be an inner product on $C(K)$ such that $C^i(K) \perp C^j(K)$ for $i \neq j$. For $\sigma \in C^j(K)$, we define $\star\sigma \in C^{n-j}$ by

$$\langle \star\sigma, \tau \rangle_C = (\sigma \cup \tau)[M] \left(= \iint_M W\sigma \wedge W\tau \right),$$

where $[M]$ denotes the fundamental class of M

Lemma (Wilson, 2007)

The following hold:
(1) $\bigstar\delta = (-1)^{j+1}\delta^*\bigstar$, *i.e.* $\bigstar$ *is a chain map.*
(2) *For* $\sigma \in C^j(K)$ *and* $\tau \in C^{n-j}$,
$\langle \bigstar\sigma, \tau \rangle_C = (-1)^{j(n-j)} \langle \sigma, \bigstar\tau \rangle_C$, *i.e.* $\bigstar$ *is (graded) skew-adjoint.*
(3) $\bigstar$ *induces isomorphisms* $\mathcal{H}C^j(K) \to \mathcal{H}C^{n-j}(K)$.

Theorem (Wilson, 2007)

There exists a positive constant C and a positive integer m, indeependent of K, such that

$$\| \star\omega - W\bigstar R\omega \|_\Omega \leq C \cdot \|(Id + \Delta)^m \omega\|_\Omega \cdot \eta(K)$$

for all C^∞ differential forms ω on M, where $\star$ is the Hodge star operator.

Dodziuk and Patodi defined the Whitney inner product on cochains and studied the relations between the smooth Hodge theory and the combinatorial Hodge theory with respect to this inner product.

Definition (the Whitney inner product)

For $\sigma, \tau \in C(K)$,

$$\langle \sigma, \tau \rangle_C = \langle W\sigma, W\tau \rangle_\Omega .$$

Now we suppose that the cochains are equipped with the Whitney inner product.

Then they showed that the approximation $WR \approx Id$ respects Hodge decompositions of $\Omega(M)$ and $C(K)$

Theorem (Dodziuk and Patodi, 1976)

Let $\omega \in \Omega^j(M)$ and $R\omega \in C^j(K)$ have Hodge decompositions

$$\begin{aligned} \omega &= d\omega_1 + \omega_2 + d^*\omega_3 \\ R\omega &= \delta a_1 + a_2 + \delta^* a_3 \end{aligned}$$

Then,

$$\begin{aligned} \|d\omega_1 - W\delta a_1\|_\Omega &\leq \lambda \cdot \|(Id + \Delta)^m \omega\|_\Omega \cdot \eta(K) \\ \|\omega_2 - W a_2\|_\Omega &\leq \lambda \cdot \|(Id + \Delta)^m \omega\|_\Omega \cdot \eta(K) \\ \|d^*\omega_3 - W\delta^* a_3\|_\Omega &\leq \lambda \cdot \|(Id + \Delta)^m \omega\|_\Omega \cdot \eta(K) \end{aligned}$$

where λ and m are independent of ω and K.

2. Holomorphic 1-forms

Let M be a closed Riemann surface of genus g.

Then we define an inner product $\langle , \rangle_\Omega$ on $\Omega(M)$ by

$$\langle \omega, \eta \rangle_\Omega = \iint_M \omega \wedge \star \overline{\eta},$$

where $^-$ means complex conjugation and $\star$ is the Hodge star operator on $\Omega(M)$.
Let $\mathcal{H}\Omega^{1,0}(M)$ be the space of holomorphic 1-forms on M.
It is known that for all $\omega \in \mathcal{H}\Omega^{1,0}(M)$, $\star\omega = -i\omega$ and $\dim \mathcal{H}\Omega^{1,0}(M) = g$.

Let $\Sigma = \{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ be a canonical homology basis. Then we define the periods of $\omega \in \mathcal{H}\Omega^{1,0}(M)$ by $\int_{a_j} \omega$ and $\int_{b_j} \omega$ for $1 \leq j \leq g$.

Theorem (Riemann's bi-linear relations)

For ω, $\eta \in \mathcal{H}\Omega^{1,0}(M)$,

$$\sum_{j=1}^{g} \left(\int_{a_j} \omega \int_{b_j} \eta - \int_{b_j} \omega \int_{a_j} \eta \right) = 0,$$

and

$$\sum_{j=1}^{g} \left(\int_{a_j} \omega \int_{b_j} \overline{\eta} - \int_{b_j} \omega \int_{a_j} \overline{\eta} \right) = \langle \star\omega, \eta \rangle_{\Omega}.$$

Then we obtain

$$\|\omega\|_{\Omega}^{2} = i \sum_{j=1}^{g} \left(\int_{a_j} \omega \int_{b_j} \overline{\omega} - \int_{b_j} \omega \int_{a_j} \overline{\omega} \right)$$

for any $\omega \in \mathcal{H}\Omega^{1,0}(M)$.

By Riemann's bi-linear relation, for a canonical homology basis $\boldsymbol{\Sigma}$, we obtain a unique basis $\{\theta_1, \cdots, \theta_g\}$ of $\mathcal{H}\Omega^{1,0}(M)$ which satisfy $\int_{a_k} \theta_j = \delta_{jk}$.
We call this basis $\{\theta_1, \cdots, \theta_g\}$ the ***canonical basis*** of $\mathcal{H}\Omega^{1,0}(M)$.

Definition (Period matrices)

*Let M be a closed Riemann surface of genus g with a canonical homology basis $\boldsymbol{\Sigma}$, and let $\{\theta_1, \cdots, \theta_g\}$ be the canonical basis of $\mathcal{H}\Omega^{1,0}(M)$. Then a **period matrix** is defined by*

$$\Pi = (\pi_{jk})_{1 \leq j,k \leq g}, \text{ where } \pi_{jk} = \int_{b_k} \theta_j.$$

Proposition (e.g. Farkas-Kra, 1991)

Period matrices are symmentric and their imaginary parts are positive definite, i.e. period matrices lie in the Siegel upper half space.

3. Holomorphic $\mathbf{1}$-cochains

Next we define holomorphic $\mathbf{1}$-cochains. To define, we need to extend some of definitions to the cochains $C(K)$ valued in $\mathbb{C}$.

We define the star operator $\bigstar$ on $C(K)$ by

$$\langle \bigstar\sigma, \tau\rangle_C = (\sigma \cup \overline{\tau})[M],$$

where the bar denotes complex cojugation and $\cup$ is extended over $\mathbb{C}$ linearly.

Just as with real coefficients, the Hodge decomposition with complex coefficient holds

$$C^1(K) = \delta C^0(K) \oplus \mathcal{H}C^1(K) \oplus \delta^* C^2(K).$$

By the properties of $\bigstar$ on cochains $\boldsymbol{C(K)}$, we have the following on $\boldsymbol{C^1(K)}$:

(1) For $\boldsymbol{\sigma},\ \boldsymbol{\tau} \in C^1(K)$, $\langle \bigstar \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_C = -\langle \boldsymbol{\sigma}, \bigstar \boldsymbol{\tau} \rangle_C$, i.e. $\bigstar$ is skew-adjoint.
(2) $\bigstar$ induces isomorphisms $\boldsymbol{\mathcal{H}C^1(K)} \to \boldsymbol{\mathcal{H}C^1(K)}$.

Now we define the combinatorial star operator $\bigstar$ on $\boldsymbol{\mathcal{H}C^1(K)}$ as an isomorphism $\boldsymbol{\mathcal{H}C^1(K)} \to \boldsymbol{\mathcal{H}C^1(K)}$ which is skew-adjoint.

Definition

Let $\langle , \rangle_C$ be a hermitian inner product on $\mathbb{C}$-valued cochains which is $\mathbb{R}$-valued on $\mathbb{R}$-cochains. We define the space $\mathcal{H}C^{1,0}(K)$ of holomorphic 1-cochains to be the span of the eigenvectors for non-positive imaginary eigenvalues of $\star$ and the space $\mathcal{H}C^{0,1}(K)$ of anti-holomorphic 1-cochains to be the span of the eigenvectors for non-negative imaginary eigenvalues of $\star$.

If σ is an eigenvector in $\mathcal{H}C^{1,0}(K) \quad \Rightarrow \quad \star\sigma = -i\lambda\sigma \ (\lambda > 0)$

If $\widetilde{\sigma}$ is an eigenvector in $\mathcal{H}C^{0,1}(K) \quad \Rightarrow \quad \star\widetilde{\sigma} = +i\widetilde{\lambda}\widetilde{\sigma} \ (\widetilde{\lambda} > 0)$

Note that since the Whitney inner product satisfies the condition ($\mathbb{R}$-valued on $\mathbb{R}$-cochains), the Whitney inner product defines $\mathcal{H}C^{1,0}(K)$ and $\mathcal{H}C^{0,1}(K)$.

Lemma (Wilson, 2008)

Let M be a closed Riemann surface of genus g and K a triangulation, and suppose that the cochains $C(K)$ are equipped with the Whitney inner product. Then the following hold:

(1) $\mathcal{H}C^1(K) = \mathcal{H}C^{1,0}(K) \oplus \mathcal{H}C^{0,1}(K)$.

(2) $\dim \mathcal{H}C^{1,0}(K) = \dim \mathcal{H}C^{0,1}(K) = g$.

(3) *Complex conjugation maps $\mathcal{H}C^{1,0}(K)$ to $\mathcal{H}C^{0,1}(K)$ and vice versa.*

Let M be a closed Riemann surface of genus g with a canonical homology basis $\Sigma = \{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ and a triangulation K. Then Wilson defined ***combinatorial periods*** as follows.

Definition (Combinatorial periods)

For $\sigma \in \mathcal{H}C^{1,0}(K)$, the combinatorial periods of σ are defined by the following complex numbers:

$$\sigma(a_j),\ \sigma(b_j)$$

for $1 \leq j \leq g$.

Wilson showed that combinatorial periods of holomorphic $\mathbf{1}$-cochains satisfy Riemann's bi-linear relations.

Theorem (Wilson, 2008)

For $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}C^{1,0}(K)$,

$$\sum_{j=1}^{g} \Big(\sigma(a_j)\tau(b_j) - \sigma(b_j)\tau(a_j) \Big) = 0,$$

and

$$\sum_{j=1}^{g} \Big(\sigma(a_j)\overline{\tau(b_j)} - \sigma(b_j)\overline{\tau(a_j)} \Big) = \langle \star\sigma, \tau \rangle_C.$$

Corollary (Wilson, 2008)

Let σ be a holomorphic 1-cochain.
(1) *If all $\sigma(a_j)$* **AND** *all $\sigma(b_j)$ are* **REAL**, *then $\sigma = 0$.*
(2) *If all $\sigma(a_j)$* **OR** *all $\sigma(b_j)$ are* **ZERO**, *then $\sigma = 0$.*

This corollary implies that for any $(c_1, \cdots, c_g) \in \mathbb{C}^g$, we obtain a unique holomorphic 1-cochain σ such that

$$\sigma(a_j) = c_j,$$

for $1 \leq j \leq g$.

By this property, we also obtain a unique basis $\{\sigma_1, \cdots, \sigma_g\}$ of $\mathcal{H}C^{1,0}(K)$ such that

$$\sigma_j(a_k) = \delta_{jk},$$

for $1 \leq j, k \leq g$.

We call this basis the ***canonical basis*** of $\mathcal{H}C^{1,0}(K)$ (with respect to a canonical homology basis Σ).

Definition (Combinatorial period matrices)

Let $\boldsymbol{M}$ *be a closed Riemann surface with a canonical homology basis* $\boldsymbol{\Sigma}$ *and a triangulation* $\boldsymbol{K}$, *and let* $\{\boldsymbol{\sigma_1}, \cdots, \boldsymbol{\sigma_g}\}$ *be the canonical basis of* $\boldsymbol{\mathcal{H}C^{1,0}(K)}$. *The* ***combinatorial period matrix*** *(CPM)* $\boldsymbol{\Pi_K}$ *is defined by*

$$\boldsymbol{\Pi_K = (\pi^K_{jk})_{1 \le j,k \le g}}, \quad \textbf{where} \quad \boldsymbol{\pi^K_{jk} = \sigma_j(b_k)}.$$

Since a canonical basis for $\boldsymbol{\mathcal{H}C^{1,0}(K)}$ is uniquely determined by a triple $\boldsymbol{(M, \Sigma, K)}$, a CPM is also uniquely determined.

Theorem (Wilson, 2008)

CPMs lie in the Siegel upper half space.

Theorem (Wilson, 2008)

Let $\boldsymbol{M}$ *be a closed Riemann surface with a canonical homology basis* $\boldsymbol{\Sigma}$ *and* $\{\boldsymbol{K_n}\}_{n\in\mathbb{N}}$ *be a sequence of triangulations with* $\lim_{n\to\infty}\boldsymbol{\eta(K_n) = 0}$*, and let* $\boldsymbol{\Pi}$ *be the period matrix of* $\boldsymbol{M}$*.*
Then we have the sequence $\{\boldsymbol{\Pi_{K_n}}\}_{n\in\mathbb{N}}$ *of CPMs corresponding to* $\{\boldsymbol{K_n}\}_{n\in\mathbb{N}}$*, i.e. each* $\boldsymbol{\Pi_{K_n}}$ *is given by* $\boldsymbol{(M, \Sigma, K_n)}$*, and this sequence converges to* $\boldsymbol{\Pi}$*:*

$$\lim_{n\to\infty}\boldsymbol{\Pi_{K_n} = \Pi}.$$

4. Main Results

Theorem

Let M be a closed Riemann surface of genus g with a canonical homology basis Σ and a triangulation K, and let $\{\theta_1, \cdots, \theta_g\}$ be the canonical basis of $\mathcal{H}\Omega^{1,0}(M)$ and $\{\sigma_1, \cdots, \sigma_g\}$ the canonical basis of $\mathcal{H}C^{1,0}(K)$. Let Π be the period matrix and Π_K the CPM of M. Then the following equation holds:

$$\Pi = \overline{\Pi_K} - \overline{\Lambda_K},$$

where

$$\Lambda_K = (\langle W\sigma_j, \star\theta_k\rangle_\Omega)_{1\le j,k\le g}.$$

Corollary

The matrix Λ_K lies in the Siegel upper half space.

Next we consider the relations between holomorphic $\mathbf{1}$-forms and holomorphic $\mathbf{1}$-cochains.

Definition

For any $\omega \in \mathcal{H}\Omega^{1,0}(M)$, we define $\iota_\omega \in \mathcal{H}C^{1,0}(K)$ which satisfies

$$\iota_\omega(a_j) = \int_{a_j} \omega$$

for $1 \leq j \leq g$.

Lemma

The map $\omega \mapsto \iota_\omega$ is an isomorphism from $\mathcal{H}\Omega^{1,0}(M)$ to $\mathcal{H}C^{1,0}(K)$.

Theorem

Let M be a closed Riemann suface of genus g with a canonical homology basis Σ, and ω arbitrary holomorphic 1-form on M. In the case of $g = 1$, for any triangulation K, we have

$$\|W\iota_\omega - \omega\|_\Omega = 0.$$

In the case of $g > 1$, for any sequence $\{K_n\}_{n\in\mathbb{N}}$ of triangulations with $\lim_{n\to\infty} \eta(K_n) = 0$, we have

$$\lim_{n\to\infty} \|W\iota_\omega^n - \omega\|_\Omega = 0,$$

where $\iota_\omega^n \in \mathcal{H}C^{1,0}(K_n)$.

To show this theorem, we prove three theorems.

Theorem 1

Let $\boldsymbol{M}$ be a closed Riemann surface of genus $\boldsymbol{g}$ with a canonical homology basis $\boldsymbol{\Sigma}$ and a triangulation $\boldsymbol{K}$, and let $\{\boldsymbol{\theta_1}, \cdots, \boldsymbol{\theta_g}\}$ be the canonical basis of $\mathcal{H}\boldsymbol{\Omega}^{1,0}(\boldsymbol{M})$ and $\{\boldsymbol{\sigma_1}, \cdots, \boldsymbol{\sigma_g}\}$ the canonical basis of $\mathcal{H}\boldsymbol{C}^{1,0}(\boldsymbol{K})$. Let $\boldsymbol{\Pi_K} = (\pi^{K}_{jk})$ be the CPM of $\boldsymbol{M}$. Then there exists a vector $\boldsymbol{\Phi_K} = (\varphi_1, \cdots, \varphi_g) \in (0,1]^g$ such that

$$\langle \bigstar \sigma_j, \sigma_j \rangle_C = \langle -i\varphi_j \sigma_j, \sigma_j \rangle_C.$$

In addition, we have

$$\|W\sigma_j - \theta_j\|_\Omega = \sqrt{2\mathrm{Im}\pi^{K}_{jj}\left(\frac{1}{\varphi_j} - 1\right)}$$

and for arbitrary holomorphic **1***-form* $\boldsymbol{\omega}$,

$$\|W\iota_\omega - \omega\|_\Omega \leq \sum_{j=1}^{g} \left(\int_{a_j} \omega\right) \cdot \sqrt{2\mathrm{Im}\pi^{K}_{jj}\left(\frac{1}{\varphi_j} - 1\right)}$$

Theorem 2

Let M be a closed Riemann surface of genus 1 (complex torus) with a canonical homology basis Σ and a triangulation K. Then

$$\Phi_K = \varphi_1 = 1.$$

Lemma

Let M be a closed Riemann surface of genus g with a canonical homology basis Σ and a triangulation K. The following are equivalent:
(1) $\Phi_K = (1, \cdots, 1)$
(2) $WR\omega = \omega$ *a.e.* on M for all $\omega \in \mathcal{H}\Omega^{1,0}(M)$
(3) $\mathcal{H}C^{1,0}(K) = \{\sigma \in \mathcal{H}C^1(K) | \star\sigma = -i\sigma\}$

On a complex torus, we obtain $WR\omega = \omega$ *a.e.* on M. Hence we colude that $\Phi_K = 1$.

Theorem 3

Let M be a closed Riemann surface of genus $g > 1$ with a canonical homology basis Σ and $\{K_n\}_{n\in\mathbb{N}}$ a sequence of triangulations with $\lim_{n\to\infty} \eta(K_n) = 0$, and let $\Phi_{K_n} = (\varphi_1^n, \cdots, \varphi_g^n)$ be the vectors in $(0,1]^g$ such that

$$\langle \star\sigma_j^n, \sigma_j^n \rangle_C = \langle -i\varphi_j^n \sigma_j^n, \sigma_j^n \rangle_C$$

for $n \in \mathbb{N}$, where $\{\sigma_1^n, \cdots, \sigma_g^n\}$ is the canonical basis of $\mathcal{H}C^{1,0}(K_n)$. Then

$$\lim_{n\to\infty} \Phi_{K_n} = (1, \cdots, 1).$$

Sketch of the proof

Let $\{\omega_1, \cdots, \omega_g\}$ be an orthogonal basis of $\mathcal{H}\Omega^{1,0}(M)$ and R^n denote the de Rham map from $\Omega(M)$ to $C(K_n)$. By the Hodge decomposition and $\mathcal{H}C^1(K_n) = \mathcal{H}C^{1,0}(K_n) \oplus \mathcal{H}C^{0,1}(K_n)$, we have

$$R^n\omega_j = \delta^* k_j^n + h_j^n + \widetilde{h}_j^n + \delta g_j^n$$

for any $n \in \mathbb{N}$, where $h_j^n \in \mathcal{H}C^{1,0}(K_n)$ and $\widetilde{h}_j^n \in \mathcal{H}C^{0,1}(K_n)$.

First, we show that we may assume that $\{h_1^n, \cdots, h_g^n\}$ is a basis of $\mathcal{H}C^{1,0}(K_n)$.

If the number of $\{K_n\}$, such that $\{h_1^n, \cdots, h_g^n\}$ is not a basis, is infinite, then there exist $j \in \{1, \cdots, g\}$ and a subsequence $\{K_m\}$ of $\{K_n\}$ such that

$$h_j^m = \sum_{p \neq j} c_{jp}^m h_p^m,$$

for all $m \in \mathbb{N}$.

Here we use the following lemma.

Lemma (Wilson, 2008)

Let $\boldsymbol{h}$ be the holomophic part of $\boldsymbol{R\omega}$, where $\boldsymbol{\omega \in \mathcal{H}\Omega^{1,0}(M)}$. Then there exists $\boldsymbol{C > 0}$, independent of $\boldsymbol{K}$, such that $\boldsymbol{\|Wh - \omega\|_\Omega \leq C \cdot \eta(K)}$.

Since $\boldsymbol{h_j^m - \sum_{p \neq j} c_{jp}^m h_p^m (= 0)}$ is the holomorphic part of $\boldsymbol{R^m(\omega_j - \sum_{p \neq j} c_{jp}^m \omega_p)}$, by this lemma,

$$\boldsymbol{\lim_{m \to \infty} \left\| \omega_j - \sum_{p \neq j} c_{jp}^m \omega_p \right\|_\Omega = 0.}$$

Also, since $\boldsymbol{\{\omega_1, \cdots, \omega_g\}}$ is an orthogonal basis,

$$\boldsymbol{\left\| \omega_j - \sum_{p \neq j} c_{jp}^m \omega_p \right\|_\Omega = \|\omega_j\|_\Omega^2 + \sum_{p \neq j} |c_{jp}^m|^2 \|\omega_p\|_\Omega^2}$$

amd therefore

$$\boldsymbol{0 \leq |c_{jp}^m|^2 \|\omega_p\|_\Omega^2 \leq \left\| \omega_j - \sum_{p \neq j} c_{jp}^m \omega_p \right\|_\Omega.}$$

Thus we obtain $\|\omega_j\|_\Omega = 0$. This is a contradiction since $\{\omega_1, \cdots, \omega_g\}$ is a basis.

Now we assume that $\{h_1^n, \cdots, h_g^n\}$ is a basis of $\mathcal{H}C^{1,0}(K_n)$ for all $n \in \mathbb{N}$. Thus we may write

$$\sigma_j^n = \sum_{\ell=1}^{g} \widetilde{c}_{j\ell}^n h_\ell^n$$

for $1 \leq j \leq g$, where $\widetilde{c}_{j\ell}^n \in \mathbb{C}$.
Then the matrix $(\widetilde{c}_{j\ell}^n)_{j,\ell}$ converges to a matrix $(s_{j\ell})_{j,\ell}$, where each $s_{j\ell}$ is determined by
$\int_{a_1} \omega_1, \cdots, \int_{a_g} \omega_1, \cdots, \int_{a_1} \omega_g, \cdots, \int_{a_g} \omega_g$.

Using Cauchy-Schwarz inequality, we have

$$\mathbf{0} \leq (\mathbf{1} - \varphi_j^n)\|\boldsymbol{\sigma}_j^n\|_C \leq \|\boldsymbol{\sigma}_j^n - i\bigstar\boldsymbol{\sigma}_j^n\|_C.$$

Since $\boldsymbol{\sigma}_j^n = \sum_{\ell=1}^{g} \widetilde{c}_{j\ell}^n \boldsymbol{h}_\ell^n$, we compute

$$\begin{aligned}
&\|\boldsymbol{\sigma}_j^n - i\bigstar\boldsymbol{\sigma}_j^n\|_C \\
= &\left\| \sum_{\ell=1}^{g} \widetilde{c}_{j\ell}^n(\boldsymbol{h}_\ell^n - i\bigstar\boldsymbol{h}_\ell^n) \right\|_C \\
\leq &\sum_{\ell=1}^{g} |\widetilde{c}_{j\ell}^n| \cdot \|W\boldsymbol{h}_\ell^n - iW\bigstar\boldsymbol{h}_\ell^n\|_\Omega \\
= &\sum_{\ell=1}^{g} |\widetilde{c}_{j\ell}^n| \cdot \|W\boldsymbol{h}_\ell^n - \omega_\ell + i\star\omega_\ell - iW\bigstar\boldsymbol{h}_\ell^n\|_\Omega \\
\leq &\sum_{\ell=1}^{g} |\widetilde{c}_{j\ell}^n| \cdot \Big(\|W\boldsymbol{h}_\ell^n - \omega_\ell\|_\Omega + \|\star\omega_\ell - W\bigstar\boldsymbol{h}_\ell^n\|_\Omega\Big).
\end{aligned}$$

Note that $\star\omega_\ell = -i\omega_\ell$.

Here we use the following lemma, again.

Lemma (Wilson, 2008)

Let h be the holomophic part of $R\omega$, where $\omega \in \mathcal{H}\Omega^{1,0}(M)$. Then there exists $C > 0$, independent of K, such that $\|Wh - \omega\|_\Omega \leq C \cdot \eta(K)$.

Also, by the original proof of this lemma, there exists $\widetilde{C} > 0$, independent of K, such that $\|W\bigstar h - \star\omega\|_\Omega \leq \widetilde{C} \cdot \eta(K)$.

Hence there exist constants C_ℓ, independendet of $\{K_n\}$, such that

$$\|Wh_\ell^n - \omega_\ell\|_\Omega + \|W\bigstar h_\ell^n - \star\omega_\ell\|_\Omega \leq C_\ell \cdot \eta(K_n).$$

This implies that $\lim_{n\to\infty} \|\sigma_j^n - i\bigstar\sigma_j^n\|_C = 0$ and so

$$\lim_{n\to\infty} (1 - \varphi_j^n)\|\sigma_j^n\|_C = 0.$$

By Riemann's bi-linear relation and $\mathbf{0} < \varphi_j^n \leq \mathbf{1}$,

$$\|\sigma_j^n\|_C^2 = \frac{2\mathrm{Im}\pi_{jj}^{K_n}}{\varphi_j^n} \geq 2\mathrm{Im}\pi_{jj}^{K_n}.$$

By Wilson's theorem $\left(\lim_{n\to\infty} \Pi_{K_n} = \Pi\right)$, we obtain

$$\lim_{n\to\infty} \|\sigma_j^n\|_C^2 \geq 2\mathrm{Im}\pi_{jj} > 0.$$

Note that $\mathrm{Im}\Pi$ is positive definite. Thus we conclude that

$$\lim_{n\to\infty} (1 - \varphi_j^n) = 1$$

for all j.