On asymptotic Jenkins-Strebel rays and a global coordinate of the Teichmüller space

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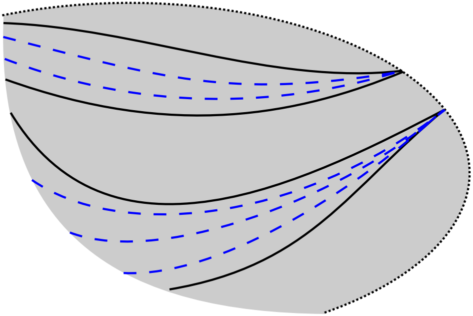
In this talk, we give a parametrization of asymptotic Jenkins-Strebel rays. It is a global coordinates of the Teichmüller space.



$\Sigma = \Sigma_{g,p}$: Riemann surface of type (g,p) s.t. $3g-3+p>0$.

Definition (Teichmüller space)

The **Teichmüller space** of Σ
$\mathscr{T} := \{(S,f) \mid S$: Riemann surface, $f:\Sigma\to S$: qc$\}/\sim$,
$(S_1,f_1)\sim(S_2,f_2) :\Leftrightarrow \exists h: S_1\to S_2$ conformal s.t. $h\circ f_1$ is homotopic to f_2. $[S,f]$: an equivalence class of (S,f).

Definition (Teichmüller distance)

The **Teichmüller distance** $d_{\mathscr{T}}$ on $\mathscr{T}$ is a complete distance;
For $\forall p_1=[S_1,f_1], \forall p_2=[S_2,f_2]\in\mathscr{T}$, define

$$d_{\mathscr{T}}(p_1,p_2) := \frac{1}{2}\log\inf_h K(h),$$

where h ranges over all qc $h: S_1\to S_2$ s.t. it homotopic to $f_2\circ f_1^{-1}$. $K(h)$: the qc dilatation of h.

Definition (holomorphic quadratic differential)

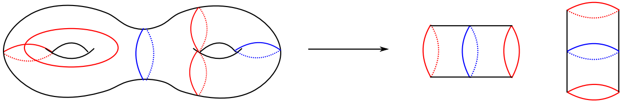
$\varphi = \varphi(z)dz^2$: a **holomorphic quadratic differential** on S, where $\varphi(z)$ is holomorphic for a local coordinate $z = x + iy$ on S.
$\exists w$: a φ-**coordinate**, φ is represented by $\varphi = dw^2$ on $S - \{\text{zeros of } \varphi\}$.
$\|\varphi\| := \iint |\varphi| dxdy < \infty \Leftrightarrow \varphi$ has at worst poles of order 1 at punctures of S.

C_φ: a union of all singularities of φ and all vertical segments for φ-coordinates which are joining singularities.
Each component of $S - C_\varphi$ is the following:

1. **annulus**: Every non-singular vertical lines for φ-coordinates are simple closed curves, they generate the annulus.
2. **minimal domain**: Every non-singular vertical lines for φ-coordinates have the recurrence.

Definition (Jenkins-Strebel differential)

φ: a **Jenkins-Strebel differential**, all components of $S - C_\varphi$ are annuli.

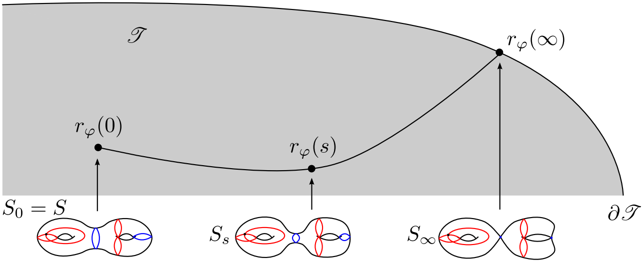


Definition (Teichmüller geodesic ray)

$[S, f] \in \mathscr{T}$, φ: a h.q.d. on S s.t. $\|\varphi\| < \infty$.
For $\forall s \geq 0$, the Beltrami coefficient $\mu_s = \tanh(s)|\varphi|/\varphi$ determines a qc $f_s : S \to S_s$.
It is represented by $z = x + iy \mapsto z_s = e^{2s}x + iy$ for a q-coordinate z.
The mapping $r_\varphi : \mathbb{R}_{\geq 0} \to \mathscr{T}$, $s \mapsto [S_s, f_s \circ f]$ is isometry, it called by a **Teichmüller geodesic ray** on $\mathscr{T}$.

Definition (Jenkins-Strebel ray and their end)

r_φ is a **Jenkins-Strebel ray**: r_φ is a Teichmüller geodesic ray and φ is J-S.
$r_\varphi(\infty)$: The end point of r_φ in $\partial\mathscr{T} := \hat{\mathscr{T}} - \mathscr{T}$.
$\hat{\mathscr{T}}$: the **augmented Teichmüller space**, it is the set of all Riemann surfaces with nodes (and markings) and elements of $\mathscr{T}$.



$\Gamma := \{\gamma_1, \ldots, \gamma_k\}$ is an admissible curve family on Σ: all curves of Γ are essential simple closed curves s.t. they are non-intersect and non-homotopic each other.
$S_+^{k-1} := \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid \sum_{j=1}^k x_j^2 = 1, x_j > 0 \text{ for } \forall j\}$.

Theorem 1 (Moduli problem, [Str84])

For $\forall\Gamma$ and $\forall\mathfrak{m} = (m_1, \ldots, m_k) \in S_+^{k-1}$, $\exists!\alpha > 0$ and $\exists\varphi$: a J-S differential on Σ whose moduli of the annuli corresponding to Γ are $\alpha\mathfrak{m} = (\alpha m_1, \ldots, \alpha m_k)$. If $\|\varphi\| = 1$, φ is unique.

$\overline{\Sigma} := \Sigma \cup \{\text{punctures of } \Sigma\}$, $x \in \overline{\Sigma}$: a puncture of Σ.
$\varphi := (a^2/z^2 + \cdots)dz^2$: a **meromorphic** quadratic differential on $\overline{\Sigma}$ s.t. z is a local coordinate of $\overline{\Sigma}$ near x with $z(x) = 0$ and $a > 0$.
a: a **leading coefficient** of φ at x.

D: the component of $\overline{\Sigma} - C_\varphi$ which contains x.
$w := \exp(1/a \int \sqrt{\varphi(z)}dz) : D \to \{0 < |w| < r\}$ conformally, and the equation $a^2dw^2/w^2 = \varphi(z)dz^2$ holds. The non-singular vertical lines of a^2dw^2/w^2 are circles of the form $\{|w| = r'\}$ for $0 < \forall r' < r$. The length of each trajectories is $2\pi a$ with respect to the metric $|\varphi(z)|^{\frac{1}{2}}|dz| = a|dw/w|$.

$\mathbb{R}_+^k$: the set of k-tuples of positive numbers.

Theorem 2 (Height problem, [Str84])

$x_1, \ldots, x_k$: punctures of $\overline{\Sigma}$. Fix $(a_1, \ldots, a_k) \in \mathbb{R}_+^k$. Then $\exists! \varphi$: h.q.d. on Σ s.t. any component of $\Sigma - C_\varphi$ is a punctured disks with the puncture x_j, and all non-singular vertical lines of φ in these punctured disks are closed and surrounds each puncture. The leading coefficient of φ at x_j is a_j.

Fix $\Gamma := \{\gamma_1, \ldots, \gamma_k\}$ on Σ, $\mathfrak{m} := (m_1, \ldots, m_k) \in S_+^{k-1}$, and $[S^*, f^*] \in \mathscr{T}$.

We apply the theorem 1 for $f^*(\Gamma)$ on S^* and $\mathfrak{m}$. Then $\exists \alpha^* > 0$ and $\exists \varphi^*$: a J-S differential on S^* s.t. the moduli of annuli determined by φ^* are $\alpha^* \mathfrak{m}$.

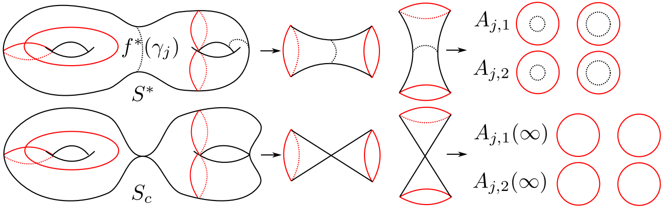
$L_1, \ldots, L_k$: lengths of closed vertical trajectories in annuli, and set $a_j := L_j / 2\pi$. We normalize φ^* s.t. its lengths (leading coefficients) $a_1^*, \ldots, a_k^*$ for φ^* satisfy $\sum_{j=1}^{k} {a_j^*}^2 = 1$. Set $\mathfrak{a}^* := (a_1^*, \ldots, a_k^*) \in S_+^{k-1}$. r_{φ^*}: a J-S ray emanating from $[S^*, f^*]$ determines a deformation $f_c : S^* \to S_c$. S_c: a Riemann surface with nodes. Define

$$\partial_\Gamma \mathscr{T} := \{[X, g] |\ g : S_c \to X \text{ is a q.c.}\}.$$

$\partial_\Gamma \mathscr{T} \subset \partial \mathscr{T}$.

Here, we consider a terminal quadratic differential on S_c introduced by φ^* and f_c.

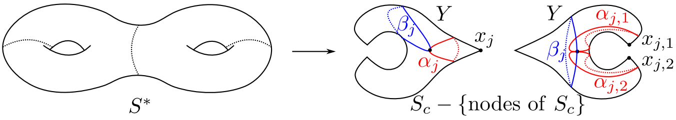
Choose a constant $c_j > 0$ s.t. $w := c_j \exp(1/a_j^* \int \sqrt{\varphi^*(z)}dz)$ maps each component of $S^* - C_{\varphi^*}$ onto $A_{j,1} = A_{j,2} := \{\exp(-\pi\alpha^* m_j) < |w| < 1\}$ conformally. The representation $\varphi^*(z)dz^2 = {a_j^*}^2 dw^2/w^2$ on $A_{j,l}$ holds. Naturally, it has same description on $A_{j,l}(\infty) := \{0 < |w| < 1\} \subset S_c$.



Then $\exists J_c$: q.d. on S_c which has the representation $(a_j^{*2}/z^2 + \cdots)dz^2$ on a neighborhood of each node. On the other hand, we apply Theorem 2 to each component of $S_c - \{\text{nodes of } S_c\}$ s.t. we assign common a_j^* to one node of S_c. The resulting differential equals to J_c.

For Y: a connected component of $S_c - \{\text{nodes of } S_c\}$, we denote punctures on Y without the original punctures of S^* as follows;

1. $x_{j,1}, x_{j,2}$: punctures which are obtained by contracting the curve $f^*(\gamma_j)$ on S^*.
2. x_j: similar as the above, but the counterpart is in the other component.



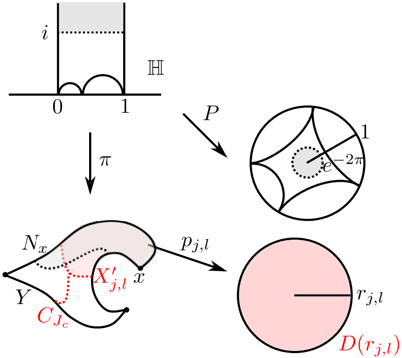
Fix $O \in Y$. $\alpha_{j,1}, \alpha_{j,2}, \beta_j$: simple closed curves on Y through O s.t. $\alpha_{j,1}, \alpha_{j,2}$ are homotopic to $x_{j,1}, x_{j,2}$ respectively, β_j is not homotopic to $\alpha_{j,1}^{\pm 1}, \alpha_{j,2}^{\pm 1}$. For x_j, we take similar α_j, β_j.

Let $[X, g] \in \partial_\Gamma \mathscr{T}$, x be a puncture $g(x_{j,l})$ (or $g(x_j)$) of X for $l = 1, 2$, and Y be a component of $X - \{\text{nodes of } X\}$ associated x. We define a Fuchsian group Γ_x associated Y as follows;

1. The lift of $g(\alpha_{j,l})$ (or $g(\alpha_j)$) corresponds to a translation $L : \zeta \mapsto \zeta + 1 \in \Gamma_x$ on $\mathbb{H}$.
2. The lift of $g(\beta_j)$ has an attracting fixed point $\zeta = 1$ on $\mathbb{H}$. (if $g(\beta_j)$ corresponds a parabolic element, $\zeta = 1$ is a unique fixed point of it.)

The Fuchsian group Γ_x is uniquely determined by Y, x, and is called a **Fuchsian equivalent**.

$\pi : \mathbb{H} \to Y(= \mathbb{H}/\Gamma_x)$: a natural projection. The image $N_x = \pi(\{\mathrm{Im}(\zeta) > 1\})$ is conformally equivalent to a punctured disk by Shimizu's lemma. For $\forall \zeta \in \mathbb{H}$, we set $P : \mathbb{H}/\langle L \rangle \to D(1)$, $z = P(\zeta) = \exp(2\pi i \zeta)$: conformal, where $D(r) = \{0 < |z| < r\}$ for $\forall r > 0$. $P \circ \pi^{-1} : N_x \to D(\exp(-2\pi))$ is conformal. We use z as the coordinate of N_x, and call it a **canonical local coordinate**.



$x_{j,1}, x_{j,2}$: punctures of S_c which are determined by $f^*(\gamma_j)$, and $[X, g] \in \partial_\Gamma \mathscr{T}$. For $\forall \mathfrak{a} = (a_1, a_2, \ldots, a_k) \in S_+^{k-1}$, we use Theorem 2 then $\exists J_c d\zeta^2$: q.d. on X s.t. for $\forall g(x_{j,l})$, $J_c d\zeta^2 = (a_j^2/\zeta^2 + \cdots)d\zeta^2$ in any local coordinate ζ with $\zeta(g(x_{j,l})) = 0$.

$X'_{j,l}$: a component of $X - C_{J_c}$ which corresponds to $g(x_{j,l})$. We use the canonical local coordinate ζ near $g(x_{j,l})$ and define $w = p_{j,l}(\zeta) = c'_{j,l} \exp(1/a_j \int \sqrt{J_c(\zeta)} d\zeta)$, where the constant $c'_{j,l}$ satisfies $(dw/d\zeta)(0) = 1$. Then, $\exists r_{j,l} = r_{j,l}([X, g], \mathfrak{a}) > 0$ s.t. $p_{j,l}(X'_{j,l}) = D(r_{j,l})$.

We would like to show the following;

Proposition

$J_c d\zeta^2$ is continuous on $\partial_\Gamma \mathscr{T} \times S_+^{k-1}$.

$[X_n, g_n] \in \partial_\Gamma \mathscr{T}, \mathfrak{a}_n := ((a_1)_n, \ldots, (a_k)_n) \in S_+^{k-1}$: an arbitrary sequences which converge to $[X, g] \in \partial_\Gamma \mathscr{T}, \mathfrak{a} := (a_1, \ldots, a_k) \in S_+^{k-1}$ respectively. By Theorem 2, $\exists (J_c)_n d\zeta^2$ q.d. corresponding to X_n and $\mathfrak{a}_n$ for $\forall n$, and $J_c d\zeta^2$ corresponding to X and $\mathfrak{a}$ are determined.

$x_n := g_n(x_{j,l})$, $N_{x_n}(r) := \{p \in N_{x_n} |\ |\zeta(p)| < r\}$, $X_n(r) := X_n - \cup_{j,l} N_{x_n}(r)$. $X'_n := X'_{j,l,n}$: a component of $X_n - C_{(J_c)_n}$ which corresponds to x_n.

Lemma

$0 < \exists R := R(\mathfrak{a}) < e^{-2\pi}$ s.t. $N_{x_n}(R) \subset X'_n$ for sufficiently large n and any x_n. Moreover, for $0 < \forall r \leq R$, $\exists M := M(\mathfrak{a}, r) > 0$ s.t. $\iint_{X_n(r)} |(J_c)_n| < M$ for sufficiently large n.

It says that $(J_c)_n d\zeta^2$ is locally uniformly bounded for fixed $\mathfrak{a}$. $(J_c)_n d\zeta^2$ and $J_c d\zeta^2$ are lifted to φ_n and φ on $\mathbb{H}$ by Fuchsian equivalents Γ_n and Γ. By the above lemma, φ_n is normal, we choose a subsequence if necessary s.t. $\varphi_n \to \exists H$ locally uniformly. By $\mathfrak{a}_n \to \mathfrak{a}$ and the uniqueness in Theorem 2, $H = \varphi$.

Theorem (A)

$\exists\hat{\Phi} : \partial_\Gamma\mathscr{T} \times S_+^{k-1} \times \mathbb{R}^k \times \mathbb{R} \to \mathscr{T}$: a homeomorphism s.t.

1. Let $\hat{\Phi}([X,g], a_1, \ldots, a_k, t_1, \ldots, t_k, s) = [R, h]$. For $\forall j = 1, \ldots, k$,

$$\hat{\Phi}([X,g], a_1, \ldots, a_k, t_1, \ldots, t_j + 2\pi, \ldots, t_k, s) = \tau_j([R,h]),$$

   where τ_j is a Dehn twist about γ_j.
2. Fix $[X,g] \in \partial_\Gamma\mathscr{T}$. $r_{\mathfrak{a},\mathfrak{t}} = \{\hat{\Phi}([X,g], \mathfrak{a}, \mathfrak{t}, s) \mid s \in \mathbb{R}_{\geq 0}\}$: a J-S ray whose moduli of associated cylinders are represented by a positive scalar multiple of $(m_1, \ldots, m_k)$, and $\lim_{s\to+\infty} r_{\mathfrak{a},\mathfrak{t}}(s) = [X,g]$. Furthermore, $\{r_{\mathfrak{a},\mathfrak{t}}\}_{\mathfrak{a},\mathfrak{t}}$ are asymptotic each other. ([Ama14])

This is also extension of the main result of [MM75].

We construct
$\Phi : \partial_\Gamma \mathscr{T} \times S_+^{k-1} \times (\mathbb{R}/2\pi\mathbb{Z})^k \times \mathbb{R} \to \mathscr{T}/\langle \tau_1, \ldots, \tau_k \rangle$.

(The criterion of $\mathfrak{t} \in (\mathbb{R}/2\pi\mathbb{Z})^k$)
S_c is the end point of the J-S ray r_{φ^*} emanating from $[S^*, f^*]$. It is constructed by $A_{j,1}(\infty) = A_{j,2}(\infty) = \{0 < |z| < 1\}$ with the appropriate gluing. $J_c = J_c(S_c, \mathfrak{a}^*)$: q.d. on S_c has the representation $a_j^{*2} dz^2/z^2$ on $A_{j,l}(\infty)$. Each component $(S_c)'_{j,l}$ of $S_c - C_{J_c}$ is conformally equivalent to $A_{j,l}(\infty)$.

$x := x_{j,l}$: a puncture of $(S_c)'_{j,l}$ corresponding to a node of S_c.
Γ_x: a Fuchsian equivalent at x.
$\pi := \pi_x : \mathbb{H} \to \mathbb{H}/\Gamma_x \supset (S_c)'_{j,l}$.
Fix $\xi_{j,l} = \xi_{j,l}(\mathfrak{a}^*) \in \mathbb{H}$ which satisfies the following conditions;

1. $\pi(\xi_{j,l}) \in \partial N_x(R(\mathfrak{a}^*)) \subset \overline{(S_c)'_{j,l}}$, that is, $|\zeta(\pi(\xi_{j,l}))| = R(\mathfrak{a}^*)$.
2. $\pi(\xi_{j,l})$ is sent on the positive real axis in $A_{j,l}(\infty)$.

Process for $\mathfrak{a}$ and S_c similarly, i.e. take $\xi_{j,l}(\mathfrak{a})$.

For $\forall[X, g] \in \partial_\Gamma \mathscr{T}$ and $\forall \mathfrak{a} \in S_+^{k-1}$, set $J_c := J_c(X, \mathfrak{a})$: a q.d. on X s.t. it has each leading coefficient a_j at $x = g(x_{j,l})$, and $X'_{j,l}$ be a component of $X - C_{J_c}$. $\pi := \pi_x : \mathbb{H} \to \mathbb{H}/\Gamma_x \supset X'_{j,l}$. Set $\xi_{j,l} := \xi_{j,l}(\mathfrak{a})$, then $\pi(\xi_{j,l}) \in \partial N_x(R(\mathfrak{a})) \subset \overline{X'_{j,l}}$.

We set $q_{j,l}(\zeta) = e^{i\theta_{j,l}} p_{j,l}(\zeta)/r_{j,l}$ where the conformal mapping $p_{j,l} : X'_{j,l} \to D(r_{j,l})$, and $\theta_{j,l}$ satisfies $q_{j,l}(\pi(\xi_{j,l})) > 0$. Then $q_{j,1} : X'_{j,1} \to A_{j,1}(\infty)$, $q_{j,2} : X'_{j,2} \to A_{j,2}(\infty)$.

These settings determine a criterion of the twist angle when we construct a Riemann surface in $\mathscr{T}$ from a Riemann surface with nodes in the boundary $\partial_\Gamma \mathscr{T}$. We notice that by J_c is continuous, so $p_{j,l}$, $r_{j,l}$, and we can take $\xi_{j,l} = \xi_{j,l}(\mathfrak{a}) \in \mathbb{H}$, $\theta_{j,l}$, and $q_{j,l}$ continuously when $\mathfrak{a}$ varies.

(Construction of Φ)
Fix $[X, g] \in \partial_\Gamma \mathscr{T}$, $\mathfrak{a} = (a_1, \ldots, a_k) \in S_+^{k-1}$, $\mathfrak{t} = (t_1, \ldots, t_k) \in (\mathbb{R}/2\pi\mathbb{Z})^k$, and $s \in \mathbb{R}$. Denote $\omega = ([X, g], \mathfrak{a}, \mathfrak{t}, s)$.

First, $\exists J_c d\zeta^2$: q.d. on X by Theorem 2 for $[X, g]$ and $\mathfrak{a}$. Each mapping $q_{j,l} : X'_{j,l} \to A_{j,l}(\infty)$ is conformal. We set $A_{j,l}(s) := A_{j,l}(\infty) - \{|\zeta| \leq \exp(-\pi m_j \exp(2s))\}$. We glue each annular set by $h_j : \zeta \mapsto \exp(it_j)\zeta$ then obtain $A_j(s)$. (The modulus of $A_j(s)$ is $m_j \exp(2s)$.) We glue $A_1(s), \ldots, A_k(s)$ each other by the original gluing mappings of X. Then, a Riemann surface S_ω is determined.

The associated homeomorphism $g_c : X - \{\text{nodes of } X\} \to S_\omega - \Gamma_\omega$ is also determined, where Γ_ω is some admissible curve family on S_ω. The composition $f_\omega := g_c \circ g \circ f_c \circ f : \Sigma - \Gamma \to S_\omega - \Gamma_\omega$ is determined, however, since Γ is degenerated by f_c, then we can find a homeomorphism of Σ onto S_ω which is homotopic to f_ω. We denote it by the same symbol f_ω, and set $\Phi(\omega) = [S_\omega, f_\omega] \in \mathscr{T}/\langle \tau_1, \ldots, \tau_k \rangle$.

(Continuity of Φ)
Confirm the continuities of Φ when each element of $\partial_\Gamma \mathscr{T} \times S_+^{k-1}$, $(\mathbb{R}/2\pi\mathbb{Z})^k$, and $\mathbb{R}$ varies.

(Existence of inverse Φ^{-1})
Then Φ is a homeomorphism.

(Lift Φ to $\hat{\Phi}$)
By $\mathbb{R}^k$ and $\mathscr{T}$ are simply connected,
$\exists \hat{\Phi} : \partial_\Gamma \mathscr{T} \times S_+^{k-1} \times \mathbb{R}^k \times \mathbb{R} \to \mathscr{T}$: a lift of Φ s.t.
$\hat{\Phi}([S_c, id], a_1^*, \dots a_k^*, 0, \dots, 0, (\log \alpha^*)/2) = [S^*, f^*]$. Moreover, for any j, the equation

$$\begin{aligned}&\hat{\Phi}([X, g], \mathfrak{a}, t_1, \dots, t_j + 2\pi, \dots, t_k, s)\\ &= \tau_j \circ \hat{\Phi}([X, g], \mathfrak{a}, t_1, \dots, t_j, \dots, t_k, s)\end{aligned}$$

holds.

Corollary (A)

$$
\begin{aligned}
d_{\mathscr{T}}(\hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t},s),\hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t},s')) &= |s-s'|,\\
d_{\mathscr{T}}(\hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t},s),\hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t}',s)) &\leq \frac{1}{2}\log\max_j K(\tau_{t_j-t'_j}),
\end{aligned}
$$

where $K(\tau_{t_j-t'_j}) = \dfrac{1+k}{1-k}$, $k = \left|\dfrac{\frac{t_j-t'_j}{2\pi m_j e^{2s}}}{2+i\frac{t_j-t'_j}{2\pi m_j e^{2s}}}\right|$.

$\exists c(\mathfrak{a},\mathfrak{a}')$ and fix $s_0 > c(\mathfrak{a},\mathfrak{a}')$. Then, for $\forall s > s_0$, $\exists C(\mathfrak{a},\mathfrak{a}',s_0)$ s.t. the following inequality

$$
d_{\mathscr{T}}(\hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t},s),\hat{\Phi}([X,g],\mathfrak{a}',\mathfrak{t},s)) \leq C(\mathfrak{a},\mathfrak{a}',s_0)
$$

holds.

Corollary (A)

On the lower estimate,

$$d_{\mathscr{T}}(\hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t},s),\hat{\Phi}([X',g'],\mathfrak{a}',\mathfrak{t}',s')) \geq |s-s'|.$$

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