Deforming affine Lorentzian transformation groups and deforming hyperbolic structures

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Abstract1

Classify properly discontinuous (affine) actions onto $\mathbb{R}^2_1$, which is not cocompact.

Only the case : action by a free group

$$\mathbb{F}_n \cong \pi_1(S_{g,b}),\, b > 0$$

Aff. deformation $\rho_{\mathbf{u}} : \pi_1(S_{g,b}) \hookrightarrow SO^0(2,1) \ltimes \mathbb{R}^2_1$, $g \mapsto (\rho_0(g), \mathbf{u}(g))$

Definition

A group Γ acts a topology space X *properly discontinuously* if for any compact set K in X, $\sharp\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \emptyset\} < \infty$.

Goal

Study the deformation space $\{\rho_{\mathbf{u}} : \text{affine deformation}\}$.

Abstract2

Let $\rho_{\mathbf{u}}(\Gamma) < SO^0(2,1) \ltimes \mathbb{R}^2_1$: discrete subgroup.

Margulis invariant $\alpha_{\mathbf{u}} : \Gamma \to \mathbb{R}$. (translation length)

The $\alpha_{\mathbf{u}}$ determines affine deformation up to a conjugation.

Main Theorem 1

The deformation space is parametrized Margulis invariant and "Affine twist cocycle" like Fenchel-Nielsen coodinates in hyperbolic geometry except some hyperbolic structures.

Abstract3

By $SO^0(2,1) \cong \mathrm{PSL}(2,\mathbb{R})$ and $\mathbb{R}^2_1 \cong \mathfrak{g} := \mathfrak{s}l_2(\mathbb{R})$,
we can consider; for $\rho_0(g) \in \rho_0(\Gamma) \subset \mathrm{SL}(2,\mathbb{R})$,

$$g_t := \exp\left(t\mathbf{u}(g)\right) \cdot \rho_0(g)$$

Lemma (Goldman-Margulis, 2000)

$$\frac{1}{2}\frac{d}{dt}\,|_{t=0}\ell(g_t) = \alpha_{\mathbf{u}}(g)$$

Theorem (M.)

$\forall h \in \pi_1(S_{g,b})$, $2\alpha_{\mathbf{AT}_g}(h) = \frac{d}{dt}|_{t=0}\ell(h_t) = \sum_{p\in g\cap h} \cos\left\{\left(\theta^h_g\right)_p\right\}$

Lorentzian spacetime

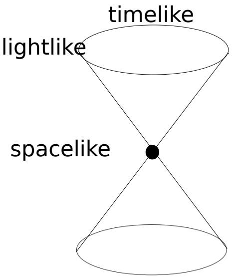
$(\mathbb{R}^2_1, B)$: $(2+1)$-dim. Lorentzian spacetime

- $\mathbb{R}^2_1$: 3-dim. vector space
- B : Lorentzian inner product

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2_1$, represented as $B(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 - x_3 y_3$.

Definition

A *light cone* (or *null cone*) is the set of lightlike vectors.



Linear action by $SO^0(2,1) \cong \mathrm{PSL}(2,\mathbb{R})$

Lorentzian transformation group

$O(2,1) > SO^0(2,1)$: preserve B, the orientations of space and time.

Consider $\mathbb{H}^2 := \{\mathbf{x} \in \mathbb{R}^2_1 \mid B(\mathbf{x},\mathbf{x}) = -1, x_3 > 0\}$ in $\mathbb{R}^2_1$. Then

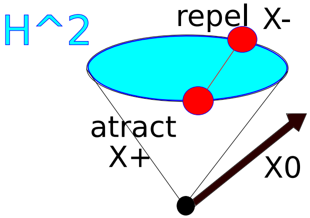
$$\begin{array}{ccc} \mathbb{R}^2_1 & \xrightarrow{SO^0(2,1)} & \mathbb{R}^2_1 \\ {\scriptstyle proj.}\downarrow & & \downarrow{\scriptstyle proj.} \\ \mathbb{H}^2 & \xrightarrow{\mathrm{PSL}(2,\mathbb{R})} & \mathbb{H}^2 \end{array}$$

Through this correspondence, we call an element of $SO^0(2,1)$ *hyperbolic*, *parabolic*, or *elliptic*.

Eigenvectors of $SO^0(2,1)$

A hyperbolic g (as $\mathrm{PSL}(2,\mathbb{R})$) has two fixed points in $\partial\mathbb{H}^2$.
As $SO^0(2,1)$, g has three eigenvectors in $\mathbb{R}^2_1$ as follows:

- $\mathbf{X}_g^-$: repelling point in $\partial\mathbb{H}^2$ with future pointing,
- $\mathbf{X}_g^+$: attracting point in $\partial\mathbb{H}^2$ with future pointing,
- $\mathbf{X}_g^0$: invariant axis of g in $\mathbb{H}^2$,
  with $B(\mathbf{X}^0,\mathbf{X}^0) = 1$ and $\det[\mathbf{X}^-,\mathbf{X}^+,\mathbf{X}^0] > 0$.



Margulis spacetime

$SO^0(2,1) \ltimes \mathbb{R}^2_1$: affine transformation group, Γ : discrete subgroup

Definition

When $\Gamma \cong$ a free group and $\Gamma \curvearrowright \mathbb{R}^2_1$ is PD, we say

- Γ is a *Margulis group* ;
- the quotient manifold $\mathbb{R}^2_1/\Gamma$ is a *Magulis spacetime*.

Theorem (Choi-Goldman, 2013)

A Margulis spacetime $\mathbb{R}^2_1/\Gamma$ *is topologically tame.*

Theorem (Danciger-Gueritaud-Kassel)

A Margulis spacetime is a principal $\mathbb{R}$ *bundle over* S.

Affine deformation of a free group

Let $S_{g,b}$ denote a hyperbolic surface ($b > 0$). $\Gamma := \pi_1(S_{g,b})$.

(1) $\rho_0 : \Gamma \to SO^0(2,1)$, holonomy, fixed

(2) $\mathbf{u} : \Gamma \to \mathbb{R}^2_1$, cocycle

(3) $\rho_{\mathbf{u}} := (\rho_0, \mathbf{u})$: affine deformation

A *cocycle* $\mathbf{u} : \Gamma \to \mathbb{R}^2_1$ is a map; for $\forall g, h \in \Gamma$, $\mathbf{u}(gh) = g\mathbf{u}(h) + \mathbf{u}(g)$.

($\mathbf{v} \in \mathbb{R}^2_1$) A cocycle $\delta_{\mathbf{v}}$ is a *coboundary* [a] if $\delta_{\mathbf{v}}(g) = \mathbf{v} - g\mathbf{v}$.

[a] corresponding to a conjugation with translation $\mathbf{v}$

$\mathrm{Z}^1(\Gamma, \mathbb{R}^2_1)$: a set of cocycles, $\mathrm{B}^1(\Gamma, \mathbb{R}^2_1)$: a set of coboundaries.

Cohomology class (with ρ_0 fixed)

$$\mathrm{H}^1(\Gamma, \mathbb{R}^2_1) := \mathrm{Z}^1(\Gamma, \mathbb{R}^2_1)/\mathrm{B}^1(\Gamma, \mathbb{R}^2_1) = \{[\mathbf{u}] \mid \mathbf{u} \in \mathrm{Z}^1(\Gamma, \mathbb{R}^2_1)\}$$

Margulis invariant

Let $\rho_{\mathbf{u}}$ be an affine deformation.

Definition (Margulis invariant)

A map $\alpha_{\mathbf{u}} : \Gamma \to \mathbb{R}$,
$\alpha_{\mathbf{u}}(g) := B(\mathbf{X}_g^0, \rho_{\mathbf{u}}(g)(x) - x) = B(\mathbf{X}_g^0, \mathbf{u}(g) + gx - x) \ , \ x \in \mathbb{R}_1^2$.

(1) independent on $x \in \mathbb{R}_1^2$,
(2) decide affine deformation up to a conj. (Drumm_Goldman),
(3) $\mathbf{u}(g)$ is represented as $\alpha_{\mathbf{u}}(g)\mathbf{X}_g^0 + c^-\mathbf{X}_g^- + c^+\mathbf{X}_g^+$,

Lemma (Margulis)

$\exists h_1, h_2 \in \Gamma$ *s.t.* $\alpha_{\mathbf{u}}(h_1) \leq 0 \leq \alpha_{\mathbf{u}}(h_2)$
$\Rightarrow \rho_{\mathbf{u}}(\Gamma) \curvearrowright \mathbb{R}_1^2$: **not** *properly discontinuous*

Cocycles inducing PD action

Theorem (Goldman-Labourie-Margulis, 2009)

Proper $\subset \mathrm{H}^1(\Gamma, \mathbb{R}^2_1)$ *consists of two* [a] *symmetric open convex cones.*

[a] Denote them by $\mathbf{Proper}_+$ and $\mathbf{Proper}_-$.

Theorem (Danciger-Gueritaud-Kassel)

$[\mathbf{u}] \in \mathbf{Proper}_+$ *if and only if*

$$\inf_{g \in \Gamma \textit{ with } \ell(g) > 0} \frac{\alpha_{\mathbf{u}}(g)}{\ell(g)} > 0$$

$\ell : \Gamma \to \mathbb{R}$, $g \to$ (length of $\rho_0(g)$ on the hyperbolic structure)

Example of affine deformations

When $\Gamma = \langle \gamma_1, \gamma_2 \rangle$, the following is canonically linearly isomorphic:

$$\mathrm{H}^1(\Gamma, \mathbb{R}^2_1) \ni [\mathbf{u}] \longleftrightarrow (\alpha_{\mathbf{u}}(\gamma_1), \alpha_{\mathbf{u}}(\gamma_2), \alpha_{\mathbf{u}}(\gamma_2^{-1}\gamma_1^{-1})) \in \mathbb{R}^3.$$

Theorem (Charette-Drumm-Goldman, 2010)

When $\Gamma = \pi_1(S_{0,3})$, $[\mathbf{u}] \in \mathbf{Proper}_+ \Leftrightarrow \alpha_{\mathbf{u}}(g_1), \alpha_{\mathbf{u}}(g_2), \alpha_{\mathbf{u}}(g_3) > 0$.

Theorem (Charette-Drumm-Goldman, 2015)

When $\pi_1(S_{1,1})$, $[\mathbf{u}] \in \mathbf{Proper}_+ \Leftrightarrow \mathbf{u}$ *is in 'Tile's.*

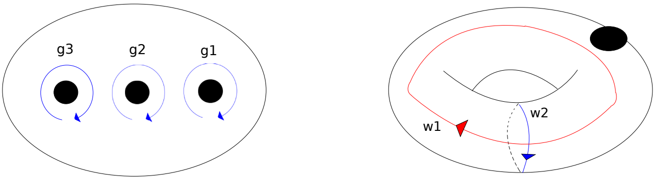


Figure: Thrice holed sphere and once holed torus

Consider $\mathbb{F}_n (n \geq 3)$

Suppose that the following surfaces satisfy : $\#\{\text{punctures}\} > 0$.

(1) Take two hyperbolic surfaces S^1, S^2
- Assume that $\exists$ boud. comp. $g^i \subset \partial S^i$ *s.t.* $\ell(g^1) = \ell(g^2) > 0$.

(2) S denote a surface glued by S_1 and S_2 along g^i.
- $g \in \pi_1(S)$, $g := g^1 = g^2$.

(3) Take $\mathbf{u}^i \in \mathrm{Z}^1(\pi_1(S^i), \mathbb{R}^2_1)$: cocycles
- Assume that $\alpha_{\mathbf{u}^1}(g^1) = \alpha_{\mathbf{u}^2}(g^2)$.

Construct special cycles

Definition (Combination of cocycles)

Define the map $\mathbf{u}^1 \#_g \mathbf{u}^2 : \pi_1(S) \to \mathbb{R}^2_1$ as follows:

$$\mathbf{u}^1 \#_g \mathbf{u}^2(h) = \begin{cases} \mathbf{u}^1(h), & (h \in \pi_1(S^1)), \\ \mathbf{u}^2(h) + \delta_{\textbf{trans.}}(h), & (h \in \pi_1(S^2)), \\ \text{by cocycle condition} & (\text{the others}), \end{cases}$$

where $\mathbf{u}^2(g) + \delta_{\textbf{trans.}}(g) = \mathbf{u}^1(g)$ holds.

Affine twist cocycle

Definition (Affine twist cocycle)

Define the map $\mathbf{AT}_g : \pi_1(S) \to \mathbb{R}^2_1$ as follows:

$$\mathbf{AT}_g(h) = \begin{cases} \mathbf{0}, & (h \in \pi_1(S^1)) \\ \mathbf{X}^0_g - h\mathbf{X}^0_g, & (h \in \pi_1(S^2)) \\ \text{by cocycle condition} & \text{(the others)}. \end{cases}$$

Theorem (M.)

$\tau \in \mathbb{R}$, *cocycles* $\mathbf{u}^1 \#_g \mathbf{u}^2 + \tau \mathbf{AT}_g$ *generate* $\mathrm{H}^1(\pi_1(S), \mathbb{R}^2_1)$.

Proof.

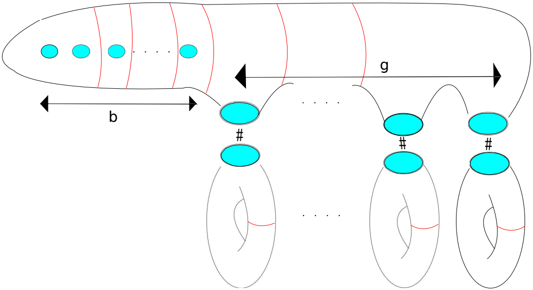
Show that their dimensions are equal. □

Decomposition of a surface

Let $S_{g,b}$ be a hyperbolic surface with boudaries.

- $\{p_i\}_i$: original boundary components ($i = 1, \ldots, b$),
- $\{q_j\}_j$: dividing curves ($i = 1, \ldots, 3g + b - 3$).

- $t_{\mathbf{u}}(q_i)$: coefficient of the affine twist along q_i



Corollary

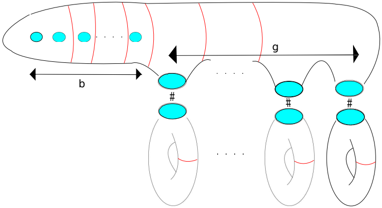
Corollary

The following correspondence is canonically linearly isomorphic.

$$\mathrm{H}_1(\pi_1(S_{g,b}), \mathbb{R}^2_1) \ni [\mathbf{u}] \leftrightarrow (\alpha, \beta, \tau) \in \mathbb{R}^{6g-6+3b},$$

where 'the angle of each torus' is not $\pi/2$, and

- $\alpha := (\alpha_{\mathbf{u}}(p_1), \ldots, \alpha_{\mathbf{u}}(p_b))$, $\beta := (\alpha_{\mathbf{u}}(q_1), \ldots, \alpha_{\mathbf{u}}(q_{3g+b-3}))$,
- $\tau := (t_{\mathbf{u}}(q_1), \ldots, t_{\mathbf{u}}(q_{3g+b-3}))$.



Angle of torus

Set $\pi_1(S_{1,1}) = \langle w_1, w_2 \rangle$.

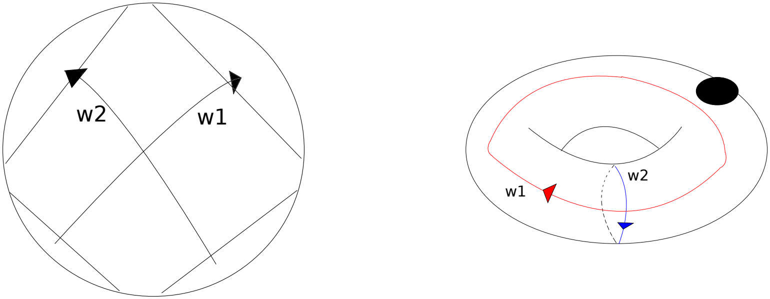


Figure: Action by w_1, w_2 on $\mathbb{H}^2$, and 'the angle of torus' $\theta_{w_1}^{w_2}$

Proof of Corollary

Let $\pi_1(S_{1,1}) = \langle w_1, w_2 \rangle$, and $K := [w_1, w_2]$.

Proof.

We would like to obtain
$\mathrm{H}^1(\pi_1(S_{1,1}), \mathbb{R}^2_1) \cong \{(\alpha_{\mathbf{u}}(w_2), \alpha_{\mathbf{u}}(K), t_{\mathbf{AT}_{w_2}}(w_2)) \in \mathbb{R}^3\}$.
For any cocycle $\mathbf{u}$ on $S_{1,1}$, we can have (by direct calculation)

$$\mathbf{u}(K) = (Id - w_1 w_2 w_1^{-1})\mathbf{u}(w_1) + (w_1 - [w_1, w_2])\mathbf{u}(w_2)$$

Next, set $\mathbf{u}(w_1) = t\mathbf{X}_1^0 + a\mathbf{X}_1^- + b\mathbf{X}_1^+$, $\mathbf{u}(w_2) = \alpha_2\mathbf{X}_2^0 + c\mathbf{X}_2^- + d\mathbf{X}_2^+$.

1. $\theta_{w_1}^{w_2} \neq \frac{\pi}{2} \Rightarrow \alpha_{\mathbf{u}}(K)$ dependes linearly on t, α_2, a, b, c, d.
2. $\theta_{w_1}^{w_2} = \frac{\pi}{2} \Rightarrow \alpha_{\mathbf{u}}(K) = 0$ for any $a, b, c, d \in \mathbb{R}$.

□

Deform hyperbolic structures by cocycle

By $\mathbb{R}^2_1 \cong \mathfrak{g} := \mathfrak{s}l_2(\mathbb{R})$, consider the deformation; for $g \in \mathrm{SL}(2, \mathbb{R})$,

$$g_t := \exp\left(t\mathbf{u}(g)\right) \cdot g$$

Lemma (Goldman-Margulis, 2000)

$$\frac{1}{2}\frac{d}{dt}\,|_{t=0}\ell(g_t) = \alpha_{\mathbf{u}}(g)$$

(Proof) Note that $\mathfrak{g}$ has Killing form $B(X, Y) = \frac{1}{2}tr(XY)$ and $tr(g) = 2\cosh(\ell(g)/2)$. By direct calculation.

Strip deformation

Let $\mathbb{A}$ be the arc complex of $S_{g,b}$, whose vertices (arcs) have end points on $\partial S_{g,}$.

(1) Fix a geodesic representation of vertex a of $\mathfrak{A}$

(2) Fix a reference point p_a on each a

(3) Fix a weight $m_a > 0$ at each p_a

Then a *strip deformation* is $F : \mathbb{A} \to \{\text{new hyperbolic surfaces}\}$ by inserting *strip* on a whose length at p_a is m_a.

An *infinitesimal strip deformation* is $\mathbf{f} : \mathbb{A} \to \mathrm{Z}^1(\pi_1(S_{g,b}), \mathbb{R}^2_1)$, which is differentiation of F with weigh $tm_a > 0$.

Properly discontinuous action

Theorem (Danciger-Gueritaud-Kassel, 2014)

Let $X \subset \mathbb{A}$, whose elements decompose $S_{g,b}$ into disks. Then (by barycentric interpolation)

$$f|_X : X \to \mathbf{Proper}_+(S_{g,b})$$

is a homeomorphism.

Corollary

$\exists$ fundamental domain bounded by pairwise disjoint crooked planes.

Twist and affine twist cocycle

Recall that the affine twist cocycle $\mathbf{AT}_g$ on $S_{g,b}(b > 0)$.
The Goldman-Margulis deformation is $h_t := \exp\left(t\mathbf{AT}_g(h)\right) \cdot h$.

Theorem (M.)

$\forall h \in \pi_1(S_{g,b})$, $2\alpha_{\mathbf{AT}_g}(h) = \frac{d}{dt}|_{t=0}\ell(h_t) = \sum_{p\in g\cap h} \cos\left\{\left(\theta_g^h\right)_p\right\}$

(Proof) $\pi_1(S_2) \cdot \pi_1(S_1) \ni h^2 h^1 = h$.
Show that $2\alpha_{\mathbf{AT}_g}(h) = \cos\{(\theta_g^h)\}_{p_1} + \cos\{(\theta_g^h)\}_{p_2}$.

$$\begin{aligned} 2\alpha_{\mathbf{AT}_g}(h) &= B(\mathbf{X}_h^0, \mathbf{AT}_g(h)) \\ &= B(\mathbf{X}_h^0, h^2\mathbf{AT}_g(h^1) + \mathbf{AT}_g(h^2)) \\ &= B(\mathbf{X}_h^0, \mathbf{AT}_g(h^2)) \\ &= B(\mathbf{X}_{h^2h^1}^0, \mathbf{X}_g^0 - h^2\mathbf{X}_g^0) \\ &= B(\mathbf{X}_{h^2h^1}^0, \mathbf{X}_g^0) - B(\mathbf{X}_{h^1h^2}^0, \mathbf{X}_g^0). \end{aligned}$$

Finally, apply the formulation $B(\mathbf{X}_a^0, \mathbf{X}_b^0) = \cos\left(\theta_a^b\right)$.