Critical exponents of normal subgroups of a Kleinian group

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Critical exponent of a Kleinian group

Denote by $(\mathbb{D}, d)$ the Poincaré ball model of hyperbolic n-space. A *Kleinian group* is a discrete subgroup of $\mathrm{Isom}^{+}(\mathbb{D}, d)$.

Definition

The critical exponent of a Kleinian group G is given by

$$\delta(G) := \inf\left\{ s > 0 \,\middle|\, \sum_{g \in G} \mathrm{e}^{-s \cdot d(0, g(0))} < \infty \right\}.$$

Fact:

$$\delta(G) = \limsup_{r \to \infty} \frac{1}{r} \log \operatorname{card}\left(\{ g \in G \mid d(0, g(0)) < r \}\right).$$

Main goal

To investigate the relation between $\delta(N)$ and $\delta(\Gamma)$ for a normal subgroup N of a Kleinian group Γ.

Lower bound of $\delta(N)$

Let Γ be a non-elementary Kleinian group, and let $\{1\} \neq N \lhd \Gamma$.

Theorem (Falk and Stratmann (2004))

$$\delta(N) \geq \delta(\Gamma)/2$$

Definition

We say that a Kleinian group Γ is of divergence type if

$$\sum_{\gamma \in \Gamma} \mathrm{e}^{-\delta(\Gamma) \cdot d(0,\gamma(0))} = \infty.$$

Theorem (Roblin (2005))

If Γ is of divergence type then $\delta(N) > \delta(\Gamma)/2$.

It is an open problem whether $\exists N \lhd \Gamma$ s.t. $\delta(N) = \delta(\Gamma)/2$.

Standing assumption – for simplicity

To make the main ideas more transparent, we assume that

$\Gamma = \langle \gamma_1, \gamma_2, \ldots \gamma_r \rangle$ is a free group with at least two generators.

An embedding lemma for abstract groups

Let $\{1\} \neq N \lhd \Gamma = \langle \gamma_1, \gamma_2, \ldots \gamma_r \rangle$.

Lemma

There exists a finite set $F \subset N$ and a map $\tau : \Gamma \to F$ such that

$$\iota : \Gamma \to N, \quad \iota(g) := g\tau(g)g^{-1}, \quad g \in \Gamma,$$

is one-to-one.

Proof.

Fix an arbitrary $h \in N \setminus \{1\}$.
For each $g \in \Gamma$ there exists $\alpha \in \{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}\}$ such that

$$g\,\alpha\,h\,\alpha^{-1}\,g^{-1} \text{ has no cancellation (exists!).}$$

Set $\tau(g) := \alpha\,h\,\alpha^{-1} \in F := \{\gamma_1^{\pm 1} h \gamma_1^{\mp 1}, \gamma_2^{\pm 1} h \gamma_2^{\mp 1}\} \subset N$. □

Triangle inequality for the hyperbolic metric

Let Γ be a Kleinian group.

Lemma

For all $g, h \in \Gamma$ and $s > 0$, we have

$$\mathrm{e}^{-sd(0,g(0))} \leq \mathrm{e}^{\frac{s}{2}d(0,h(0))}\mathrm{e}^{-\frac{s}{2}d\left(0,g^{-1}hg(0)\right)}.$$

Proof.

Triangle inequality implies that for all $g, h \in \Gamma$,

$$\begin{aligned} & d\left(0, g^{-1}hg(0)\right) \\ \leq\ & d\left(0, g^{-1}(0)\right) + d\left(g^{-1}(0), g^{-1}h(0)\right) + d\left(g^{-1}h(0), g^{-1}hg(0)\right) \\ =\ & d\left(0, g^{-1}(0)\right) + d\left(0, h(0)\right) + d\left(0, g(0)\right) \\ =\ & 2d\left(g(0), 0\right) + d\left(h(0), 0\right). \end{aligned}$$

Hence, $d\left(g(0), 0\right) \geq -\frac{1}{2}d\left(h(0), 0\right) + \frac{1}{2}d\left(0, g^{-1}hg(0)\right)$. □

Weak inequality $\delta(N) \geq \delta(\Gamma)/2$.

Proof.

Let $s > 0$. First using the triangle-inequality-lemma and then the embedding lemma gives

$$\begin{aligned}
\sum_{g \in \Gamma} \mathrm{e}^{-sd(0,g(0))} &\leq \sum_{g \in \Gamma} \mathrm{e}^{\frac{s}{2} d(0,\tau(g)(0))} \mathrm{e}^{-\frac{s}{2} d\left(0,g^{-1}\tau(g)g(0)\right)}. \\
&\leq \max_{h \in F} \mathrm{e}^{\frac{s}{2} d(0,h(0))} \sum_{g \in \Gamma} \mathrm{e}^{-\frac{s}{2} d\left(0,g^{-1}\tau(g)g(0)\right)}. \\
&\leq \max_{h \in F} \mathrm{e}^{\frac{s}{2} d(0,h(0))} \sum_{\rho \in N} \mathrm{e}^{-\frac{s}{2} d(0,\rho(0))}.
\end{aligned}$$

Let $\varepsilon > 0$. Putting $s := \delta(\Gamma) - \varepsilon$ in the above shows that $\sum_{\rho \in N} \mathrm{e}^{-\frac{\delta(\Gamma)-\varepsilon}{2} d(0,\rho(0))} = \infty$. Hence, $\delta(N) \geq (\delta(\Gamma) - \varepsilon)/2$. □

Strict inequality $\delta(N) > \delta(\Gamma)/2$ for Γ of divergence type

To prove the strict inequality we use the following result:

Matsuzaki/Yabuki: If N is of divergence type, then $\delta(N) = \delta(\Gamma)$.

Proof.

We prove that $\delta(N) > \delta(\Gamma)/2$ if Γ is of divergence type.
Recall that we have

$$\sum_{g\in\Gamma} \mathrm{e}^{-\delta(\Gamma) d(0,g(0))} \leq \max_{h\in F} \mathrm{e}^{\frac{\delta(\Gamma)}{2} d(0,h(0))} \sum_{\rho\in N} \mathrm{e}^{-\frac{\delta(\Gamma)}{2} d(0,\rho(0))}.$$

Since Γ is of divergence type, $\sum_{\rho\in N} \mathrm{e}^{-\frac{\delta(\Gamma)}{2} d(0,\rho(0))} = \infty$.
Suppose for a contradiction that $\delta(N) = \delta(\Gamma)/2$. Then N is of divergence type, which implies that $\delta(N) = \delta(\Gamma)$. Hence, $\delta(\Gamma)/2 = \delta(\Gamma)$ contradicting $\delta(\Gamma) > 0$. □

Conformal measure

To prove the result of Matsuzaki/Yabuki, we need some facts from the Patterson-Sullivan theory.

Definition

The limit set $\Lambda(G)$ of a Kleinian group G is given by

$$\Lambda(G) := \overline{G(0)} \setminus G(0).$$

Note that $\Lambda(G) \subset \partial\mathbb{D} = \mathbb{S}$. The complement of $\Lambda(G)$ is the largest open subset of $\overline{\mathbb{D}}$ such that G acts properly discontinuously.

Definition (Patterson 1976, Sullivan 1979)

A Borel probability measure μ on $\Lambda(G)$ is called *s-conformal* if

$$\mu(g(E)) = \int_E |g'|^s \, d\mu, \quad \forall g \in G, \quad E \subset \Lambda(G) \text{ Borel}.$$

Conformal measures and Hausdorff dimension of the limit set

Denote by $\Lambda_c(G) \subset \Lambda(G)$ the conical limit set of G.

Theorem (Bishop/Jones 1997)

For every Kleinian group Γ we have

$$\delta(\Gamma) = \dim_H(\Lambda_c(\Gamma)).$$

Construction principle of Patterson-Sullivan measure implies

$$\inf\{s > 0 \mid \exists s\text{-conformal measure}\} \leq \delta(\Gamma).$$

Using a shadow lemma of Sullivan for an s-conformal measure, one can show that

$$\dim_H(\Lambda_c) \leq s.$$

Hence, by combining with $\delta(\Gamma) = \dim_H(\Lambda_c(\Gamma))$ we see that

$$\inf\{s > 0 \mid \exists s\text{-conformal measure}\} = \delta(\Gamma).$$

Conformal measures and observation point

For a probability measure μ on $\Lambda(G)$, $z \in \mathbb{D}$ and $s > 0$ we define

$$\mu_z := \left|k_z'\right|^s d\mu,$$

where k_z is a Moebius transformation satisfying $k_z(z) = 0$.

Criterion

μ is s-conformal for G if for every $g \in G$ and $z \in \mathbb{D}$,

$$\mu_{g(z)} \circ g = \mu_z.$$

We also need the following fact.

Theorem

If G is of divergence type then there exists one and only one $\delta(G)$-conformal measure.

Proof of Matsuzaki/Yabuki result

Theorem

If N is of divergence type and $\{1\} \neq N \lhd \Gamma$, then $\delta(N) = \delta(\Gamma)$.

Sketch of Proof.

We prove that the unique $\delta(N)$-conformal measure μ for N is $\delta(N)$-conformal for Γ. Hence, $\delta(\Gamma) \leq \delta(N)$.
With μ_z as above we have $\mu_{h(z)} \circ h = \mu_z$, for all $h \in N$ and $z \in \mathbb{D}$.
Fix $g \in \Gamma$ and put $\nu_z := \mu_{g(z)} \circ g$. For $h \in N$ there exists $\tilde{h} \in N$ such that $gh = \tilde{h}g$. Hence,

$$\nu_{h(z)} \circ h = \mu_{g(h(z))} \circ g \circ h = \mu_{\tilde{h}g(z)} \circ \tilde{h} \circ g = \mu_{g(z)} \circ g = \nu_z.$$

From the uniqueness of $\delta(N)$-conformal measure for N, one can deduce that $\nu_z = \mu_z$. Hence, $\mu_{g(z)} \circ g = \mu_z$ and μ is $\delta(N)$-conformal for Γ. □

Lower bound is sharp

Theorem (Matsuzaki/Taylor/Bonfert-Taylor 2012)

If the Fuchsian group Γ uniformizes a closed hyperbolic surface, then there exists a sequence of normal subgroups $N_k \triangleleft \Gamma$ such that $\delta(N_k) \searrow 1/2 = \delta(\Gamma)/2$ as $k \to \infty$.

Idea of proof: Define N_k such that $\mathbb{D}/N_k$ is planar and lengths of all closed geodesics greater than k. By Gauss-Bonnet we estimate $A(W)/\ell(\partial W)$ for geodesic compact subsurface $W \subset \mathbb{D}/N_k$:

$$\frac{A(W)}{\ell(\partial W)} < \frac{-2\pi(2-n)}{n \cdot k} < \frac{2\pi}{k} \to 0, \qquad \text{as } k \to \infty.$$

Hence, the isoperimetric const. satisfy $h(\mathbb{D}/N_k) \searrow 1$ (Matsuzaki 2005). By Cheeger's inequality, bottom of Laplace spectrum $\lambda_0(\mathbb{D}/N_k) \nearrow 1/4$. By Elstrodt-Patterson-Sullivan, $\delta(\mathbb{D}/N_k) \searrow 1/2$.

Cayley graph setting - critical exponent

Denote by F_n the free group of rank $n \geq 2$ and by T_n the Cayley graph of F_n with respect to a free set of generators S. Denote by d the word metric on F_n.

For $G < F_n$ the critical exponent of G is given by

$$\delta(G) = \inf \left\{ s > 0 \mid \sum_{g \in G} \mathrm{e}^{-sd(1,g)} < \infty \right\}.$$

We have

$$\delta(G) = \limsup_{n \to \infty} \frac{1}{R} \log \operatorname{card} \{ g \in G \mid d(1,g) \leq R \}.$$

For $\{1\} \neq N \triangleleft F_n$ the cogrowth of F_n/N is given by

$$\eta(F_n/N) := \frac{\delta(N)}{\delta(F_n)} = \frac{\delta(N)}{\log(2n-1)}.$$

Cayley graph setting – Results

Theorem (Roblin, 2005)

If $G < F_n$ is of divergence type, then

$$\forall N \lhd G \text{ non-trivial}: \qquad \delta(N) > \delta(G)/2.$$

In particular, $\eta(F_n/N) > 1/2$ (Grigorchuk, 1980).
For $\{1\} \neq N \lhd F_n$ we denote by Γ_N the quotient graph F_n/N.
The growth of Γ_N is given by

$$\operatorname{growth}(\Gamma_N) = \limsup_{n\to\infty} \operatorname{card}\{\gamma \in \Gamma_N \mid d(1,\gamma) \leq n\}^{1/n}.$$

Theorem (Grigorchuk- de la Harpe, 1997)

For every $N \lhd F_n$ non-trivial with $|F_n : N| = \infty$ we have

$$\operatorname{growth}(\Gamma_N) < \operatorname{growth}(T_n) = 2n - 1.$$

Cayley graph setting – New result

For $\{1\} \neq N \triangleleft F_n$ denote by ℓ_N the injectivity radius

$$\ell_N := \min\{\ell \in \mathbb{N} \mid \exists h \in N \text{ s.t. } d(1,h) = \ell\}.$$

Theorem (J.-Matsuzaki)

Let (N_k) be a sequence of normal subgroups of F_n such that Γ_{N_k} is planar. If $\ell_{N_k} \to \infty$ then

$$\lim_{n\to\infty} \delta(N_k) = \frac{1}{2}\delta(F_n) \quad \textit{and} \quad \lim_{n\to\infty} \operatorname{growth}(\Gamma_{N_k}) = \operatorname{growth}(T_n).$$

Corollary

Let $F_n = \langle\langle g_1, \dots, g_n \rangle\rangle$ and $N_k := \langle\langle g_1^{k_1}, g_2^{k_2}, \dots, g_s^{k_s} \rangle\rangle$. If $\min\{k_i \mid i \leq s\} \to \infty$ then the assertion of the theorem holds.

The corollary also follows from results of Grigorchuk and Shukhov.

Sketch of proof

The strategy is the same as for Fuchsian groups.

Lemma

If $\Gamma_{N_k} := F_n/N_k$ is planar and $\ell_{N_k} \to \infty$ then

$$i(\Gamma_{N_k}) := \inf \frac{1}{2n} \frac{\mathrm{card}(\partial C)}{\mathrm{card}(C)} \to \frac{n-1}{n}, \quad \text{as } k \to \infty,$$

where the infimum is taken over all finite subgraphs of Γ_{N_k}.

We use a reduction to core subgraphs: The core subgraph of a connected finite graph is the minimal subgraph for which the inclusion is a homotopy equivalence.
To prove the lemma, we estimate using planarity

$$\mathrm{card}(C) \geq (-\chi(C)+2)2\ell_{N_k}.$$

Sketch of proof – cont.

The discrete Laplacian $\triangle : \ell^2(\mathscr{V}(\Gamma)) \to \ell^2(\mathscr{V}(\Gamma))$ is given by

$$\triangle f(x) := f(x) - \frac{1}{2n} \sum_{y \sim x} f(y), \quad x \in \mathscr{V}(\Gamma).$$

Denote by $\lambda_0(\Gamma)$ the bottom of the spectrum of $\triangle$.

Theorem (Mohar, 1988)

$$i(\Gamma) \leq \sqrt{1-(1-\lambda_0(\Gamma)^2}.$$

It follows that, if $i(\Gamma_{N_k}) \to \frac{n-1}{n}$ then $\lambda_0(\Gamma_{N_k}) \to \lambda_0(T_n)$.
Finally, by Grigorchuk's cogrowth formula,

$$\delta(N_k) \to \delta(F_n)/2, \quad \text{as } k \to \infty.$$

Weighted critical exponents

Let $F_n = \langle S \rangle$ denote the free group and let

$$\partial F_n = \Sigma := \left\{ \omega = (\omega_1, \omega_2, \dots) \in (S \cup S^{-1})^{\mathbb{N}} \mid \omega_i \omega_{i+1} \neq 1 \right\}.$$

The left shift is given by

$$\sigma : \Sigma \to \Sigma, \qquad \sigma(\omega_1, \omega_2, \dots) = (\omega_2, \omega_3, \dots).$$

Let $\varphi : \Sigma \to \mathbb{R}$ be a (weight) function.
For $g \in F_n$, $g = \omega_1 \dots \omega_k$ we set

$$S_{\omega_1 \dots \omega_k} \varphi := \sup_x \sum_{j=0}^{k-1} \varphi \circ \sigma^j(\omega_1, \dots, \omega_k, x).$$

For $G < F_n$ the critical exponent of G with respect to φ is given by

$$\delta(G, \varphi) = \inf \left\{ s > 0 \mid \sum_{g \in G} \mathrm{e}^{-s S_g \varphi} < \infty \right\}.$$

Lower bounds for weighted critical exponents

Let φ be symmetric Hölder continuous weight, where φ is called symmetric if

$$\exists C > 0 \text{ s.t. } \forall g \in G : \quad S_g\varphi \leq C \cdot S_{g^{-1}}\varphi.$$

We say that $G < F_n$ is of $\delta(G,\varphi)$-divergence type if

$$\sum_{g \in G} \mathrm{e}^{-\delta(G,\varphi) S_g \varphi} = \infty.$$

Theorem (J. 2014)

Let $\{1\} \neq N \triangleleft F_n$. If N is of $\delta(N,\varphi)$-divergence type then

$$\delta(N,\varphi) = \delta(F_n,\varphi).$$

Theorem (J. 2014)

Let $\{1\} \neq N \triangleleft F_n$. Then $\delta(N,\varphi) > \delta(F_n,\varphi)/2$.

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