The period matrix of the hyperelliptic curve $w^2 = z^{2g+1} - 1$

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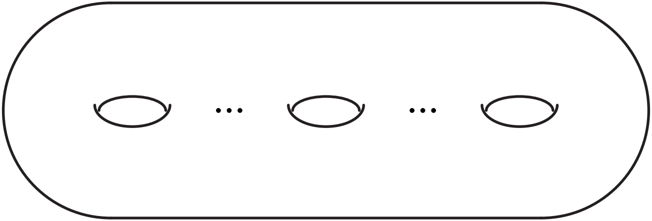
Overview

Riemann surface is an important object from analytic, algebraic, geometric, and topological viewpoints.

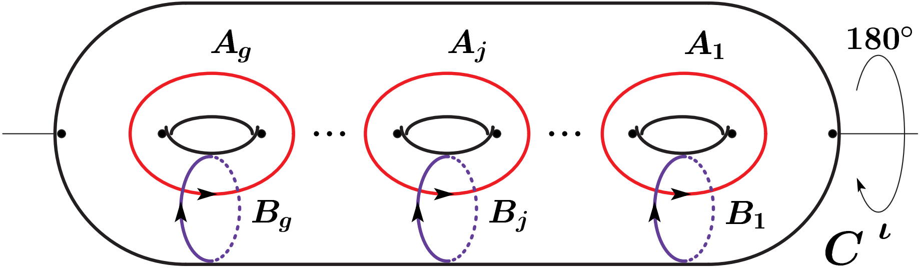


We put emphasis on a complex analytic invariant, **Period matrix**

Overview

First part

- C_g: hyperelliptic curve $w^2 = z^{2g+1} - 1$ of genus $g \geq 2$.
- $\{A_i, B_i\}_{i=1,\ldots,g} \subset H_1(C_g;\mathbb{Z})$: a fixed symplctic basis(natural type)



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- $\{A_i, B_i\}_{i=1,\dots,g} \subset H_1(C_g; \mathbb{Z})$: a fixed symplctic basis(natural type)
- τ_g: period matrix of C_g with respect to $\{A_i, B_i\}$
  A complex analytic invariant of Riemann surfaces

$\Longrightarrow$ **We explicitly determine τ_g**

Overview

Second part

- $X_{p,l,m}$: compact Riemann surface $w^p = z^l(1-z)^m$ of genus $g = (p-1)/2$.
- $F = F_N$: Fermat curve $w^N = 1 - z^N$ of genus $g = (N-1)(N-2)/2$.

$\Longrightarrow$ **We made a program which computes**

$$(p, l, m) \longrightarrow \text{period matrix of } X_{p,l,m}$$
$$N \longrightarrow \text{period matrix of } F$$

Definition of Period matrix

- X : a compact Riemann surface of genus $g \geq 1$
- $\{\omega_1, \ldots, \omega_g\}$: a basis of $H^{1,0}(X) \cong \mathbb{C}^g$
- $\{a_i, b_i\}_{i=1,\ldots,g}$:a symplectic basis of $H_1(X; \mathbb{Z})$
- $\Omega_A = \left(\int_{a_j} \omega_i\right), \ \Omega_B = \left(\int_{b_j} \omega_i\right)$: Periods

$$\tau_X := \Omega_A^{-1}\Omega_B \in M_g(\mathbb{C})$$

Properties of τ_X

- A complex analytic invariant of X.
- It depends only on the choice of a symplectic basis of $H_1(X;\mathbb{Z})$.
- It is symmetric and its imaginary part is positive definite.
  $\tau_X \in \mathcal{H}_g$: Siegel upper halfspace

period map $\varphi : \mathbb{M}_g \to \mathrm{Sp}_{2\mathrm{g}}(\mathbb{Z}) \backslash \mathcal{H}_g$

Motivation

1. Torelli's theorem
   - X, Y : compact Riemann surfaces of genus g
   - $J(X) = \mathbb{C}^g/(\mathbb{Z}^g + \tau_X \mathbb{C}^g)$: its Jacobian varieties

   $\boldsymbol{X \cong Y \Leftrightarrow J(X) \cong J(Y)}$ as p.p.a.v.

2. For generic genus, few examples of period matrices are known.
   The difficulty is in finding a symp. basis
   - only three types of hyperelliptic curves C
   - no examples of **nonhyperelliptic curves** (for generic genus)

   We are trying to compute these examples using our program.

Motivation

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    The difficulty is in finding a symp. basis
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    - no examples of **nonhyperelliptic curves** (for generic genus)

    We are trying to compute these examples using our program.

Schindler's results(1993)

Method: Action of $\mathrm{Aut}\, C$, $(z, w) \mapsto (\zeta z, w)$

① $\boldsymbol{C : \omega^2 = z^{2g+2} - 1}$

$(\zeta = \zeta_{2g+2} = \exp(2\pi\sqrt{-1}/(2g+2)))$

$$\tau_X = \left(\frac{1}{g+1} \sum_{k=1}^{g} \frac{\zeta^k(\zeta^{-2ik} - 1)(\zeta^{2kj} - 1)}{1 - \zeta^{2k}} \right)_{i,j}$$

Schindler's results(1993)

② $C \cong C_g : \omega^2 = z(z^{2g+1} - 1)$
$(\zeta = \zeta_{2g+1})$

Schindler's results(1993)

$$② \, \boldsymbol{C \cong C_g : \omega^2 = z(z^{2g+1} - 1)}$$

$(\zeta = \zeta_{2g+1})$

$$\begin{cases} t_1 = (-1)^g \zeta^{g^2}, \quad t_2 = t_1 \zeta / (1+\zeta), \\ t_{i+1} = t_1 \left(1 - \sum_{k=2}^{i} \zeta^{g-i+k-1} t_k t_{i-k+2} \right) \Big/ (1 + \zeta^{-i}) \end{cases}$$

Schindler's results(1993)

② $C \cong C_g : \omega^2 = z(z^{2g+1} - 1)$
$(\zeta = \zeta_{2g+1})$

Theorem ((i,j)-th entry of τ_g^S)

$$s_{i,j} = 1 - \sum_{k=1}^{i} t_k t_{j-i+k}/t_1$$

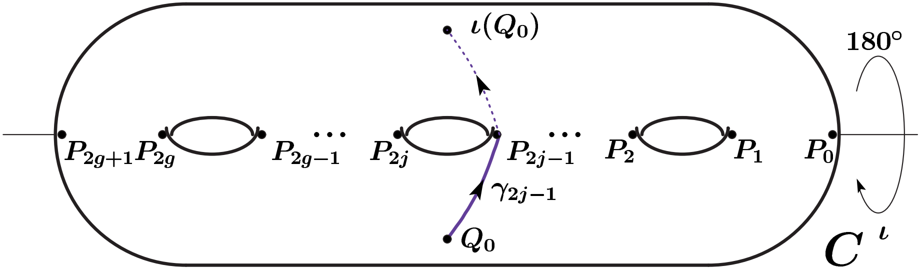
for $1 \le i \le j \le g$ *and* $s_{j,i}$ *for* $g \ge i > j \ge 1$.

recurrence expression

③ $C : \omega^2 = z(z^{2g} - 1)$ more complex expression

A symplectic basis of hyperelliptic curves

$C \xrightarrow{2:1} \mathbb{C}P^1$: a hyperelliptic curve, ι: its involution,
$\gamma_j : [0,1] \to C$: path from Q_0 to $\iota(Q_0)$

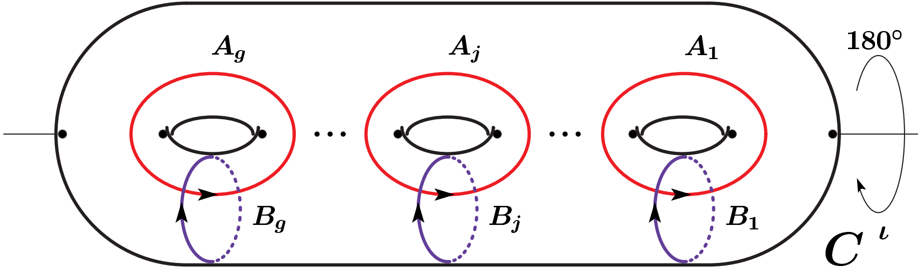


$$\begin{cases} A_i := \gamma_{2i-1} \cdot \gamma_{2i}^{-1} \\ B_i := \gamma_{2i-1} \cdot \gamma_{2i-2}^{-1} \cdot \cdots \cdot \gamma_1 \cdot \gamma_0^{-1} \end{cases}$$

A symplectic basis of hyperelliptic curves

$$\begin{cases} A_i & := & \gamma_{2i-1} \cdot \gamma_{2i}^{-1} \\ B_i & := & \gamma_{2i-1} \cdot \gamma_{2i-2}^{-1} \cdot \cdots \cdot \gamma_1 \cdot \gamma_0^{-1} \end{cases}$$

$\Rightarrow \{A_i, B_i\}_{i=1,2,\ldots,g}$: a symp. basis of $H_1(C;\mathbb{Z})$



Periods

- C_g: hyperelliptic curve $w^2 = z^{2g+1} - 1$ $(g \geq 2)$.
- $\{A_i, B_i\}_{i=1,\dots,g} \subset H_1(C_g;\mathbb{Z})$: the fixed symp. basis
- $\{\omega_i = z^{i-1}dz/w\}_{i=1,\dots,g} \subset H^{1,0}(C_g)$: a basis
- $\zeta := \zeta_{2g+1} = \exp(2\pi\sqrt{-1}/(2g+1))$
- τ_g: period matrix of C_g with respect to $\{A_i, B_i\}$

$\Omega_A = \left(\int_{A_j} \omega_i\right)$, $\Omega_B = \left(\int_{B_j} \omega_i\right)$ were obtained by Tashiro, Yamazaki, Ito, and Higuchi(1996).
Moreover $\det \Omega_A$ and $\det \Omega_B$ too.

Periods

Case: $g = 3$

$$\Omega_A =$$
$$\begin{pmatrix} 1-\zeta & 1-\zeta+\zeta^2-\zeta^3 & 1-\zeta+\zeta^2-\zeta^3+\zeta^4-\zeta^5 \\ 1-\zeta^2 & 1-\zeta^2+\zeta^4-\zeta^6 & 1-\zeta^2+\zeta^4-\zeta^6+\zeta^8-\zeta^{10} \\ 1-\zeta^3 & 1-\zeta^3+\zeta^6-\zeta^9 & 1-\zeta^3+\zeta^6-\zeta^9+\zeta^{12}-\zeta^{15} \end{pmatrix}$$

$$\Omega_B =$$
$$\begin{pmatrix} 1-\zeta^2 & 1-\zeta+\zeta^2-\zeta^4 & 1-\zeta+\zeta^2-\zeta^3+\zeta^4-\zeta^6 \\ 1-\zeta^4 & 1-\zeta^2+\zeta^4-\zeta^8 & 1-\zeta^2+\zeta^4-\zeta^6+\zeta^8-\zeta^{12} \\ 1-\zeta^6 & 1-\zeta^3+\zeta^6-\zeta^{12} & 1-\zeta^3+\zeta^6-\zeta^9+\zeta^{12}-\zeta^{18} \end{pmatrix}$$

Periods

$$\Omega_A =$$

$$\begin{pmatrix} 1-\zeta & 1-\zeta+\zeta^2-\zeta^3 & 1-\zeta+\zeta^2-\zeta^3+\zeta^4-\zeta^5 \\ 1-\zeta^2 & 1-\zeta^2+\zeta^4-\zeta^6 & 1-\zeta^2+\zeta^4-\zeta^6+\zeta^8-\zeta^{10} \\ 1-\zeta^3 & 1-\zeta^3+\zeta^6-\zeta^9 & 1-\zeta^3+\zeta^6-\zeta^9+\zeta^{12}-\zeta^{15} \end{pmatrix}$$

$$= \begin{pmatrix} -1+\zeta & & \\ & -1+\zeta^2 & \\ & & -1+\zeta^3 \end{pmatrix}$$

$$\begin{pmatrix} \zeta & & \\ & \zeta^2 & \\ & & \zeta^3 \end{pmatrix} \begin{pmatrix} 1 & \zeta^2 & \zeta^4 \\ 1 & \zeta^4 & \zeta^8 \\ 1 & \zeta^6 & \zeta^{12} \end{pmatrix}$$

Key lemma(Knuth's book)

- $a_1, \ldots, a_n$: distinct complex constants.
- $V_n = \left(a_i^{j-1}\right)_{i,j}$: A Vandermonde matrix.
- $\sigma_i(a_1, a_2, \ldots, a_n)$: i-th symmetric polynomial.

$$\Rightarrow V_n^{-1} = \left((-1)^{i-1} \frac{\sigma_{n-i}(a_1, \ldots, \widehat{a_j}, \ldots, a_n)}{\prod_{m=1, m\neq j}^{n} (a_m - a_j)} \right)_{i,j}$$

Key lemma(Knuth's book)

Case: $n = 3$

$$\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{bc}{(b-a)(c-a)} & -\frac{ac}{(a-b)(c-b)} & \frac{ab}{(a-c)(b-c)} \\ \frac{b+c}{(b-a)(c-a)} & -\frac{a+c}{(a-b)(c-b)} & \frac{a+b}{(a-c)(b-c)} \\ \frac{1}{(b-a)(c-a)} & -\frac{1}{(a-b)(c-b)} & \frac{1}{(a-c)(b-c)} \end{pmatrix}$$

Result(2014)

Theorem ((i,j)-th entry of τ_g)

$$\sum_{k=1}^{g} \frac{(-1)^{i+g}}{2g+1}(1-\zeta^{2kj})\sigma_{g-i}(\zeta^2,\ldots,\widehat{\zeta^{2j}},\ldots,\zeta^{2g}) \prod_{m=g-k+1}^{2g-k}(1-\zeta^{2m})$$

Result(2014)

$$\tau_3 = \begin{pmatrix} -\zeta^5 & -2-\zeta^2-\zeta^4-\zeta^5 & \zeta+\zeta^3+\zeta^5 \\ -2-\zeta^2-\zeta^4-\zeta^5 & \zeta+2\zeta^3-\zeta^4+\zeta^5 & 1+\zeta^2+\zeta^3+\zeta^5 \\ \zeta+\zeta^3+\zeta^5 & 1+\zeta^2+\zeta^3+\zeta^5 & \zeta^2 \end{pmatrix}$$

Result(2014)

A relation between Schindler's result and τ_g

$$L_g = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & \iddots & & \\ -1 & & & \end{pmatrix} \in M_g(\mathbb{Z})$$

$$\Rightarrow \tau_g^S = L_g \tau_g L_g$$

∵) See the symplectic basis for Schindler's period matrix

Algorithms and programs

Algorithm

- Tretkoff and Tretkoff
  Hurwitz system and Frobenius method
- Kamata $\subset$ T.T. **for Fermat type curves**
- Ours $\subset$ T.T. **Chord slide method for $X_{p,l,m}$**

Programs

Ours	Maple `algcurves`
$X_{p,l,m}$	$f(x, y) = 0$
$\mathbb{Q}(\zeta)$	Approximate value
elementary	complex

A program

- p : prime, $0 < l, m < p-1$: coprime
- $X_{p,l,m} := \{w^p = z^l(1-z)^m\}$:
  a compact Riemann surface of $g = (p-1)/2$.
- π : $X_{p,l,m} \ni (z,w) \mapsto z \in \mathbb{C}P^1$:
  p-cyclic covering branched over $0, 1, \infty \subset \mathbb{C}P^1$

Using "Chord Slide Method(CSM)", we obtain a geometric algorithm for finding symp. basis of $X_{p,l,m}$'s.

A program

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Using "Chord Slide Method(CSM)", we obtain a mathematica program for calculating period matrices of $X_{p,l,m}$'s.

Demonstration

- $X_{p,l,m}$: compact Riemann surface $w^p = z^l(1-z)^m$ of genus $g = (p-1)/2$.

$\Longrightarrow$ **We made a program which computes**

$$(p, l, m) \longrightarrow \text{period matrix of } X_{p,l,m}$$

Intersection matrix(Outline)

- $\sigma(z,w) = (z, \zeta w)$: automorphism with order p.
- Define $c_i : [0,1] \to X_{p,l,m}$ $(i = 1, 2, \ldots, 2g)$ paths

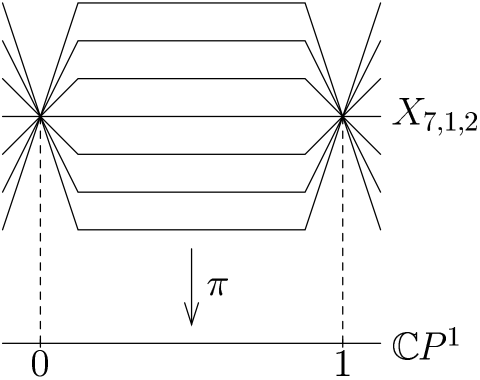
$\Rightarrow A = (c_i \cdot c_j)$ intersection matrix

1. p-cyclic covering of $\mathbb{C}P^1$
2. Dessin d'enfants
3. Chord diagram on S^1

Sample: $X_{7,1,2} = \{w^7 = z(1-z)^2\}$: Klein quartic
$K_4 := \{X^3Y + Y^3Z + Z^3X = 0\} \subset \mathbb{C}P^2$
$(z = X^3Y^{-2}Z^{-1} + 1,\ w = -XY^{-1})$

Intersection matrix(Details)

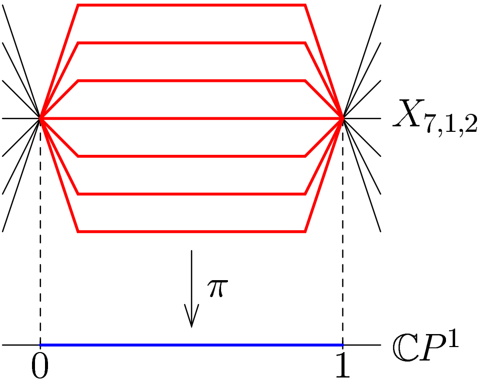
p-cyclic covering of $\mathbb{C}P^1$→DD→CD



$\pi : X_{p,l,m} \ni (z, w) \mapsto z \in \mathbb{C}P^1$, p-cyclic covering

Intersection matrix(Details)

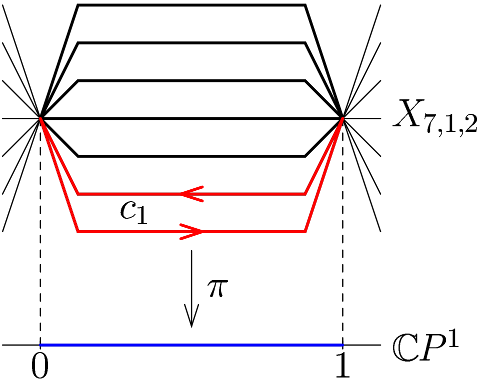
p-cyclic covering of $\mathbb{C}P^1$→DD→CD



lifts of $[0, 1] \subset \mathbb{C}P^1$

Intersection matrix(Details)

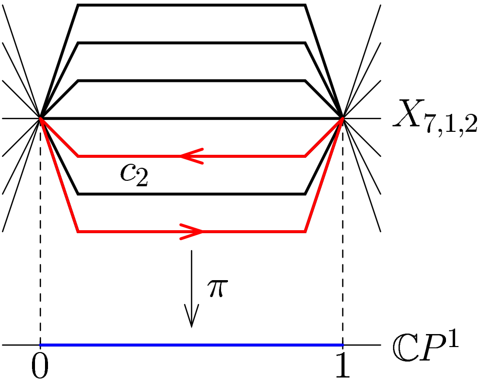
p-cyclic covering of $\mathbb{C}P^1$→DD→CD



$$c_1 = I_0 \cdot \sigma(I_0)^{-1}$$

Intersection matrix(Details)

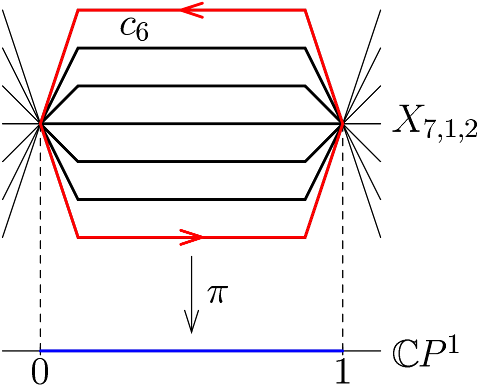
p-cyclic covering of $\mathbb{C}P^1$→DD→CD



$c_2 = I_0 \cdot \sigma^2(I_0)^{-1}$

Intersection matrix(Details)

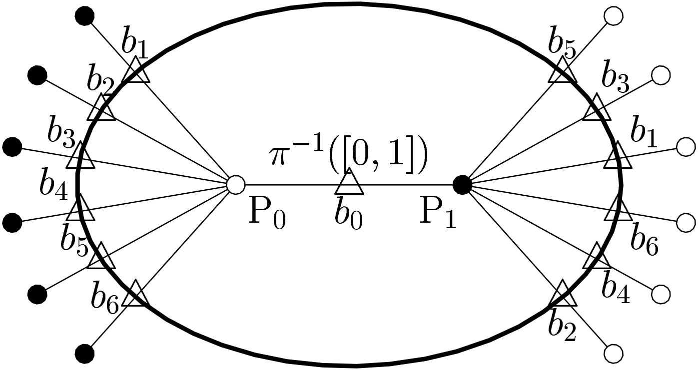
p-cyclic covering of $\mathbb{C}P^1$→DD→CD



$$c_6 = I_0 \cdot \sigma^6 (I_0)^{-1}$$

Intersection matrix(Details)

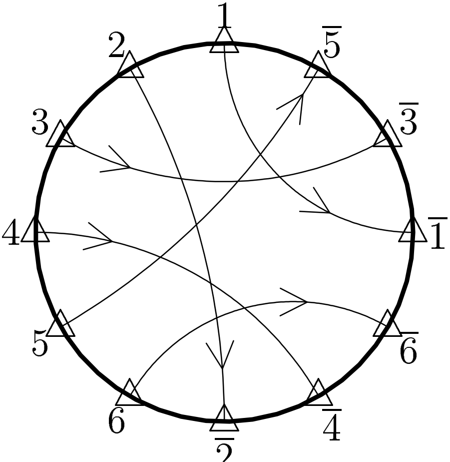
CC→Dessin d'enfants→CD



A dessin d'enfants of $C_{7,2}$

Intersection matrix(Details)

CC→DD→Chord diagram



A chord diagram of $C_{7,2}$

Intersection matrix(Details)

We obtain the intersection matrix $A = (c_i \cdot c_j)$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

Chord diagram methods

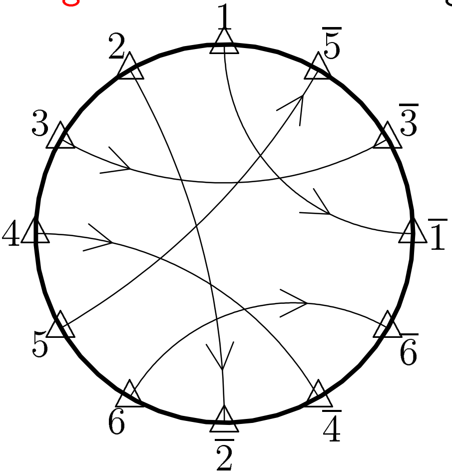
Find $T \in M_{2g}(\mathbb{Z})$ s.t. $TA^tT = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

Then, we have a symplectic basis

$$(a_1, \ldots, a_g, b_1, \ldots, b_g) = (c_1, c_2, \ldots, c_{2g})^tT$$

Chord diagram methods

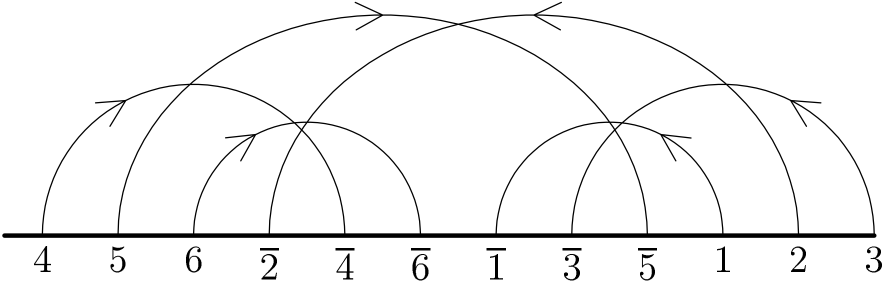
Chord diagram→Linear Chord Diagrams



A chord diagram of $C_{7,2}$

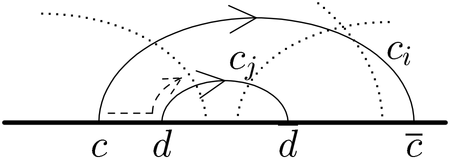
Chord diagram methods

Chord diagram→Linear Chord Diagrams



A linear chord diagram of $C_{7,2}$

Chord diagram methods



CSM(Base Change)

$$c'_k = \begin{cases} c_i - c_j & (k = i), \\ c_k & (k \neq i). \end{cases}$$

The advantage of CSM is its applicability to other curves for generic genus. In fact, we obtain another symplectic basis of C_g different to $\{A_i, B_i\}$.

Chord diagram methods

$$T \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$TA^tT \qquad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

A basis of holomorphic 1-forms

- $\alpha_l = \lfloor nl/p \rfloor$, $\alpha_m = \lfloor nm/p \rfloor$
- $d_n = \lfloor n(l+m)/p \rfloor - \alpha_l - \alpha_m - 1$
- $\omega_{n,d} = z^{\alpha_l}(1-z)^{\alpha_m} z^d dz/w^n$
- $S := \{(n,d) \colon \; 0 \leq d \leq d_n$ and $1 \leq n \leq p-1\}$

Theorem (Bennama(1998))

$\{\omega_{n,d}\}_{(n,d)\in S}$: *a basis of* $H^{1,0}(X)$

(n,d)	$(3,0)$	$(5,0)$	$(6,0)$
$\omega_{n,d}$	dz/w^3	$(1-z)dz/w^5$	$(1-z)dz/w^6$

Periods

- $\Omega_A = \left(\int_{a_j} \omega_i\right)$, $\Omega_B = \left(\int_{b_j} \omega_i\right)$: Periods

$$\Omega_A = \begin{pmatrix} 1-\zeta & \zeta-\zeta^2 & 1-\zeta^2+\zeta^3-\zeta^5 \\ 1-\zeta^2 & \zeta^2-\zeta^4 & 1-\zeta^4+\zeta^6-\zeta^{10} \\ 1-\zeta^4 & \zeta^4-\zeta^8 & 1-\zeta^8+\zeta^{12}-\zeta^{20} \end{pmatrix}$$

$$\Omega_B = \begin{pmatrix} 1-\zeta^3 & 1-\zeta^4 & \zeta-\zeta^2+\zeta^4-\zeta^6 \\ 1-\zeta^6 & 1-\zeta^8 & \zeta^2-\zeta^4+\zeta^8-\zeta^{12} \\ 1-\zeta^{12} & 1-\zeta^{16} & \zeta^4-\zeta^8+\zeta^{16}-\zeta^{24} \end{pmatrix}$$

Period matrix

- $\tau_g = \Omega_A^{-1}\Omega_B$
- $A^{-1} = \underbrace{\frac{1}{\det A}}_{\text{Euclidean Algorithm}} \operatorname{adj} A$

$$\tau_g = \begin{pmatrix} 6+3\xi & 4+2\xi & -2-\xi \\ 4+2\xi & 4+4\xi & -2\xi \\ -2-\xi & -2\xi & 2+3\xi \end{pmatrix}$$

where $\zeta = \zeta_7$ and $\xi = \zeta + \zeta^2 + \zeta^4 = (-1+\sqrt{-7})/2$.

Summary

- We explicitly determine τ_g by the affine equation $w^2 = z^{2g+1} - 1$, its entries being elements of the $\mathbb{Q}(\zeta_{2g+1})$
- We made a program which computes

$$(p, l, m) \longrightarrow \text{period matrix of } X$$

Summary

- We explicitly determine τ_g by the affine equation $w^2 = z^{2g+1} - 1$, its entries being elements of the $\mathbb{Q}(\zeta_{2g+1})$
- We made a program which computes

$$(p, l, m) \longrightarrow \text{period matrix of } X$$

Find an explicit expression of period matrices of other curves for generic genus!!

Summary

Thank you very much!
ありがとうございました!