Analytic Study of Singular Curves

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Feb. 16, 2015
Introduction

Generalized Jacobi varieties for singular curves were algebraically defined by Rosenlicht in 1954. Since then, the theory has been developed extremely. A generalized Jacobi variety is analytically considered as a complex Lie group. We generalize the analytic theory for compact Riemann surfaces to singular curves. We expect to get some analytic properties of generalized Jacobi varieties from our treatment.
1 Construction of singular curves

$X$: an irreducible non-singular complex projective algebraic curve (i.e. a compact Riemann surface)

$\mathcal{O}_X$: the structure sheaf on $X$

$S \subset X$: a finite subset

$R$: an equivalent relation on $S$

$\overline{S} := S/R$

$\overline{X} := (X \setminus S) \cup \overline{S}$

$\rho: X \to \overline{X}$ the canonical projection

We use notations according to J. -P. Serre, Groupes algébriques et corps de classes, Hermann, Paris, 1959.

**Definition 1.** $m$: a modulus with support $S$

$\iff \forall P \in S, \ m(P) > 0$ integer

We may assume $\deg m \geq 2$.

$\operatorname{Mer}(X)$: the field of meromorphic functions on $X$

$\forall f \in \operatorname{Mer}(X), \forall P \in X, \operatorname{ord}_P(f)$: the order of $f$ at $P$
Definition 2. \( f, g \in \text{Mer}(X) \)

\[ f \equiv g \mod m \]

if \( \text{ord}_P(f - g) \geq m(P) \) for any \( P \in S \).

\( \rho_* \mathcal{O}_X \) : the direct image of \( \mathcal{O}_X \) by \( \rho \)

\( \forall Q \in \overline{S} \)

\( \mathcal{I}_Q \) : the ideal of \( (\rho_* \mathcal{O}_X)_Q \) formed by the function \( f \) with \( \text{ord}_P(f) \geq m(P), \forall P \in \rho^{-1}(Q) \)

We define a sheaf \( \mathcal{O}_m \) on \( \overline{X} \) by

\[
\mathcal{O}_{m,Q} := \begin{cases} 
(\rho_* \mathcal{O}_X)_Q = \mathcal{O}_{X,Q} & \text{if } Q \in X \setminus S \\
\mathbb{C} + \mathcal{I}_Q & \text{if } Q \in \overline{S}.
\end{cases}
\]

\( (\overline{X}, \mathcal{O}_m) \) : 1-dimensional compact reduced complex space

We denote it by \( X_m \).

Conversely, any reduced and irreducible singular curve is obtained as above.

2 Genus of \( X_m \)

\( \forall Q \in X_m \)

\[ \delta_Q := \dim((\rho_* \mathcal{O}_X)_Q/\mathcal{O}_{m,Q}) \]

\[ \delta := \sum_{Q \in X_m} \delta_Q = \deg m - \# \overline{S}. \]
\( g \): the genus of \( X \)
\( \pi := g + \delta \): the genus of \( X_m \)
\[
\dim H^1(X_m, \mathcal{O}_m) = \pi
\]

3 Riemann-Roch Theorem

**Definition 3.** A divisor \( D \) on \( X \) is said to be prime to \( S \) if \( D(P) = 0 \) for \( P \in S \).

\( \text{Div}(X_m) \): the group of divisors prime to \( S \)

\( \text{Mer}(X_m) \): the field of meromorphic functions on \( X_m \)

\[
\rho^*\text{Mer}(X_m) \subset \text{Mer}(X)
\]

\( f \in \text{Mer}(X_m) \)
\[
(f) = \sum_{Q \in X_m} \text{ord}_Q(f)Q,
\]
where \( \text{ord}_Q(f) = \sum_{P \in \rho^{-1}(Q)} \text{ord}_P(f \circ \rho) \).

**Definition 4.** \( D_1, D_2 \in \text{Div}(X_m) \)

\[
D_1 \sim D_2 \iff \exists f \in \text{Mer}(X_m) \text{ s.t. } D_1 - D_2 = (f)
\]

\( \overline{\text{Div}(X_m)} := \text{Div}(X_m)/\sim, \quad \overline{\text{Div}^0(X_m)} := \text{Div}^0(X_m)/\sim \)
\( D \in \text{Div}(X_m) \subset \text{Div}(X) \)
\( L(D) := \{ f \in \text{Mer}(X); (f) \geq -D \} \)
\( \mathcal{L}(D) : \text{sheafication of } L(D) \)

\[
\mathcal{L}_m(D)_Q := \begin{cases} 
\mathcal{O}_{m,Q} & \text{if } Q \in \overline{S} \\
\mathcal{L}(D)_Q & \text{if } Q \in X \setminus S.
\end{cases}
\]

**Theorem 1** (Riemann-Roch Theorem). Let \( X, S, m, X_m \) be as above.

Let \( D \in \text{Div}(X_m) \). Then, \( H^0(X_m, \mathcal{L}_m(D)) \) and \( H^1(X_m, \mathcal{L}_m(D)) \) are finite dimensional, and we have

\[
\dim H^0(X_m, \mathcal{L}_m(D)) - \dim H^1(X_m, \mathcal{L}_m(D)) = \deg D + 1 - \pi.
\]

4 Serre duality

\( U \subset X_m \): an open set

\( \Omega_m(U) := \{ \text{a mero. } 1\text{-form } \omega \text{ on } \rho^{-1}(U) \text{ satisfying the condition (\ast)} \} \)

The condition (\ast):

\[
\forall Q \in U, \forall f \in \mathcal{O}_{m,Q} \sum_{P \in \rho^{-1}(Q)} \text{Res}_P(\rho^*f\omega) = 0.
\]

\( \Omega_m \): the sheaf defined by \( \{ \Omega_m(U), r^U_V \} \) (the duality sheaf on \( X_m \))

\( \Omega \): the sheaf of germs of hol. 1-forms on \( X \)
\( D \in \text{Div}(X_m) \subset \text{Div}(X) \)

\( W \subset X \): an open subset

\( \Omega(D)(W):= \{ \text{a meromorphic 1-form } \eta \text{ on } W \text{ with } (\eta) \geq -D \text{ on } W \} \).

\( \Omega(D) \): the sheaf on \( X \) defined by \( \{ \Omega(D)(W), r_W^W \} \)

We define a sheaf \( \Omega_m (D) \) on \( X_m \) by

\[
\Omega_m (D)_Q := \begin{cases} 
\Omega_m .Q & \text{if } Q \in \overline{S} \\
\Omega(D)_Q & \text{if } Q \in X \setminus S.
\end{cases}
\]

**Theorem 2** (Serre duality). For any \( D \in \text{Div}(X_m) \) we have

\[
H^0(X_m, \Omega_m (-D)) \cong H^1(X_m, \mathcal{L}_m (D))^* ,
\]

where \( H^1(X_m, \mathcal{L}_m (D))^* \) is the dual space of \( H^1(X_m, \mathcal{L}_m (D)) \).

For a completely analytic proof of Theorem 2, we need special sheaves \( \mathcal{E}_m^{(1,0)} \) and \( \mathcal{E}_m^{(2)} \), some modifications of the proof of non-singular case. However we omit details.

Using Theorem 2, we can rewrite the Riemann-Roch Theorem as follows

**Theorem 3** (Riemann-Roch Theorem (second version)). For any \( D \in \text{Div}(X_m) \) we have

\[
\dim H^0(X_m, \mathcal{L}_m (D)) - \dim H^0(X_m, \Omega_m (-D)) = \deg D + 1 - \pi.
\]
Rosenlich first formulated and proved a generalized Abel’s theorem for a singular curve which was considered algebraically.

Jambois tried to treat it analytically. However, we think Jambois’ argument was incomplete.


Rosenlicht and Jambois considered functions $f$ satisfying

\[ f \equiv 1 \mod m. \]

This means that $f$ takes the common value 1 at all singular points.

Then it is a special function for the number of singular points $\neq 1$ in general.

We assign a non-zero constant $c_Q$ to each point $Q$ in $\overline{S}$. We call

\[ c(\overline{S}) := (c_Q)_{Q \in \overline{S}} \]

a multiconstant on $\overline{S}$. 
**Definition 5.** \( f \in \operatorname{Mer}(X) \), \( c(S) \): a multiconstant on \( S \)

We write

\[
f \equiv c(S) \mod m
\]

if \( \text{ord}_P(f - c_Q) \geq m(P) \) for any \( P \in S \) with \( \rho(P) = Q \) at any \( Q \in S \).

Our formulation of a generalized Abel’s theorem is the following.

**Theorem 4.** \( D \in \operatorname{Div}(X_m) \) with \( \deg D = 0 \)

\[
\exists f \in \operatorname{Mer}(X) \text{ with } f \equiv c(S) \mod m \text{ for some } c(S) \text{ such that } D = (f)
\]

\[\iff\]

\[
\exists 1\text{-chain } c \in C_1(X \setminus S) \text{ with } \partial c = D \text{ such that }
\]

\[
\int_c \rho^* \omega = 0, \quad \forall \omega \in H^0(X_m, \Omega_m)
\]

6 Proof of Theorem 4

\( D \in \operatorname{Div}(X_m) \), \( X_D := \{ P \in X ; D(P) \geq 0 \} \)

**Definition 6.** A \( C^\infty \) function \( f \) on \( X_D \) is called a weak solution of \( D \)

if it satisfies the following condition:

\( \forall P \in X \)

\[
\exists (U, z) : \text{ a coordinate nbd. of } P \text{ with } z(P) = 0
\]

\[
\exists \psi : C^\infty \text{ function on } U \text{ with } \psi(P) \neq 0 \text{ such that }
\]

\[
f = \psi z^{D(P)} \text{ on } U \cap X_D
\]
Sheaf $\mathcal{E}_m^{(1)}$

$U \subset X_m$: an open set We define

$$\mathcal{E}_m^{(1)}(U) := \{a C^\infty 1\text{-form } \omega \text{ on } U \setminus (U \cap S) \text{ satisfying the condition } (**)\}.$$  

The condition (**):
Let $Q \in U \cap S$. We set $\rho^{-1}(Q) = \{P_1, \ldots, P_k\}$. Let $V \subset U$ be an open neighbourhood of $Q$ such that

$$\rho^{-1}(V) = \bigsqcup_{i=1}^k V_i \ (P_i \in V_i),$$

$(V_i, z_i)$ is a coordinate neighbourhood of $P_i$ with $z_i(P_i) = 0$ and there exist $C^\infty$ functions $\varphi_i$ and $\psi_i$ on $V_i \setminus \{P_i\}$ with

$$\rho^* \omega = \varphi_i dz_i + \psi_i d\overline{z}_i \text{ on } V_i \setminus \{P_i\}.$$  

Then limits

$$\lim_{P \to P_i} \varphi_i(P) z_i(P)^m(P_i) \text{ and } \lim_{P \to P_i} \psi_i(P) \overline{z}_i(P)^m(P_i)$$

exist.

Then a presheaf $\{\mathcal{E}_m^{(1)}(U), \rho_U\}$ defines a sheaf $\mathcal{E}_m^{(1)}$ on $X_m$.
Lemma 1. Suppose that $c : [0, 1] \rightarrow X \setminus S$ is a curve and $U$ is a relatively compact open neighbourhood of $c([0, 1])$ in $X \setminus S$. Then there exists a weak solution $f$ of $\partial c$ with $f|(X \setminus U) = 1$ such that for every 1-form $\omega \in H^0(X_m, \mathcal{E}_m^{(1)})$ with $d\omega = 0$ we have

$$\frac{1}{2\pi\sqrt{-1}} \int_\Gamma \int_X \frac{df}{f} \wedge \rho^*\omega = \int_c \rho^*\omega.$$ 

Lemma 2. For any $D \in \text{Div}(X_m)$ the following two conditions are equivalent.

(1) There exists a meromorphic function $g$ on $X$ such that $D = (g)$ and we have a branch $f$ of $\log g$ defined in a neighbourhood of $S$ with the property

$$\sum_{P \in \rho^{-1}(Q)} \text{Res}_P(f\omega) = 0$$

for any point $Q \in \overline{S}$ and for any $\omega \in H^0(X, \rho^*\Omega_m)$.

(2) There exist a meromorphic function $g$ on $X$ and a multiconstant $c(\overline{S})$ such that

$$D = (g) \quad \text{and} \quad g \equiv c(\overline{S}) \mod m.$$ 

Proof of Theorem 4 (Necessity)

Assumption

$$\exists 1 \text{-chain } c \in C_1(X \setminus S) \text{ with } \partial c = D \text{ s.t.}$$

$$\int_c \rho^*\omega = 0, \quad \forall \omega \in H^0(X_m, \Omega_m)$$
By Lemma 1

\[ \exists f : \text{a weak solution of } D = \partial c \text{ s.t. } f|(X \setminus U) = 1 \text{ and} \]

\[ \frac{1}{2\pi \sqrt{-1}} \iint_X \frac{df}{f} \wedge \rho^* \omega = \int_c \rho^* \omega \]

for every \( \omega \in H^0(X_m, \mathcal{C}^{(1)}_m) \) with \( d\omega = 0 \), where \( U \) is an open neighbourhood of the support of \( c \) with \( U \subset \subset X \setminus S \).

Since \( H^0(X_m, \Omega_m) \subset H^0(X_m, \mathcal{C}^{(1)}_m) \), we obtain for every \( \omega \in H^0(X_m, \Omega_m) \)

\[ 0 = \int_c \rho^* \omega = \frac{1}{2\pi \sqrt{-1}} \iint_X \frac{df}{f} \wedge \rho^* \omega = \frac{1}{2\pi \sqrt{-1}} \iint_X \frac{\overline{\partial f}}{f} \wedge \rho^* \omega \]

by the assumption.

\[ \sigma := \frac{\overline{\partial f}}{f} : C^\infty (0,1)\text{-form on } X \]

Since \( H^0(X, \Omega) \subset \rho^* H^0(X_m, \Omega_m) \),

\[ \frac{1}{2\pi \sqrt{-1}} \iint_X \sigma \wedge \eta = 0, \quad \forall \eta \in H^0(X, \Omega) \]

\[ \exists g : C^\infty \text{ function on } X \text{ s.t. } \overline{\partial} g = \sigma = \frac{\overline{\partial f}}{f} \]

\( F := e^{-g} f \) is also a weak solution of \( D \), and meromorphic on \( X \). Since \( f = 1 \) on a neighborhood of \( S \), \( F = e^{-g} \) there. Hence, \( -g \) is a branch of \( \log F \) on a neighborhood of \( S \).

For any \( \omega \in H^0(X_m, \Omega_m) \) we have

\[ \frac{1}{2\pi \sqrt{-1}} \iint_X \overline{\partial} g \wedge \rho^* \omega = \frac{1}{2\pi \sqrt{-1}} \iint_X \frac{\overline{\partial f}}{f} \wedge \rho^* \omega = \int_c \rho^* \omega = 0. \]
$Q \in \overline{S}$, $\rho^{-1}(Q) = \{P_1, \ldots, P_N\}$

$B_j(\varepsilon)$: a small disc centered at $P_j$ with radius $\varepsilon > 0$

Since

$$\frac{1}{2\pi \sqrt{-1}} \iint_X \partial g \wedge \rho^* \omega = \lim_{\varepsilon \to 0} \frac{1}{2\pi \sqrt{-1}} \iint_{X \setminus \bigcup_{j=1}^N B_j(\varepsilon)} \partial g \wedge \rho^* \omega$$

$$= \lim_{\varepsilon \to 0} \left( \sum_{j=1}^N \frac{1}{2\pi \sqrt{-1}} \int_{\partial B_j(\varepsilon)} (-g) \rho^* \omega \right)$$

$$= \sum_{P \in \rho^{-1}(Q)} \text{Res}_P ( (-g) \rho^* \omega ),$$

we obtain

$$\sum_{P \in \rho^{-1}(Q)} \text{Res}_P ( (-g) \rho^* \omega ) = 0.$$ 

This is the condition (1) in Lemma 2. Then the condition (2) in Lemma 2 is satisfied: i.e.

$\exists h \in \text{Mer}(X), \exists c(\overline{S}) : $ multiconstant s.t.

$$D = (h) \quad \text{and} \quad h \equiv c(\overline{S}) \mod \mathfrak{m}$$

(Sufficiency)

Assumption

$\exists f \in \text{Mer}(X)$ s.t.

$$D = (f) \quad \text{and} \quad f \equiv c(\overline{S}) \mod \mathfrak{m} \text{ for some multiconstant } c(\overline{S})$$

$F : X \rightarrow \mathbb{P}^1$ holomorphic map defined by $f$

$$\forall \omega \in H^0(X_\mathfrak{m}, \Omega_\mathfrak{m})$$
Trace($\rho^* \omega$): the trace of $\rho^* \omega$ by $F$

Trace($\rho^* \omega$) is a meromorphic 1-form on $\mathbb{P}^1$.

$F(S) := \{cQ; Q \in \overline{S}\}$

It is obvious that Trace($\rho^* \omega$) is holomorphic on $\mathbb{P}^1 \setminus F(S)$.

By a careful investigation at a point in $F(S)$, we see it is holomorphic on the whole of $\mathbb{P}^1$.

Then Trace($\rho^* \omega$) = 0.

Therefore we can apply the usual argument.

7 Albanese varieties

$X_m$: a singular curve of genus $\pi = g + \delta$

$\{\omega_1, \ldots, \omega_\pi\}$: a basis of $H^0(X_m, \Omega_m)$ s.t.

$\{\rho^* \omega_1, \ldots, \rho^* \omega_g\}$: a basis of $H^0(X, \Omega)$

$\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$: a canonical homology basis of $X$.

$S = \{P_1, \ldots, P_s\}$

$\gamma_j$: a small circle centered at $P_j$ with anticlockwise direction

$\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_s\}$: a basis of $H_1(X \setminus S, \mathbb{Z}) = H_1(X_m \setminus \overline{S}, \mathbb{Z})$

$A := H^0(X_m, \Omega_m)^*/H_1(X_m \setminus \overline{S}, \mathbb{Z})$. 

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$\Gamma$: a discrete subgroup generated by the following $2g + s$ vectors over $\mathbb{Z}$

$$
\left( \int_{\alpha_i} \rho^* \omega_1, \ldots, \int_{\alpha_i} \rho^* \omega_\pi \right), \quad i = 1, \ldots, g,
$$

$$
\left( \int_{\beta_i} \rho^* \omega_1, \ldots, \int_{\beta_i} \rho^* \omega_\pi \right), \quad i = 1, \ldots, g,
$$

$$
\left( \int_{\gamma_j} \rho^* \omega_1, \ldots, \int_{\gamma_j} \rho^* \omega_\pi \right), \quad j = 1, \ldots, s
$$

$A = H^0(X_m, \Omega_m)^*/H_1(X \setminus S, \mathbb{Z}) \cong \mathbb{C}^\pi / \Gamma$ as a complex Lie group

We write it $\text{Alb}^{an}(X_m)$ emphasizing its analytic structure.

We define a period map $\varphi$ with base point $P_0 \in X \setminus S$ by

$$
\varphi : X \setminus S \rightarrow \text{Alb}^{an}(X_m), \quad P \mapsto \left[ \left( \int_{P_0}^P \rho^* \omega_1, \ldots, \int_{P_0}^P \rho^* \omega_\pi \right) \right].
$$

$G$: a commutative complex Lie group

$\psi : X \setminus S \rightarrow G$: a holomorphic map, $\forall D \in \text{Div}(X_m)$

$$
\psi(D) := \sum_{P \in X \setminus S} D(P) \psi(P)
$$

$g \in \text{Mer}(X)$ with $g \equiv c(S) \mod m$ for some $c(S)$

$$
\psi((g)) := \sum_{P \in X \setminus S} \text{ord}_{P}(g) \psi(P) \quad \text{well-defined}
$$
Definition 7. A holomorphic map $\psi : X \setminus S \longrightarrow G$ admits $m$ for a modulus

$\iff \psi((f)) = 0, \forall f \in \text{Mer}(X) \text{ with } f \equiv c(S) \mod m \text{ for some } c(S)$

Remark. In [R] and [Ser], $f \equiv 1 \mod m$ is considered.


Proposition 1. The period map $\varphi : X \setminus S \longrightarrow \text{Alb}^{an}(X_m)$ defined above admits $m$ for a modulus. Furthermore, it is a holomorphic embedding if $g \geq 1$.

Theorem 5. The map $\varphi : (X \setminus S)^{(\pi)} \longrightarrow \text{Alb}^{an}(X_m)$ is surjective.

$(X \setminus S)^{(\pi)}$: the $\pi$-symmetric product of $X \setminus S$

Corollary 1. $\text{Div}^0(X_m) \cong \text{Alb}^{an}(X_m)$ as groups

Theorem 6. The map $\varphi : (X \setminus S)^{(\pi)} \longrightarrow \text{Alb}^{an}(X_m)$ is bimeromorphic.
\[ \text{Alb}^{an}(X_m) = \mathbb{C}^p \times (\mathbb{C}^*)^q \times \mathfrak{Q} \]

\(\mathfrak{Q}\) : an \(r\)-dimensional quasi-abelian variety of kind 0, \(p + q + r = \pi\)

\(\mathfrak{Q} = \mathbb{C}^r / \Gamma_0\), rank \(\Gamma_0 = r + s\)

\(\mathfrak{Q} \rightarrow A_0\) : principal \((\mathbb{C}^*)^{r-s}\)-bundle over an abelian variety \(A_0\)

\(\overline{\mathfrak{Q}}\) : the standard compactification of \(\mathfrak{Q}\)

\(\overline{\text{Alb}^{an}(X_m)} := (\mathbb{P}^1)^{p+q} \times \overline{\mathfrak{Q}}\) : the standard compactification of \(\text{Alb}^{an}(X_m)\)

Remark. The map \(\varphi : X \setminus S \rightarrow \text{Alb}^{an}(X_m)\) does not extend to a holomorphic map \(\overline{\varphi} : X \rightarrow \overline{\text{Alb}^{an}(X_m)}\).

**Theorem 7** (Universality Property). Let \(G\) be a commutative complex Lie group, and let \(P_0\) be the base point of the map \(\varphi : X \setminus S \rightarrow \text{Alb}^{an}(X_m)\). Then, for any holomorphic map \(\psi : X \setminus S \rightarrow G\) which admits \(m\) for a modulus there exists uniquely a homomorphism \(\Psi : \text{Alb}^{an}(X_m) \rightarrow G\) between complex Lie groups such that \(\psi = \Psi \circ \varphi + g_0\), where \(g_0 = \psi(P_0)\).
8 The reason why $\text{Div}(X_m)$ is sufficient

$D \in \text{Div}(X_m) \iff D$ : a divisor prime to $S$

We should consider divisors on the whole $X_m$.

$\mathcal{M}_m$: the quotient sheaf of $\mathcal{O}_m$

The divisor sheaf $\mathcal{D}_m$ on $X_m$ is

$$\mathcal{D}_m = \mathcal{M}_m^*/\mathcal{O}_m^*.$$  

An element in $H^0(X_m, \mathcal{D}_m)$ is identified with a divisor

$$D = \sum_{Q \in X_m} D(Q)Q$$

$$D(Q) = \sum_{P \in \rho^{-1}(Q)} n_P, n_P \in \mathbb{Z} \text{ with } |n_P| \geq m(P) \text{ and } n_P n_{P'} > 0,$$

$\forall P, P' \in \rho^{-1}(Q)$ if $Q \in \overline{S}$ and $D(Q) \neq 0$,

$D(Q) \in \mathbb{Z}$ if $Q \notin X_m \setminus \overline{S}$.

The number of points with $D(Q) \neq 0$ is finite.

$\widetilde{\text{Div}}_m(X_m)$: the group of all such divisors

$\forall f \in \text{Mer}(X_m), f \neq 0$

$$(f) := \sum_{Q \in X_m} \text{ord}_Q(f)Q \in \widetilde{\text{Div}}_m(X_m)$$

**Definition 8.** $D_1, D_2 \in \widetilde{\text{Div}}_m(X_m)$

$$D_1 \sim_m D_2 \iff \exists f \in \text{Mer}(X_m) \text{ s.t. } D_1 - D_2 = (f)$$
Lemma 3. $\forall D \in \widetilde{\text{Div}_m}(X_m)$

$\exists f \in \text{Mer}(X_m)$ s.t. $\widetilde{D} - (f) \in \text{Div}(X_m)$

Proof. Assume: $Q \in \mathcal{S}$, $M := \widetilde{D}(Q) \neq 0$

It suffices to consider the case $M > 0$.

$\rho^{-1}(Q) = \{P_1, \ldots, P_N\}$

$\forall P_i$, $\exists n_i \in \mathbb{N}$ with $n_i \geq m(P_i)$ s.t. $M = \sum_{i=1}^{N} n_i$

$z_i$: a local coordinate at $P_i$

$r_i(z_i) := z_i^{n_i}$

$\forall P \in S \setminus \{P_1, \ldots, P_N\}$, $r_P(z_P) := 1 + z_P^{m(P)}$

$z_P$: a local coordinate at $P$

$\exists f \in \text{Mer}(X)$ s.t.

$$\begin{cases} 
\text{ord}_P(f - r_i) > n_i & \text{if } P = P_i \text{ for some } i = 1, \ldots, N, \\
\text{ord}_P(f - r_P) > m(P) & \text{if } P \in S \setminus \{P_1, \ldots, P_N\}
\end{cases}$$

$\exists g \in \text{Mer}(X_m)$ s.t. $f = \rho^* g$

$\widetilde{D} - (g) = 0$ at $Q$ \hfill $\Box$

$$\left[\widetilde{\text{Div}_m}(X_m)\right] := \widetilde{\text{Div}_m}(X_m) / \sim_m$$

$$\widetilde{\text{Div}_m^0}(X_m) := \{\widetilde{D} \in \widetilde{\text{Div}_m}(X_m) ; \deg \widetilde{D} = 0\}$$

$$\left[\widetilde{\text{Div}_m^0}(X_m)\right] := \widetilde{\text{Div}_m^0}(X_m) / \sim_m$$
Proposition 2.

\[
\left[ \overline{\text{Div}_m(X_m)} \right] \cong \overline{\text{Div}(X_m)} \\
\left[ \overline{\text{Div}^0_m(X_m)} \right] \cong \overline{\text{Div}^0(X_m)}
\]
Let $Q \in \mathcal{S}$

$\forall P \in \rho^{-1}(Q), \ m(\geq m(P))$: the multiplicity of $F$ at $P$

$\exists t$: a local coordinate at $c_Q$

$\exists w$: a local coordinate at $P$ s.t.

$$F \text{ is represented as } t = w^m.$$

$\exists h(w)$: meromorphic function in a nbd. of $P$ s.t.

$$\rho^*\omega = h(w)dw \quad \text{and} \quad h(w) = \sum_{n \geq -m(P)} c_n w^n.$$

By $dt = mw^{m-1}dw, \ \rho^*\omega = \frac{h(w)}{mw^{m-1}}dt$

$\zeta^i w (i = 0, 1, \ldots, m - 1)$: the preimages of $t = w^m$

$(\zeta = \exp(\sqrt{-1} \frac{2\pi}{m}))$

Then

$$\sum_{i=0}^{m-1} \frac{h(\zeta^i w)}{mw^{m-1}} dt$$

$$= \frac{1}{m} \sum_{n \geq -m(P)} c_n \left( \sum_{i=0}^{m-1} \zeta^{i(n-m+1)} \right) w^{n-m+1} dt \hspace{1cm} (\ast)$$
If \( n - m + 1 \neq km \), then \( \sum_{i=0}^{m-1} \zeta^{i(n-m+1)} = 0 \).

Since \( n \geq -m(P) \) and \( m \geq m(P) \), we have

\[
(*) = \sum_{k \geq 0} c_{km-1} t^{k-1} dt.
\]

Noting \( c_{-1} = \text{Res}_P(\rho^* \omega) \), we obtain the expression of \( \text{Trace}(\rho^* \omega) \) at \( c_Q \) as follows:

\[
\text{Trace}(\rho^* \omega) = \left( \left( \sum_{P \in \rho^{-1}(Q)} \text{Res}_P(\rho^* \omega) \right) \frac{1}{t} + \text{holomorphic part} \right) dt
\]

\[
\sum_{P \in \rho^{-1}(Q)} \text{Res}_P(\rho^* \omega) = 0 \quad \text{for} \quad \omega \in H^0(X_m, \Omega_m)
\]

Then \( \text{Trace}(\rho^* \omega) \) is holomorphic at \( c_Q \).