Analytic Study of Singular Curves

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Introduction

Generalized Jacobi varieties for singular curves were algebraically defined by Rosenlicht in 1954. Since then, the theory has been developed extremely. A genaralized Jacobi variety is analytically considered as a complex Lie group. We generalize the analytic theory for compact Riemann surfaces to singular curves. We expect to get some analytic properties of generalized Jacobi varieties from our treatment.

1 Construction of singular curves

X: an irreducible non-singular complex projective algebraic curve (i.e. a compact Riemann surface)

$\mathcal{O}_X$: the structure sheaf on X

$S \subset X$: a finite subset

R : an equivalent relation on S

$\overline{S} := S/R$

$\overline{X} := (X \setminus S) \cup \overline{S}$

$\rho : X \longrightarrow \overline{X}$ the canonical projection

We use notations according to

J. -P. Serre, Groupes algébriques et corps de classes, Hermann, Paris, 1959.

Definition 1. $\mathfrak{m}$: a modulus with support S

$\underset{\text{def}}{\Longleftrightarrow}$ $\forall P \in S$, $\mathfrak{m}(P) > 0$ integer

We may assume $\deg \mathfrak{m} \geqq 2$.

$\mathrm{Mer}(X)$: the field of meromorphic functions on X

$\forall f \in \mathrm{Mer}(X)$, $\forall P \in X$, $\mathrm{ord}_P(f)$: the order of f at P

Definition 2. $f, g \in \mathrm{Mer}(X)$

$$f \equiv g \mod \mathfrak{m}$$

if $\mathrm{ord}_P(f-g) \geq \mathfrak{m}(P)$ for any $P \in S$.

$\rho_* \mathcal{O}_X$: the direct image of $\mathcal{O}_X$ by ρ

$\forall Q \in \overline{S}$

$\mathcal{I}_Q$: the ideal of $(\rho_* \mathcal{O}_X)_Q$ formed by the function f with

$\mathrm{ord}_P(f) \geq \mathfrak{m}(P)$, $\forall P \in \rho^{-1}(Q)$

We define a sheaf $\mathcal{O}_{\mathfrak{m}}$ on $\overline{X}$ by

$$\mathcal{O}_{\mathfrak{m},Q} := \begin{cases} (\rho_* \mathcal{O}_X)_Q = \mathcal{O}_{X,Q} & \text{if } Q \in X \setminus S \\ \mathbb{C} + \mathcal{I}_Q & \text{if } Q \in \overline{S}. \end{cases}$$

$(\overline{X}, \mathcal{O}_{\mathfrak{m}})$: 1-dimensional compact reduced complex space

We denote it by $X_{\mathfrak{m}}$.

Conversely, any reduced and irreducible singular curve is obtained as above.

2 Genus of $X_{\mathfrak{m}}$

$\forall Q \in X_{\mathfrak{m}}$

$$\delta_Q := \dim((\rho_* \mathcal{O}_X)_Q / \mathcal{O}_{\mathfrak{m},Q})$$

$$\delta := \sum_{Q \in X_{\mathfrak{m}}} \delta_Q = \deg \mathfrak{m} - \#\overline{S}.$$

g: the genus of X

$\pi := g + \delta$: the genus of $X_{\mathfrak{m}}$

$$\dim H^1(X_{\mathfrak{m}}\ , \mathcal{O}_{\mathfrak{m}}\ ) = \pi$$

3 Riemann-Roch Theorem

Definition 3. A divisor D on X is said to be prime to S if $D(P) = 0$ for $P \in S$.

$\mathrm{Div}(X_{\mathfrak{m}}\ )$: the group of divisors prime to S

$\mathrm{Mer}(X_{\mathfrak{m}}\ )$: the field of meromorphic functions on $X_{\mathfrak{m}}$

$$\rho^* \mathrm{Mer}(X_{\mathfrak{m}}\ ) \subset \mathrm{Mer}(X)$$

$f \in \mathrm{Mer}(X_{\mathfrak{m}}\ )$

$$(f) = \sum_{Q \in X_{\mathfrak{m}}} \mathrm{ord}_Q(f) Q,$$

where $\mathrm{ord}_Q(f) = \sum_{P \in \rho^{-1}(Q)} \mathrm{ord}_P(f \circ \rho)$.

Definition 4. $D_1, D_2 \in \mathrm{Div}(X_{\mathfrak{m}}\ )$

$$D_1 \sim D_2 \quad \underset{\mathrm{def}}{\Longleftrightarrow} \quad \exists f \in \mathrm{Mer}(X_{\mathfrak{m}}\ )\ s.t.\ D_1 - D_2 = (f)$$

$$\overline{\mathrm{Div}(X_{\mathfrak{m}}\ )} := \mathrm{Div}(X_{\mathfrak{m}}\ )/\sim, \quad \overline{\mathrm{Div}^0(X_{\mathfrak{m}}\ )} := \mathrm{Div}^0(X_{\mathfrak{m}}\ )/\sim$$

$D \in \mathrm{Div}(X_{\mathfrak{m}}) \subset \mathrm{Div}(X)$

$L(D) := \{f \in \mathrm{Mer}(X); (f) \geqq -D\}$

$\mathcal{L}(D)$: sheafication of $L(D)$

$$\mathcal{L}_{\mathfrak{m}}(D)_Q := \begin{cases} \mathcal{O}_{\mathfrak{m},Q} & \text{if } Q \in \overline{S} \\ \mathcal{L}(D)_Q & \text{if } Q \in X \setminus S. \end{cases}$$

Theorem 1 (Riemann-Roch Theorem)**.** *Let $X, S, \mathfrak{m}, X_{\mathfrak{m}}$ be as above. Let $D \in \mathrm{Div}(X_{\mathfrak{m}})$. Then, $H^0(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D))$ and $H^1(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D))$ are finite dimensional, and we have*

$$\dim H^0(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D)) - \dim H^1(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D)) = \deg D + 1 - \pi.$$

4 Serre duality

$U \subset X_{\mathfrak{m}}$: an open set

$\Omega_{\mathfrak{m}}(U) := \{$a mero. 1-form ω on $\rho^{-1}(U)$ satisfying the condition $(*)\}$

The condition $(*)$:

$\forall Q \in U$, $\forall f \in \mathcal{O}_{\mathfrak{m},Q}$

$$\sum_{P \in \rho^{-1}(Q)} \mathrm{Res}_P(\rho^* f\omega) = 0.$$

$\Omega_{\mathfrak{m}}$: the sheaf defined by $\{\Omega_{\mathfrak{m}}(U), r_V^U\}$ (the duality sheaf on $X_{\mathfrak{m}}$)

Ω : the sheaf of germs of hol. 1-forms on X

$D \in \mathrm{Div}(X_{\mathfrak{m}}) \subset \mathrm{Div}(X)$

$W \subset X$: an open subset

$\Omega(D)(W) := \{$ a meromorphic 1-form η on W with $(\eta) \geq -D$ on $W\}$.

$\Omega(D)$: the sheaf on X defined by $\{\Omega(D)(W), r^W_{W'}\}$

We define a sheaf $\Omega_{\mathfrak{m}}(D)$ on $X_{\mathfrak{m}}$ by

$$\Omega_{\mathfrak{m}}(D)_Q := \begin{cases} \Omega_{\mathfrak{m},Q} & \text{if } Q \in \overline{S} \\ \Omega(D)_Q & \text{if } Q \in X \setminus S. \end{cases}$$

Theorem 2 (Serre duality)**.** *For any $D \in \mathrm{Div}(X_{\mathfrak{m}})$ we have*

$$H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}}(-D)) \cong H^1(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D))^*,$$

where $H^1(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D))^*$ is the dual space of $H^1(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D))$.

For a completely analytic proof of Theorem 2, we need special sheaves $\mathcal{E}^{(1,0)}_{\mathfrak{m}}$ and $\mathcal{E}^{(2)}_{\mathfrak{m}}$, some modifications of the proof of non-singular case. However we omit details.

Using Theorem 2, we can rewrite the Riemann-Roch Theorem as follows

Theorem 3 (Riemann-Roch Theorem (second version))**.** *For any $D \in \mathrm{Div}(X_{\mathfrak{m}})$ we have*

$$\dim H^0(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}(D)) - \dim H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}}(-D)) = \deg D + 1 - \pi.$$

5 Generalized Abel's Theorem

Rosenlich first formulated and proved a generalized Abel's theorem for a singular curve which was considered algebraically.

Jambois tried to treat it analytically. However, we think Jambois' argument was imcomplete.

(T. Jambois, The theorem of Torelli for singular curves, Trans. Amer. Math. Soc., **239** (1978),123–146)

Rosenlicht and Jambois considered functions f satisfying

$$\boxed{f \equiv 1 \bmod \mathfrak{m}}.$$

This means that f takes the common value 1 at all singular points. Then it is a special function for the number of singular points $\neq 1$ in general.

We assign a non-zero constant c_Q to each point Q in $\overline{S}$. We call

$$c(\overline{S}) := (c_Q)_{Q \in \overline{S}}$$

a multiconstant on $\overline{S}$.

Definition 5. $f \in \mathrm{Mer}(X)$, $c(\overline{S})$: a multiconstant on $\overline{S}$

We write

$$f \equiv c(\overline{S}) \bmod \mathfrak{m}$$

if $\mathrm{ord}_P(f - c_Q) \geq \mathfrak{m}\ (P)$ for any $P \in S$ with $\rho(P) = Q$ at any $Q \in \overline{S}$.

Our formulation of a generalized Abel's theorem is the following.

Theorem 4. $D \in \mathrm{Div}(X_{\mathfrak{m}}\ )$ *with* $\deg D = 0$

$\exists f \in \mathrm{Mer}(X)$ *with* $f \equiv c(\overline{S}) \bmod \mathfrak{m}$ *for some* $c(\overline{S})$ *such that* $D = (f)$

$\Longleftrightarrow$

$\exists$ 1-*chain* $c \in C_1(X \setminus S)$ *with* $\partial c = D$ *such that*

$$\int_c \rho^* \omega = 0, \quad \forall \omega \in H^0(X_{\mathfrak{m}}\ , \Omega_{\mathfrak{m}}\ )$$

6 Proof of Theorem 4

$D \in \mathrm{Div}(X_{\mathfrak{m}}\ )$, $X_D := \{P \in X; D(P) \geqq 0\}$

Definition 6. A C^∞ function f on X_D is called a weak solution of D if it satisfies the following condition:

$\forall P \in X$

$\exists (U, z)$: a coordinate nbd. of P with $z(P) = 0$

$\exists \psi$: C^∞ function on U with $\psi(P) \neq 0$ such that

$$f = \psi z^{D(P)} \quad \text{on} \quad U \cap X_D$$

Sheaf $\mathcal{E}_{\mathfrak{m}}^{(1)}$

$U \subset X_{\mathfrak{m}}$: an open set We define

$\mathcal{E}_{\mathfrak{m}}^{(1)}(U) := \{$a C^∞ 1-form ω on $U \setminus (U \cap \overline{S})$ satisfying the condition $(\star\star)\}$.

The condition $(\star\star)$:

Let $Q \in U \cap \overline{S}$. We set $\rho^{-1}(Q) = \{P_1, \ldots, P_k\}$. Let $V \subset U$ be an open neighbourhood of Q such that

$$\rho^{-1}(V) = \bigsqcup_{i=1}^{k} V_i \quad (P_i \in V_i),$$

(V_i, z_i) is a coordinate neighbourhood of P_i with $z_i(P_i) = 0$ and there exist C^∞ functions φ_i and ψ_i on $V_i \setminus \{P_i\}$ with

$$\rho^*\omega = \varphi_i dz_i + \psi_i d\overline{z}_i \quad \text{on} \quad V_i \setminus \{P_i\}.$$

Then limits

$$\lim_{P \to P_i} \varphi_i(P) z_i(P)^{\mathfrak{m}(P_i)} \quad \text{and} \quad \lim_{P \to P_i} \psi_i(P) \overline{z_i(P)}^{\mathfrak{m}(P_i)}$$

exist.

Then a presheaf $\{\mathcal{E}_{\mathfrak{m}}^{(1)}(U), r_V^U\}$ defines a sheaf $\mathcal{E}_{\mathfrak{m}}^{(1)}$ on $X_{\mathfrak{m}}$.

Lemma 1. *Suppose that* $c : [0,1] \longrightarrow X \setminus S$ *is a curve and* U *is a relatively compact open neighbourhood of* $c([0,1])$ *in* $X \setminus S$*. Then there exists a weak solution* f *of* ∂c *with* $f|(X \setminus U) = 1$ *such that for every* 1*-form* $\omega \in H^0(X_{\mathfrak{m}}, \mathcal{E}^{(1)}_{\mathfrak{m}})$ *with* $d\omega = 0$ *we have*

$$\frac{1}{2\pi\sqrt{-1}} \iint_X \frac{df}{f} \wedge \rho^*\omega = \int_c \rho^*\omega.$$

Lemma 2. *For any* $D \in \mathrm{Div}(X_{\mathfrak{m}})$ *the following two conditions are equivalent.*

(1) There exists a meromorphic function g *on* X *such that* $D = (g)$ *and we have a branch* f *of* $\log g$ *defined in a neighbourhood of* S *with the property*

$$\sum_{P \in \rho^{-1}(Q)} \mathrm{Res}_P(f\omega) = 0$$

for any point $Q \in \overline{S}$ *and for any* $\omega \in H^0(X, \rho^*\Omega_{\mathfrak{m}})$*.*

(2) There exist a meromorphic function g *on* X *and a multiconstant* $c(\overline{S})$ *such that*

$$D = (g) \quad \textit{and} \quad g \equiv c(\overline{S}) \bmod \mathfrak{m}.$$

Proof of Theorem 4 (Necessity)

$\boxed{\text{Assumption}}$

$\exists 1 - chain \;\; c \in C_1(X \setminus S)$ with $\partial c = D$ s.t.

$$\int_c \rho^*\omega = 0, \quad \forall \omega \in H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$$

By Lemma 1

$\exists f$: a weak solution of $D = \partial c$ s.t. $f|(X \setminus U) = 1$ and

$$\frac{1}{2\pi\sqrt{-1}} \iint_X \frac{df}{f} \wedge \rho^*\omega = \int_c \rho^*\omega$$

for every $\omega \in H^0(X_{\mathfrak{m}}, \mathcal{E}^{(1)}_{\mathfrak{m}})$ with $d\omega = 0$, where U is an open neighbourhood of the support of c with $U \subset\subset X \setminus S$.

Since $H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}}) \subset H^0(X_{\mathfrak{m}}, \mathcal{E}^{(1)}_{\mathfrak{m}})$, we obtain for every $\omega \in H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$

$$0 = \int_c \rho^*\omega = \frac{1}{2\pi\sqrt{-1}} \iint_X \frac{df}{f} \wedge \rho^*\omega = \frac{1}{2\pi\sqrt{-1}} \iint_X \frac{\overline{\partial} f}{f} \wedge \rho^*\omega$$

by the assumption.

$\sigma := \dfrac{\overline{\partial} f}{f}$: C^∞ (0,1)-form on X

Since $H^0(X, \Omega) \subset \rho^* H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$,

$$\frac{1}{2\pi\sqrt{-1}} \iint_X \sigma \wedge \eta = 0, \quad \forall \eta \in H^0(X, \Omega)$$

$\exists g$: C^∞ function on X s.t. $\overline{\partial} g = \sigma = \dfrac{\overline{\partial} f}{f}$

$F := e^{-g} f$ is also a weak solution of D, and meromorphic on X. Since $f = 1$ on a neighborhood of S, $F = e^{-g}$ there. Hence, $-g$ is a branch of $\log F$ on a neighborhood of S.

For any $\omega \in H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$ we have

$$\frac{1}{2\pi\sqrt{-1}} \iint_X \overline{\partial} g \wedge \rho^*\omega = \frac{1}{2\pi\sqrt{-1}} \iint_X \frac{\overline{\partial} f}{f} \wedge \rho^*\omega = \int_c \rho^*\omega = 0.$$

$Q \in \overline{S}$, $\rho^{-1}(Q) = \{P_1, \ldots, P_N\}$

$B_j(\varepsilon)$: a small disc centered at P_j with radius $\varepsilon > 0$

Since

$$\begin{aligned}
\frac{1}{2\pi\sqrt{-1}} \iint_X \overline{\partial} g \wedge \rho^* \omega &= \lim_{\varepsilon \to 0} \frac{1}{2\pi\sqrt{-1}} \iint_{X \setminus \left(\cup_{j=1}^N B_j(\varepsilon)\right)} \overline{\partial} g \wedge \rho^* \omega \\
&= \lim_{\varepsilon \to 0} \left(\sum_{j=1}^N \frac{1}{2\pi\sqrt{-1}} \int_{\partial B_j(\varepsilon)} (-g) \rho^* \omega \right) \\
&= \sum_{P \in \rho^{-1}(Q)} \mathrm{Res}_P \left((-g) \rho^* \omega \right),
\end{aligned}$$

we obtain

$$\sum_{P \in \rho^{-1}(Q)} \mathrm{Res}_P \left((-g) \rho^* \omega \right) = 0.$$

This is the condition (1) in Lemma 2. Then the condition (2) in Lemma 2 is satisfied: i.e.

$\exists h \in \mathrm{Mer}(X)$, $\exists c(\overline{S})$: multiconstant s.t.

$$D = (h) \quad \text{and} \quad h \equiv c(\overline{S}) \bmod \mathfrak{m}$$

(Sufficiency)

$\boxed{\text{Assumption}}$

$\exists f \in \mathrm{Mer}(X)$ s.t.

$$D = (f) \quad \text{and} \quad f \equiv c(\overline{S}) \bmod \mathfrak{m} \text{ for some multiconstant } c(\overline{S})$$

$F : X \longrightarrow \mathbb{P}^1$ holomorphic map defined by f

$\forall \omega \in H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$

Trace($\rho^*\omega$): the trace of $\rho^*\omega$ by F

Trace($\rho^*\omega$) is a meromorphic 1-form on $\mathbb{P}^1$.

$F(S) := \{c_Q; Q \in \overline{S}\}$

It is obvious that Trace($\rho^*\omega$) is holomorphic on $\mathbb{P}^1 \setminus F(S)$.

By a careful investigation at a point in $F(S)$, we see it is holomorphic on the whole of $\mathbb{P}^1$.

Then Trace($\rho^*\omega$) $= 0$.

Therefore we can apply the usual argument.

7 Albanese varieties

$X_{\mathfrak{m}}$: a singular curve of genus $\pi = g + \delta$

$\{\omega_1, \ldots, \omega_\pi\}$: a basis of $H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$ s.t.

$\{\rho^*\omega_1, \ldots, \rho^*\omega_g\}$: a basis of $H^0(X, \Omega)$

$\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$: a canonical homology basis of X.

$S = \{P_1, \ldots, P_s\}$

γ_j : a small circle centered at P_j with anticlockwise direction

$\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_s\}$: a basis of $H_1(X \setminus S, \mathbb{Z}) = H_1(X_{\mathfrak{m}} \setminus \overline{S}, \mathbb{Z})$

$$A := H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})^* / H_1(X_{\mathfrak{m}} \setminus \overline{S}, \mathbb{Z}).$$

Γ: a discrete subgroup generated by the following $2g+s$ vectors over $\mathbb{Z}$

$$\left(\int_{\alpha_i} \rho^*\omega_1, \ldots, \int_{\alpha_i} \rho^*\omega_\pi\right), \quad i = 1, \ldots, g,$$
$$\left(\int_{\beta_i} \rho^*\omega_1, \ldots, \int_{\beta_i} \rho^*\omega_\pi\right), \quad i = 1, \ldots, g,$$
$$\left(\int_{\gamma_j} \rho^*\omega_1, \ldots, \int_{\gamma_j} \rho^*\omega_\pi\right), \quad j = 1, \ldots, s$$

$$A = H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})^*/H_1(X \setminus S, \mathbb{Z}) \cong \mathbb{C}^\pi/\Gamma \quad \text{as a complex Lie group}$$

We write it $\mathrm{Alb}^{an}(X_{\mathfrak{m}})$ emphasizing its analytic structure.

We define a period map φ with base point $P_0 \in X \setminus S$ by

$$\varphi : X \setminus S \longrightarrow \mathrm{Alb}^{an}(X_{\mathfrak{m}}), \quad P \longmapsto \left[\left(\int_{P_0}^{P} \rho^*\omega_1, \ldots, \int_{P_0}^{P} \rho^*\omega_\pi\right)\right].$$

G : a commutative complex Lie group

$\psi : X \setminus S \longrightarrow G$: a holomorphic map, $\forall D \in \mathrm{Div}(X_{\mathfrak{m}})$

$$\psi(D) := \sum_{P \in X \setminus S} D(P)\psi(P)$$

$g \in \mathrm{Mer}(X)$ with $g \equiv c(\overline{S}) \bmod \mathfrak{m}$ for some $c(\overline{S})$

$$\psi((g)) := \sum_{P \in X \setminus S} \mathrm{ord}_P(g)\psi(P) \quad \text{well-defined}$$

Definition 7. A holomorphic map $\psi : X \setminus S \longrightarrow G$ admits $\mathfrak{m}$ for a modulus

$$\underset{\text{def}}{\Longleftrightarrow} \quad \psi((f)) = 0, \forall f \in \mathrm{Mer}(X) \text{ with } f \equiv c(\overline{S}) \bmod \mathfrak{m} \text{ for some } c(\overline{S})$$

Remark. *In [R] and [Ser],* $\boxed{f \equiv 1 \bmod \mathfrak{m}}$ *is considered.*

[R] M. Rosenlicht, Generalized jacobian varieties, Ann. of Math., **59** (1954), 505–530.

[Ser] J.-P. Serre, Groupes algébriques et corps de classes, Hermann, Paris, 1959.

Proposition 1. *The period map* $\varphi : X \setminus S \longrightarrow \mathrm{Alb}^{an}(X_{\mathfrak{m}})$ *defined above admits* $\mathfrak{m}$ *for a modulus. Furthermore, it is a holomorphic embedding if* $g \geq 1$.

Theorem 5. *The map* $\varphi : (X \setminus S)^{(\pi)} \longrightarrow \mathrm{Alb}^{an}(X_{\mathfrak{m}})$ *is surjective.*

$(X \setminus S)^{(\pi)}$: the π-symmetric product of $X \setminus S$

Corollary 1. $\overline{\mathrm{Div}^0(X_{\mathfrak{m}})} \cong \mathrm{Alb}^{an}(X_{\mathfrak{m}})$ *as groups*

Theorem 6. *The map* $\varphi : (X \setminus S)^{(\pi)} \longrightarrow \mathrm{Alb}^{an}(X_{\mathfrak{m}})$ *is bimeromorphic.*

$$\mathrm{Alb}^{an}(X_{\mathfrak{m}}) = \mathbb{C}^p \times (\mathbb{C}^*)^q \times \mathfrak{Q}$$

$\mathfrak{Q}$: an r-dimensional quasi-abelian variety of kind 0, $p+q+r=\pi$

$\mathfrak{Q} = \mathbb{C}^r/\Gamma_0$, rank $\Gamma_0 = r+s$

$\mathfrak{Q} \longrightarrow A_0$: principal $(\mathbb{C}^*)^{r-s}$-bundle over an abelian variety A_0

$\overline{\mathfrak{Q}}$: the standard compactification of $\mathfrak{Q}$

$\overline{\mathrm{Alb}^{an}(X_{\mathfrak{m}})} := (\mathbb{P}^1)^{p+q} \times \overline{\mathfrak{Q}}$: the standard compactification of $\mathrm{Alb}^{an}(X_{\mathfrak{m}})$

Remark. *The map $\varphi : X \setminus S \longrightarrow \mathrm{Alb}^{an}(X_{\mathfrak{m}})$ does not extend to a holomorphic map $\overline{\varphi} : X \longrightarrow \overline{\mathrm{Alb}^{an}(X_{\mathfrak{m}})}$.*

Theorem 7 (Universality Property)**.** *Let G be a commutative complex Lie group, and let P_0 be the base point of the map $\varphi : X \setminus S \longrightarrow \mathrm{Alb}^{an}(X_{\mathfrak{m}})$. Then, for any holomorphic map $\psi : X \setminus S \longrightarrow G$ which admits $\mathfrak{m}$ for a modulus there exists uniquely a homomorphism $\Psi : \mathrm{Alb}^{an}(X_{\mathfrak{m}}) \longrightarrow G$ between complex Lie groups such that $\psi = \Psi \circ \varphi + g_0$, where $g_0 = \psi(P_0)$.*

8 The reason why $\mathrm{Div}(X_{\mathfrak{m}})$ is sufficient

$D \in \mathrm{Div}(X_{\mathfrak{m}}) \iff D$: a divisor prime to S

We should consider divisors on the whole $X_{\mathfrak{m}}$.

$\mathcal{M}_{\mathfrak{m}}$: the quotient sheaf of $\mathcal{O}_{\mathfrak{m}}$

The divisor sheaf $\mathcal{D}_{\mathfrak{m}}$ on $X_{\mathfrak{m}}$ is

$$\mathcal{D}_{\mathfrak{m}} = \mathcal{M}_{\mathfrak{m}}^* / \mathcal{O}_{\mathfrak{m}}^*.$$

An element in $H^0(X_{\mathfrak{m}}, \mathcal{D}_{\mathfrak{m}})$ is identified with a divisor

$$D = \sum_{Q \in X_{\mathfrak{m}}} D(Q)Q$$

$D(Q) = \sum_{P \in \rho^{-1}(Q)} n_P$, $n_P \in \mathbb{Z}$ with $|n_P| \geqq \mathfrak{m}(P)$ and $n_P n_{P'} > 0$,

$\forall P, P' \in \rho^{-1}(Q)$ if $Q \in \overline{S}$ and $D(Q) \neq 0$,

$D(Q) \in \mathbb{Z}$ if $Q \notin X_{\mathfrak{m}} \setminus \overline{S}$.

The number of points with $D(Q) \neq 0$ is finite.

$\widetilde{\mathrm{Div}}_{\mathfrak{m}}(X_{\mathfrak{m}})$: the group of all such divisors

$\forall f \in \mathrm{Mer}(X_{\mathfrak{m}})$, $f \neq 0$

$$(f) := \sum_{Q \in X_{\mathfrak{m}}} \mathrm{ord}_Q(f) Q \in \widetilde{\mathrm{Div}}_{\mathfrak{m}}(X_{\mathfrak{m}})$$

Definition 8. $D_1, D_2 \in \widetilde{\mathrm{Div}}_{\mathfrak{m}}(X_{\mathfrak{m}})$

$$D_1 \sim_{\mathfrak{m}} D_2 \quad \underset{\mathrm{def}}{\iff} \quad \exists f \in \mathrm{Mer}(X_{\mathfrak{m}}\ )\ s.t.\ D_1 - D_2 = (f)$$

Lemma 3. $\forall \widetilde{D} \in \widetilde{\mathrm{Div}}_{\mathfrak{m}}(X_{\mathfrak{m}})$

$\exists f \in \mathrm{Mer}(X_{\mathfrak{m}})$ *s.t.* $\widetilde{D} - (f) \in \mathrm{Div}(X_{\mathfrak{m}})$

Proof. Assume: $Q \in \overline{S}$, $M := \widetilde{D}(Q) \neq 0$

It suffices to consider the case $M > 0$.

$\rho^{-1}(Q) = \{P_1, \ldots, P_N\}$

$\forall P_i$, $\exists n_i \in \mathbb{N}$ with $n_i \geqq \mathfrak{m}(P_i)$ s.t. $M = \sum_{i=1}^{N} n_i$

z_i: a local coordinate at P_i

$r_i(z_i) := z_i^{n_i}$

$\forall P \in S \setminus \{P_1, \ldots, P_N\}$, $r_P(z_P) := 1 + z_P^{\mathfrak{m}(P)}$

z_P: a local coordinate at P

$\exists f \in \mathrm{Mer}(X)$ s.t.

$$\begin{cases} \mathrm{ord}_P(f - r_i) > n_i & \text{if } P = P_i \text{ for some } i = 1, \ldots, N, \\ \mathrm{ord}_P(f - r_P) > \mathfrak{m}(P) & \text{if } P \in S \setminus \{P_1, \ldots, P_N\} \end{cases}$$

$\exists g \in \mathrm{Mer}(X_{\mathfrak{m}})$ s.t. $f = \rho^* g$

$\widetilde{D} - (g) = 0$ at Q $\square$

$$\left[\widetilde{\mathrm{Div}}_{\mathfrak{m}}(X_{\mathfrak{m}})\right] := \widetilde{\mathrm{Div}}_{\mathfrak{m}}(X_{\mathfrak{m}}) / \sim_{\mathfrak{m}}$$

$$\widetilde{\mathrm{Div}}^0_{\mathfrak{m}}(X_{\mathfrak{m}}) := \{\widetilde{D} \in \widetilde{\mathrm{Div}}_{\mathfrak{m}}(X_{\mathfrak{m}}); \deg \widetilde{D} = 0\}$$

$$\left[\widetilde{\mathrm{Div}}^0_{\mathfrak{m}}(X_{\mathfrak{m}})\right] := \widetilde{\mathrm{Div}}^0_{\mathfrak{m}}(X_{\mathfrak{m}}) / \sim_{\mathfrak{m}}$$

Proposition 2.

$$\left[\widetilde{\mathrm{Div}}_{\mathfrak{m}}(X_{\mathfrak{m}})\right] \cong \overline{\mathrm{Div}(X_{\mathfrak{m}})}$$

$$\left[\widetilde{\mathrm{Div}}^{0}_{\mathfrak{m}}(X_{\mathfrak{m}})\right] \cong \overline{\mathrm{Div}^{0}(X_{\mathfrak{m}})}$$

9 Appendix

p.14 lines 5–6

Let $Q \in \overline{S}$

$\forall P \in \rho^{-1}(Q)$, $m(\geqq \mathfrak{m}(P))$: the multiplicity of F at P

$\exists t$: a local coordinate at c_Q

$\exists w$: a local coordinate at P s.t.

$$F \text{ is represented as } t = w^m.$$

$\exists h(w)$: meromorphic function in a nbd. of P s.t.

$$\rho^*\omega = h(w)dw \quad \text{and} \quad h(w) = \sum_{n \geqq -\mathfrak{m}(P)} c_n w^n.$$

By $dt = mw^{m-1}dw$, $\rho^*\omega = \dfrac{h(w)}{mw^{m-1}}dt$

$\zeta^i w$ $(i = 0, 1, \ldots, m-1)$: the preimages of $t = w^m$

$(\zeta = \exp(\sqrt{-1}\frac{2\pi}{m}))$

Then

$$\begin{aligned} &\sum_{i=0}^{m-1} \frac{h(\zeta^i w)}{mw^{m-1}} dt \\ &= \frac{1}{m} \sum_{n \geqq -\mathfrak{m}(P)} c_n \left(\sum_{i=0}^{m-1} \zeta^{i(n-m+1)} \right) w^{n-m+1} dt \end{aligned} \quad (*)$$

If $n-m+1 \neq km$, then $\sum_{i=0}^{m-1} \zeta^{i(n-m+1)} = 0$.

Since $n \geqq -\mathfrak{m}(P)$ and $m \geqq \mathfrak{m}(P)$, we have

$$(*) = \sum_{k \geqq 0} c_{km-1} t^{k-1} dt.$$

Noting $c_{-1} = \mathrm{Res}_P(\rho^*\omega)$, we obtain the expression of $\mathrm{Trace}(\rho^*\omega)$ at c_Q as follows:

$$\mathrm{Trace}(\rho^*\omega) = \left(\left(\sum_{P \in \rho^{-1}(Q)} \mathrm{Res}_P(\rho^*\omega) \right) \frac{1}{t} + \text{holomorphic part} \right) dt$$

$$\sum_{P \in \rho^{-1}(Q)} \mathrm{Res}_P(\rho^*\omega) = 0 \quad \text{for} \quad \omega \in H^0(X_{\mathfrak{m}}, \Omega_{\mathfrak{m}})$$

Then $\mathrm{Trace}(\rho^*\omega)$ is holomorphic at c_Q.